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SIMPLE SMALE FLOWS ON S^3 USING TWISTED 4-BAND TEMPLATES

ANTHONY SLOAN AND MICHAEL SULLIVAN

Abstract. We investigate the linking structures of attractor - repeller pairs in simple Smale flows on S^3 in which the chaotic saddle sets are modeled by four-band templates with twisted bands. We obtain new theorems which illustrate that the dynamics of simple Smale flows are sensitive to half-twists in the bands of the embedded template. Haynes and Sullivan showed that the attractor-repeller pair $a \cup r$ in a simple Smale flow with chaotic saddle set modeled by embedded template U^+ is either a Hopf link or a trefoil and meridian. We modify template U^+ by placing half-twists in selected bands. For simple Smale flows on S^3 with chaotic saddle sets modeled by these modified templates, we find that such simple Smale flows are realizable and that $a \cup r$ must be a Hopf link, a figure-8 knot and meridian, a trefoil and meridian, or a cinquefoil and meridian.

1. Introduction

We study simple Smale flows on the 3-sphere S^3 whose chaotic saddle sets are modeled by templates with twisted bands. A simple Smale flow is a structurally stable flow with a one dimensional chain recurrent set whose stable and unstable manifolds satisfy a transversality conditon. The chain recurrent set decomposes into three disjoint basic sets: an attracting closed orbit, a repelling closed orbit, and a chaotic saddle set consisting of infinitely many orbits. The chaotic saddle set is modeled by a compact two dimensional branched manifold with semi-flow which is known as a template. J. Birman and R. Williams [3, 4] studied knotted

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periodic orbits in dynamical systems and developed the theory of templates. The possibly knotted attracting and repelling closed orbits are embeddings of the circle S^1 in the three dimensional sphere S^3.

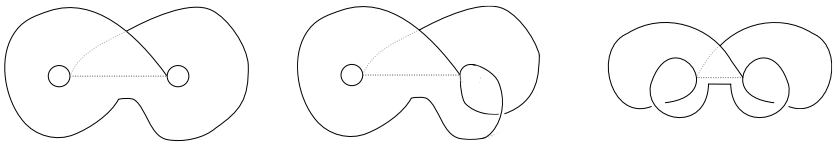
The aim of this paper is to explore the linking structure of a pair of knots $a \cup r$ where a is the attracting closed orbit and r is the repelling closed orbit of a simple Smale flow in S^3. In particular, we determine precisely the knot types of a and r, how they are linked, and how they are embedded in S^3. In section three, we prove two a new theorem that characterizes the linking structure of $a \cup r$ in a realizable simple Smale flow on S^3 whose chaotic saddle set is modeled by four-band template with a half-twist placed in single band.

The second author [17] initiated a program of visualizing simple Smale flows. In that paper, Sullivan studied simple Smale flows on S^3 with chaotic saddle set modeled by the two-band Lorenz template. The flows are called Lorenz-Smale flows. In particular, Sullivan proved the following theorem classifying all configurations of a Lorenz-Smale flow in S^3 which are realizable.

Theorem 1.1. (Sullivan [17]) *For a Lorenz-Smale flow in S^3 the following and only the following configurations are realizable. The link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the saddle set is modeled by a standardly embedded Lorenz template, i.e. both bands are unknotted, untwisted, and unlinked, with the core of each band a meridian of the trefoil component of $a \cup r$. In the former case, there are three possibilities: (1) The saddle set is standardly embedded. (2) One band is twisted with n full-twists for any n, but remains unknotted and unlinked to the other band, which must be unknotted and untwisted. (3) One band is a (p, q) torus knot, for any pair of coprime integers, with twist $p + q - 1$. The other band is unknotted, untwisted, and unlinked to the knotted one.*

Prior to Sullivan's work, Franks [5] classified nonsingular Smale flows on S^3 using the notion of a Lyapunov graph. However, in that scheme, (as pointed out in [17]) one cannot see the Smale flows as there are no explicit details concerning the embeddings of the basic sets of the flow. Wada [18] has given a complete classification of the simpler Morse-Smale flows on S^3 which does give explicit details regarding the embeddings of the basic sets of the flow.

