# Lebesgue Integration: A non-rigorous introduction

## What is wrong with Riemann integration?

**Example.** Let $f(x) = \begin{cases} 0 & \text{for} \quad x \in \mathbb{Q} \\ 1 & \text{for} \quad x \notin \mathbb{Q}. \end{cases}$

The upper integral is 1, while the lower integral is 0. Yet, the function equals 1 almost everywhere. Shouldn't the integral be 1 rather than be undefined? Remember Riemann integrable functions that differ only on a set of measure zero have equal integrals. Why should this function be nonintegrable?

An approach to this problem is called **Lebesgue integration**. The set of Lebesgue integrable functions will turn out to be larger than the Riemann integrable functions. In fact, although we won't show this here, they turn out to be the completion of the Riemann integrable functions in the same sense that $\mathbb{R}$ is the completion of $\mathbb{Q}$.

The basic idea is to partition the $y$-axis instead of the $x$-axis. This requires studying the sets in the domain, the $x$-axis, that can be realized as inverse images of intervals in the $y$-axis.

# Size Matters

**Definition.** Let $E \subset \mathbb{R}$ be the disjoint union of $E_1, \ldots, E_n$. Let $a_1, \ldots, a_n$ be real numbers. Let

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i}.$$

Functions that can be defined this way are called **simple functions**. (There is an additional criteria that I will mention later.)

We want to define

$$\int_E \phi = \sum_{i=1}^{n} a_i \text{ size } (E_i).$$

But what do we mean by “size”?

**Example.** If we take $\text{size}\big([0,1] \cap \mathbb{Q}\big) = 0$ and $\text{size}\big([0,1] \cap (\mathbb{R} - \mathbb{Q})\big) = 1$, then

$$\int_{[0,1]} \chi_{\mathbb{R}-\mathbb{Q}} = 1.$$

It will turn out that limits of simple functions behave well under this integral and we can define Lebesgue integration as a limit. But, first we have to get real about what “size” should mean.

Ideally, we want a mapping $m$ from subsets of $\mathbb{R}$ to $[0, \infty]$ such that the following hold.

o. $m(E)$ is defined for all $E \subset \mathbb{R}$.
i. $m([a,b]) = m((a,b]) = m([a,b)) = m((a,b)) = b - a$.
ii. If $\{E_i\}_{i=1}^{\infty}$ is disjoint then $m(\cup E_i) = \sum m(E_i)$.
iii. If $E \subset \mathbb{R}$, $a \in \mathbb{R}$ and $E_a = \{x + a \,|\, x \in \mathbb{R}\}$, then $m(E) = m(E_a)$.

But, it is known that there is no such function! It is turns out that (o) is usually dropped, that is, we will have to accept the existence of **nonmeasurable sets**[1]. Criteria (ii) is called **countable additivity**. Criteria (iii) is **translation invariance** and is sometimes dropped.

The collection of subsets of $\mathbb{R}$ that we will be able to measure will be referred to a the **measurable** subsets and denoted $\mathcal{M}$. We still want $\mathcal{M}$ to be a large as possible and to include all reasonable subsets. This leads to the idea of a $\sigma$-algebra.

**Definition.** A collection of subsets $\mathcal{A}$ of a set $X$ is called a **$\sigma$-algebra** if the following hold.

a. $X \in \mathcal{A}$.
b. $S \in \mathcal{A} \implies X - S \in \mathcal{A}$.
c. $S_n \in \mathcal{A} \implies \cup S_n \in \mathcal{A}$.

It follows that $\emptyset \in \mathcal{A}$ and that $\mathcal{A}$ is closed under countable intersections.

**Fact.** If $\mathcal{F}$ is any collection of subsets of $X$, there exists a smallest $\sigma$-algebra that contains $\mathcal{F}$.

**Definition.** Let $(X, \mathcal{T})$ be a topological space. The smallest $\sigma$-algebra containing $\mathcal{T}$ is called the **Borel $\sigma$-algebra** of $(X, \mathcal{T})$ and is usually denoted $\mathcal{B}$.

**Note.** A set is a $G_\delta$ set if it is a countable intersection of open sets. A set is an $F_\sigma$ set if it is a countable union of closes sets. All $G_\delta$ and $F_\sigma$ sets are contained in the Borel sets.

We want our measurable sets to contain the Borel sets.