A complete classification of non-singular Smale flows on S^3 seems untenable at this time. Even classifying simple Smale flows on S^3 is daunting. Sullivan [15] has conjectured that any two knots K_1 and K_2 can be realized as the attractor-repeller pair in a simple Smale flow on S^3. Since Sullivan's seminal paper on visualizing Smale flows, many related results have appeared. Bin Yu [19] studied Lorenz-like Smale flows in



(A) $L(0,0)$ (B) $L(0,1)$ (C) $L(1,1)$

FIGURE 1. Lorenz Template (A) and Lorenz-like Templates (B) and (C)

which the chaotic saddle set was modeled by a Lorenz-like template. A Lorenz-like template is a two-band template in which one or both bands have a half-twist (see Figure 1). Yu proved the following two theorems.

Theorem 1.2. (Yu [19]) *For an $L(0,1)$ Lorenz like Smale flow on S^3, the following and only the following configurations are realizable. The link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the saddle set is modeled by a standard L(0, 1) Lorenz like template, i.e. the saddle set is modeled by embedded L(0, 1) and the cores of the bands are unknotted and unlinked each other. In the former case, there are three possibilities: (1) The saddle set is standardly embedded. (2) The saddle set is modeled by embedded L(2p + 2q-2, 2p +2q - 1). The cores of two bands are twp parallel (p,q) torus knots. p, q are any coprime integers. (3) The saddle set is modeled by L(0, 2p + 2q - 1). The core of the twisted band is a (p,q) torus knot, the core of the other band is unknotted and unlinked with the former one. p, q are any coprime integers.*

Theorem 1.3. (Yu [19]) *For an $L(1,1)$ Lorenz like Smale flow on S^3, the following and only the following configurations are realizable. The link $a \cup r$ is composed of a trivial knot and a $(p,3)$ torus knot in the boundary of a solid torus neighborhood of the trivial knot. Here p is any integer such that p, 3 are coprime. The saddle set is modeled by embedded L(2n + 1, 4n + 1) for any n. The linking number of the cores of these two bands is $2n$, the core of one band is unknotted and the core of the other band is a (2, 2n + 1) torus knot.*

E. Haynes and M. Sullivan [11] studied simple Smale flows in which the chaotic saddle set is modeled by four-band templates including Ghrist's [7] so called universal template U. Sullivan [16] studied simple Smale flows on S^3 such that the chaotic saddle set is a suspension of a full n-shift. He proved among other things that for $n \geq 2$ and linking number of 1, there exists a simple Smale flow on S^3 such that the pair $a \cup r$ is a Hopf link, a trefoil and meridian, or a figure-8 knot and meridian. For the $n = 2$ case explicit examples are known as per theorems in this section.

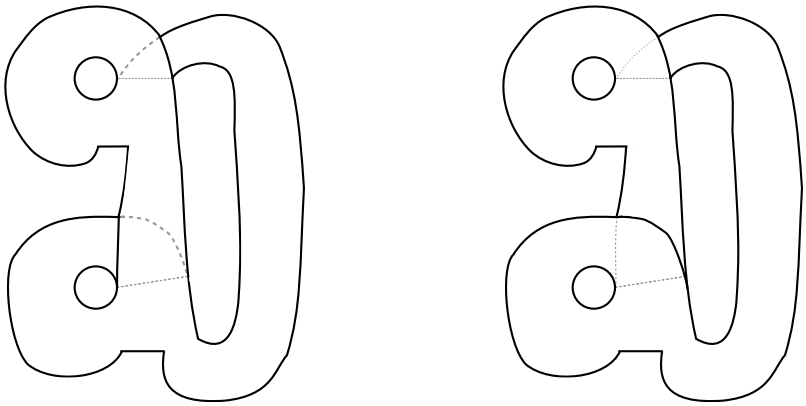


FIGURE 2. Four Band Templates U (left) and U^+ (right)

It follows, however, from Theorem 3.1 proved in this paper that these are not the only possibilities for the $n = 2$ case. M. Adhikari and M. Sullivan [2] studied simple Smale flows whose chaotic saddle sets are modeled by different four band templates. Adhikari [1] also studied simple Smale flows on three-band templates. We record these theorems below.

Theorem 1.4. (Haynes and Sullivan [11]) *For a simple Smale flow on S^3 with saddle set modeled by U the link $a \cup r$ is either a Hopf link or figure-8 knot and meridian. In the latter case the bands are untwisted, unknotted, and unlinked. In the Hopf link case one or two bands may form (p, q) torus knots about a or r; however the two looped bands on the left of Fig. 2 cannot both be knotted, twisted, linked.*

Theorem 1.5. (Haynes and Sullivan [11]) *For a simple Smale flow on S^3 with saddle set modeled by U^+, the link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the bands are untwisted, unknotted, and unlinked. In the Hopf link case one or two bands may form (p, q) torus knots about a or r; however the two looped bands on the left of Fig. 2 cannot both be knotted, twisted, linked.*

Theorem 1.6. (Adhikari and Sullivan [2]) *For a simple Smale flow on S^3 with the saddle set modeled by H, the link $a \cup r$ is either a Hopf link or a figure-8 knot and its meridian. In the latter case the saddle set is modeled bt standardly embedded template H where the bands are untwisted, unlinked, and unknotted. In the Hopf link case, (a) the saddle set is standardly embedded template or (b) one or two bands may form (p, q) torus knot about a or r but not both of them twisted, knotted, or linked.*

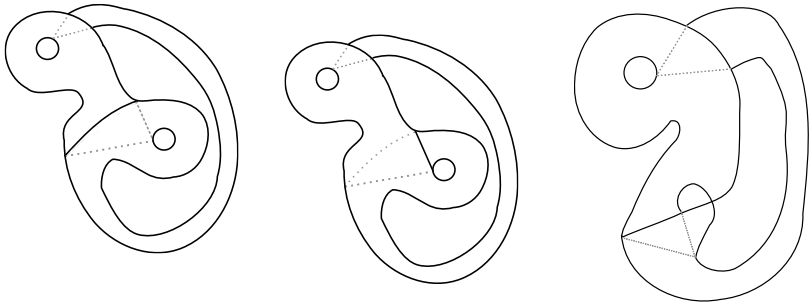


FIGURE 3. Four Band Templates H (left) and H^+ (middle) and Three Band Template V (right)

Theorem 1.7 (Adhikari and Sullivan [2]) *For a simple Smale flow on S^3 with the saddle set modeled by H^+, the link $a \cup r$ is a Hopf link.*

Theorem 1.8. (Adhikari [1]) *For a simple Smale flow on S^3 where the saddle set is modeled by template V, the attractor and repeller are two disjoint unknots.*

Haynes [10] and Yu [19] have also studied which other closed, orientable three-manifolds besides S^3 that admit Lorenz and Lorenz like Smale flows. Ghrist [7] and others [8] have studied knotting on the templates themselves.

In this work, we modify the four-band template U^+ from Theorem 1.5 by placing half-twists in selected bands. We then find realizations of simple Smale flows on S^3 whose chaotic saddle sets are modeled by these modified templates. This work extends the works of Adhikari, Haynes, and Sullivan [[2], [11], [17], [16]] by continuing the study of simple Smale flows on S^3 whose chaotic saddle sets are modeled by four-band templates. We also extend the work of Yu [19] by placing half-twists in a four-band template as opposed to a two-band template. Much of this work appeared in [13].

2. Background

In this section, we formally define Smale flows and simple Smale flows. We also introduce the notion of templates and record the necessary Template theorem. More extensive background material is available in [8, 13, 15] and references therein.

Definition 2.1. A C^r *flow* on a manifold M is a C^r mapping $\Phi : \mathbb{R} \times M \longrightarrow M$ such that $\phi_t = \Phi(t, -)$ satisfies:

1. $\phi_0(x) = x$ for all $x \in M$
2. $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ for all $x \in M$ and $s, t \in \mathbb{R}$.

The *set* $\Gamma_x = \{\Phi(t,x) \in M : t \in \mathbb{R}\}$ is said to be an *orbit* of the flow Φ on the manifold M. Therefore a flow on a manifold M decomposes M into a collection of disjoint orbits. We say that an orbit is *periodic* of period $T > 0$ provided $\Phi(t+T,x) = \Phi(t,x)$ for all $t \in \mathbb{R}$. Such orbits are also called *closed orbits*. It turns out that the orbits of a C^0 flow on a Hausdorff space are limited to three types: a point, a homeomorphism of S^1, or the image of $\mathbb{R}$ under a continuous bijection [12].

A *knot* is an embedding of S^1 into $\mathbb{R}^3$ or S^3. A *link* L:$\coprod S^1 \hookrightarrow S^3$ is a finite, disjoint collection of knots The image of each copy of S^1 is called a component of the link L. The *knot group* $G(K)$ of a knot K is defined to be the fundamental group of $S^3 - K$.