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[1]See Royden, Chapter 3, Section 4. The construction uses the Axiom of Choice.

# The Outer Measure

To define our measure on $\mathbb{R}$ we first define the **outer measure**.

**Definition.** Let $A \subset \mathbb{R}$. Let $\mathcal{C}$ be the collection of all countable collections of open intervals that cover $A$. That is $\{I_n\} \in \mathcal{C}$ means $A \subset \cup I_n$. The **outer measure** of $A$ is

$$m^*(A) = \inf_{\mathcal{C}} \sum l(I_n),$$

where $l(I_n)$ is the Euclidean length of $I_n$, and infinity is allowed.

**Fact.** (o), (i), & (iii) hold.

It can also be shown that $m^*$ is **countably subadditive**, meaning if $\{E_n\}$ is a countable collection of disjoint sets, then

$$m^*(\cup E_n) \leq \sum m^*(E_n).$$

However, there are examples where equality fails. Thus (ii) does not hold.

To get around this we throw out some "bad" subsets of $\mathbb{R}$ and dismiss them as "nonmeasurable".

# Lebesgue Measure

**Definition.** Let $E \subset \mathbb{R}$. Then we say $E$ is **measurable** if for every $X \subset \mathbb{R}$ we have

$$m^*(X) = m^*(E \cap X) + m^*(E^c \cap X).$$

The collection of measurable subsets of $\mathbb{R}$ will be denoted by $\mathcal{M}$. We define $m$ to be the restriction of $m^*$ to $\mathcal{M}$. It is called the **Lebesgue measure**.

This may seem a bit odd at first, but think of it this way. If $E$ is a reasonable set, then if we use it to partition other sets into two pieces, the measure of the pieces should equal the measure of the whole.

**Facts.** $\mathcal{M}$ is a $\sigma$-algebra that contains all the Borel sets and $m$ satisfies (i), (ii) & (iii), but (o) fails.

$m(\emptyset) = 0$.

If $m^*(A) = 0$ then $A \in \mathcal{M}$.

If $A \subset B$ then $m^*(A) \leq m^*(B)$ and if both are measurable $m(A) \leq m(B)$.

**Theorem.** For $E \subset \mathbb{R}$ the following are equivalent.

a. $E \in \mathcal{M}$.
b. $\forall \epsilon > 0$, $\exists$ an open set $O$, such that $E \subset O$ and $m^*(O - E) < \epsilon$.
c. $\forall \epsilon > 0$, $\exists$ an closed set $C$, such that $C \subset E$ and $m^*(E - C) < \epsilon$.
d. $\exists$ a $G_\delta$ set $G$, such that $E \subset G$ and $m^*(G - E) = 0$.
e. $\exists$ a $F_\sigma$ set $F$, such that $F \subset E$ and $m^*(E - F) = 0$.

# Measurable Functions

**Definition.** Let $X$ be any set with $\mathcal{V}$ a $\sigma$-algebra for some measure $\mu$ (meaning $\mu : \mathcal{V} \to [0, \infty]$ and i, ii & iii hold). Let $Y$ be a topological space and $f : X \to Y$. Then $f$ is a **measurable function** provided for any open set $U \subset Y$ we have $f^{-1}(U) \in \mathcal{V}$. In words, $f$ inverse maps the topology of $Y$ into the measurable sets of $X$.

**Theorem.** Let $D \subset \mathbb{R}$ be measurable. Let $f : D \to \mathbb{R}$. If $f$ is measurable all the classes of set below are measurable and if any one of these is measurable, then $f$ is measurable.

$$f^{-1}((a, \infty)),\ \ f^{-1}([a, \infty)),\ \ f^{-1}((-\infty, a)),\ \ f^{-1}((-\infty, a]),$$
$$f^{-1}((a, b)),\ \ f^{-1}([a, b)),\ \ f^{-1}((a, b]),\ \ f^{-1}([a, b])$$

**Theorem.** Let $D \subset \mathbb{R}$. If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are measurable then $f + g$ and $f \cdot g$ are measurable. Constant functions are measurable.

**Theorem.** Let $D \subset \mathbb{R}$. If $f : D \to \mathbb{R}$ is measurable and $f = g$, a.e., then $g$ is measurable.