Definition 2.2. Let ϕ_t be a flow on a manifold M. A subset $\Lambda \subset M$ is said to be an *invariant set* of the flow ϕ_t provided $\phi_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$.

Definition 2.3. Let M be a smooth manifold with a smooth flow ϕ_t. An invariant set $\Lambda \subset M$ is said to be *hyperbolic* provided the tangent bundle TM restricted to Λ is a Whitney sum

$$TM_\Lambda = E^s_\Lambda \oplus E^c_\Lambda \oplus E^u_\Lambda$$

of three subbundles each invariant under $D\phi_t$ where E^c_x is spanned by $\frac{d\phi_t}{dt}(x)\,|_{t=0}$ for $x \in \Lambda$ and there are constants $C > 0$ and $\lambda > 1$ such that

1. $\|D\phi_t(v)\| \leq Ce^{-\lambda t}\|v\|$ for every $v \in E^s_\Lambda$ and for every $t > 0$
2. $\|D\phi_{-t}(v)\| \geq Ce^{-\lambda t}\|v\|$ for every $v \in E^u_\Lambda$ and for every $t > 0$.

We call E^s_Λ the stable bundle, E^u_Λ the unstable bundle, and E^c_Λ the center bundle. The one-dimensional "center" direction E^c_x is tangent to the orbit itself at each point.

Definition 2.4. Let M be a manifold. A point $p \in M$ is said to be *chain recurrent* for a flow ϕ_t on M if for any $\epsilon > 0$ there are finite sequences $\{p = p_1, p_2, \cdots, p_m = p\} \subset M$ and $\{t_1, t_2, \cdots, t_{m-1}\} \subset \mathbb{R}$ such that $t_i > 1$ and $d(\phi_{t_i}(p_i) - p_{i+1}) < \epsilon$ for all $1 \leq i \leq m-1$. The set of all chain recurrent points of M with respect to the flow ϕ_t is called the *chain recurrent set* and denoted by $\mathcal{R}(\phi)$.

The chain recurrent set contains the essential dynamics of a flow including all the closed orbits and fixed points of a flow.

Theorem 2.5. (**Smale's Spectral Decomposition Theorem** [14]) *If a flow on a compact manifold M has hyperbolic chain recurrent set $\mathcal{R}$, then $\mathcal{R}$ is a finite disjoint union of compact invariant sets Λ_j, $1 \leq j \leq n$ each of which contains a dense orbit.*

These compact invariant sets are referred to as *basic sets* of the flow.

Definition 2.6. Let Λ be an invariant set of a flow ϕ_t on a manifold M. The sets

$$\begin{aligned} W^s(X) &= \left\{ y \in M : \lim_{t\to\infty} d(\phi_t(X), \phi_t(y)) = 0 \right\} \\ W^u(X) &= \left\{ y \in M : \lim_{t\to-\infty} d(\phi_t(X), \phi_t(y)) = 0 \right\} \end{aligned}$$

are called the *stable set* and *unstable set*, respectively, of $X \subset \Lambda \subset M$.

By the Stable Manifold Theorem, the stable and unstable sets of a hyperbolic invariant set are manifolds. If we add the condition that for some $\epsilon > 0$, $d(\phi_t(X), \phi_t(y)) < \epsilon$ for all $t \geq 0$, then $W^s(X)$ is called a local stable manifold. Similarly, we obtain a local unstable manifold.

Definition 2.7. A flow ϕ_t on a manifold M is called a *Smale flow* provided:

1. The chain recurrent set $\mathcal{R}$ is hyperbolic
2. The stable manifold of any orbit in $\mathcal{R}$ meets the unstable manifold of any other orbit in $\mathcal{R}$ transversely
3. The basic sets of $\mathcal{R}$ are zero-dimensional or one-dimensional

A flow that does not have any fixed point orbits is called a *non-singular flow*. A *simple Smale flow* is a non-singular Smale flow whose hyperbolic chain recurrent set consists of exactly three basic sets: an attracting closed orbit a, a repelling closed orbit r, and a chaotic saddle set. Our primary interest will be simple Smale flows on S^3.

Next we briefly introduce templates and their relationship to flows. We also record the very important Template theorem. The relevant template theory can be found in [4, 8].