**Theorem.** If $E \in \mathcal{M}$, then $\chi_E$ is a measurable function.

# From Simple Functions to the Lebesgue Integral

We redefine simple functions. Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, where $E_1, \ldots, E_n$ are disjoint, but now we require $a_i \geq 0$ and, more importantly, each $E_i \in \mathcal{M}$. Let $E \in \mathcal{M}$. Then these are called the simple functions. Define

$$\int_E \phi = \sum_{i=1}^{n} a_i m(E \cap E_i).$$

Let $f : E \to \mathbb{R}$ be bounded, nonnegative, $E \in \mathcal{M}$ and $m(E) < \infty$. Consider the two numbers,

$$\inf_{f \leq \phi} \int_E \phi \qquad \& \qquad \sup_{f \geq \phi} \int_E \phi,$$

where the infimum and supremum are taken over all simple functions that are greater than or equal to $f$ or less than or equal to $f$, respectively. These are analogous to the upper and lower integrals we used in defining the Darboux integral. (But, $\pm\infty$ are allowed.)

When these two numbers are equal they are called the **Lebesgue integral** of $f$. This holds if and only if $f$ is a measurable function.

We can generalize to functions that aren't nonnegative as follows. Let $f : E \to \mathbb{R}$ be bounded, $m(E) < \infty$. Let

$$f^+(x) = \max\{f(x), 0\} \qquad \& \qquad f^-(x) = \max\{-f(x), 0\}.$$

It can be shown these are measurable and obviously $f = f^+ - f^-$. Define $\int_E f = \int_E f^+ - \int_E f^-$.

All Riemann integrable functions are Lebesgue integrable and the integrals are equal. We use the same symbol for both when no confusion will arise.

# Unbounded Joy

Let $f : E \to [0, \infty)$, where $E \in \mathcal{M}$. Let $H$ be the set of nonnegative bounded measurable functions from $E$ to $\mathbb{R}$ that are less than or equal to $f$, and such that $m\big(\{x \,|\, h(x) \neq 0\}\big) < \infty$. Thus for $h \in H$ the integral $\int_E h$ is defined. Then we define the Lebesgue integral of $f$ over $E$ by

$$\int_E f = \sup \left\{ \int_E h \ \middle|\, h \in H \right\}.$$

This can be extended to not necessarily nonnegative functions as before.

We say $f : E \to \mathbb{R}$ is **Lebesgue integrable** if $f$ is measurable and $\int_E |f| < \infty$. Let $L^1(E)$ be the set of Lebesgue integrable functions over $E$. It is a vector space under addition of functions and multiplication by real constants. The Lebesgue integral is linear as a map from $L^1(E)$ to $\mathbb{R}$.

We also have the following. Assume $A, B, E$ are measurable and $f$ and $g$ are $L^1$ functions.

- If $A \cap B = \emptyset$, then $\int_{A \cup B} f = \int_A f + \int_B f$.
- If $f = g$ a.e., then $\int_E f = \int_E g$.
- If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.
- If $f$ is nonnegative and $\int_E f = 0$, then $f = 0$ a.e.
- If $f$ is nonnegative and $A \subset B$, then $\int_A f \leq \int_B f$.

# Limit Theorems

Assume $E$ is a measurable set and all functions below are from $E$ to $\mathbb{R}$.

**Fatou's Lemma.** Let $(f_n)$ be a sequence of nonnegative measurable functions and suppose $f_n(x) \to f(x)$ for a.e. $x \in E$. Then

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

**The Monotone Convergence Theorem.** Let $(f_n)$ be an *increasing* sequence of nonnegative measurable functions and suppose $f_n(x) \to f(x)$ for a.e. $x \in E$. Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

**Corollary.** Let $(f_n)$ be an sequence of nonnegative measurable functions and suppose $f_n(x) \to f(x)$ for a.e. $x \in E$. If $f_n \leq f$ on $E$ for each $n$ then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

**The Dominated Convergence Theorem.** Let $g$ be integrable over $E$. (It need not be bounded.) Let $(f_n)$ be an sequence of measurable functions and suppose $f_n(x) \to f(x)$ for a.e. $x \in E$. If $|f_n(x)| \leq g(x)$ for a.e. $x \in E$, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

**Example.** Let $(q_i)$ be an enumeration of $[0,1] \cap \mathbb{Q}$. Let $f_i = \chi_{q_i}$ and $g_k = \sum_{i=1}^{k} f_i$. Then on [0,1], $g_k \to \chi_{\mathbb{Q}}$. For Lebesgue integration we have

$$\int_{[0,1]} g_k = 0 \to 0 = \int_{[0,1]} \chi_{\mathbb{Q}}.$$

This fails for Riemann integration since $\int_{[0,1]} \chi_{\mathbb{Q}}$ does not exist.

# The Fundamental Theorem of Calculus

The FTC takes the following form.

**Definition.** Let $g : [a, b] \to \mathbb{R}$. Then we say $g$ is **absolutely continuous** if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that whenever $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are disjoint intervals in $[a, b]$ we have

$$\sum \beta_i - \alpha_i < \delta \implies \sum g(\beta_i) - g(\alpha_i) < \epsilon.$$

**Fact.**

$$\text{Absolute continuity} \implies \text{uniform continuity} \implies \text{continuity}$$

but the reverse implications are false. The Devil's staircase function is uniformly continuous on [0,1] but can be shown to not be absolutely continuous.

**FTC.** Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable. Then

a. $F(x) = \int_{[a,x]} f$ is absolutely continuous.
b. $F'(x) = f(x)$ for a.e. $x \in [a, b]$.
c. If $G : [a, b] \to \mathbb{R}$ is absolutely continuous and $G'(x) = f(x)$ for a.e. $x \in [a, b]$, then $F = G + C$, a constant, a.e.

**Fact.** $F(x)$ is of the form $\int_{[a,x]} f$ iff $F(x)$ is absolutely continuous.

# The $L^p$ spaces

Let $p > 0$ and $E \subset \mathbb{R}$ be measurable. (Usually, $E$ is a compact interval.) Define

$$L^p(E) = \left\{ f \in \mathcal{M} \,\middle|\, \int_E |f|^p < \infty \right\}.$$

It is easy to show that $L^p$ is a vector space. It is given the norm

$$||f||_p = \sqrt[p]{\int_E |f|^p}.$$

We define $L^\infty(E) =$ all a.e. bounded, measurable functions on $E$ and norm

$$||f||_\infty = \inf \{ M \in \mathbb{R} \mid m\{x \in E \mid f(x) > M\} = 0 \} .$$

It is a normed vector space.

**The Minkowski Inequalities.**

For $0 < p < 1$ we have $||f + g||_p \geq ||f||_p + ||g||_p$.
For $1 \leq p \leq \infty$ we have $||f + g||_p \leq ||f||_p + ||g||_p$.

**Holder's Inequality.** Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Let $f \in L^p$ and $g \in L^q$. Then $f \cdot g \in L^1$ and

$$\int_E |f \cdot g| \leq ||f||_p ||g||_q.$$

**Note.** In the $L^p$ spaces we usually declare $f \sim g$ whenever $f = g$ a.e. Thus, we are really working with equivalence classes of functions.

**Definition.** Let $(X, ||\cdot||)$ be a normed vector space. A map $F : X \to \mathbb{R}$ is called a **linear functional** if it is linear. A linear functional is **bounded** if for some $\mu \geq 0$ we have

$$|F(f)| \leq \mu ||f||, \forall f \in X.$$

Define

$$||F|| = \sup_{f \in X, ||f|| \neq 0} \frac{|F(f)|}{||f||}.$$

For every $g \in L^q(E)$ define $G : L^p(E) \to \mathbb{R}$ by $G(f) = \int_E f \cdot g$. Then $||G|| = ||g||_q$.

**Riesz Representation Theorem.** Let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. Let $G : L^p(E) \to \mathbb{R}$ be a bounded linear functional. Then $\exists!$ $g \in L_q(E)$ such that

$$G(f) = \int_E f \cdot g \qquad \& \qquad ||G|| = ||g||_q.$$

## Convergence in Mean and Completeness

**Definition.** Let $(X, ||\cdot||)$ be a normed vector space. Let $f \in X$. If $(f_n)$ is a sequence in $X$ and $||f - f_n|| \to 0$ then we say $(f_n)$ **converges in the norm** to $f$. This is different from point-wise convergence. If all Cauchy sequences converge in the norm we say $X$ is *complete.* A complete, normed vector space is call a **Banach space**.

**The Facts.** The $L^p$ spaces for $p \geq 1$ are complete. Here convergence in norm is often called **convergence in the mean of order p**. The space $L^1$ is the completion of the subset of Riemann integrable functions. The Lebesgue integral is the unique extension of the Riemann integral – in a certain sense.