Definition 2.8. Consider the pair $(I \times I, \Phi)$ where Φ is a semi-flow on $I \times I$ with $\Phi(t, (x, y)) = (x, t + y)$. A *splitting chart* is the set $(I \times I) - \left\{(x, y) : x \in \left(\frac{1}{3}, \frac{2}{3}\right), y \in \left[0, \frac{1}{2}\right)\right\}$ together with the associated semi-flow. The quotient $((I \times I) \times (I \times I)) / \left\{(x, y) = (x, y) : y \leq \frac{1}{2}\right\}$ together with the associated semi-flow is called a *joining chart*.

A joining chart has two incoming strips and one outgoing strip which meet tangentially at the *branch line* $\left\{(x, y) : y = \frac{1}{2}\right\}$. The set T of finitely many joining and splitting charts glued linearly along free ends so that no free ends remain is a compact branched 2-manifold with boundary.

Definition 2.9. A pair (T, Φ), where T is a compact branched 2-manifold whose atlas consists of joining charts and splitting charts and semi-flow Φ, is said to be a *template*.

It must be the case that a template is composed of an equal number of joining charts and splitting charts. The simplest template is the Lorenz template consisting of one joining chart and one splitting chart.

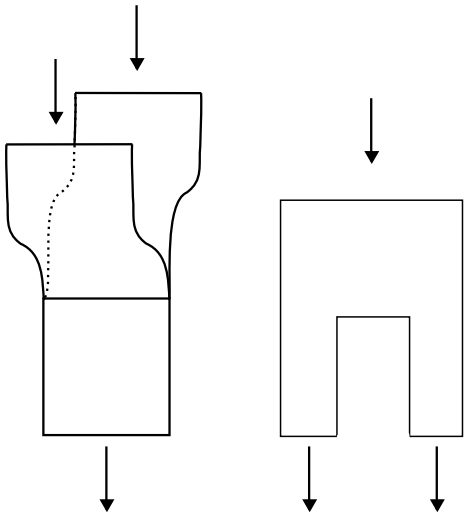


FIGURE 4. Joining Chart and Splitting Chart

Definition 2.10. Let Λ be a closed invariant set of a flow ϕ_t in a manifold M. We say that a set U is an *isolating neighborhood* of Λ provided $\Lambda \subset int\,(U)$ and $\Lambda = \bigcap\limits_{t\in\mathbb{R}} \phi_t\,(U)$.

Theorem 2.11. (**The Template Theorem** [4]) *Let ϕ_t be a non-singular Smale flow on a 3-manifold M with a basic set Λ of the chaotic saddle type. There is an isolating neighborhood N of Λ such that when local stable manifolds of points in Λ are collapsed, the resulting quotient space is a template (T, Φ) such that the link of periodic orbits of the flow ϕ_t are in bijective correspondance with the link of periodic orbits of the semi-flow Φ and T is embedded in M.*

Based on the Template Theorem, we may regard a thickened template to be an isolating neighborhood of a chaotic saddle set of a non-singular Smale flow on a 3-manifold. The thickened template recovers local stable manifolds of the points of the chaotic saddle set. A thickened template is a solid g-handlebody whose boundary is divided into an entrance set and an exit set whose closures intersect in a finite collection of boundary loops. The flow is tangent to these boundary loops. The set of points in the boundary of the thickened template in which the flow is transverse inward is called the *entrance set*. The set of points in the boundary of the thickened template where the flow is transverse outward is called the *exit set*

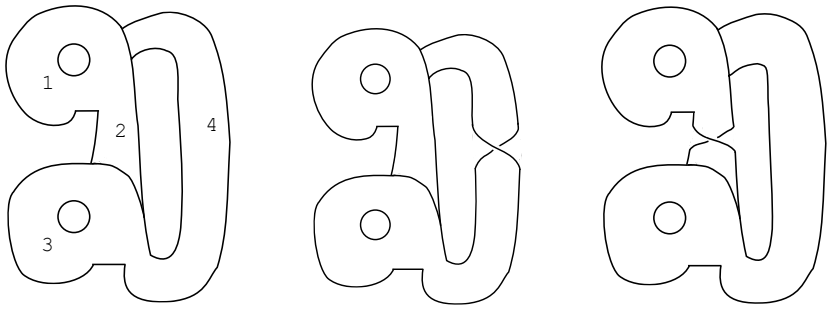


FIGURE 5. Four Band Templates U^+ (left), $U^+(0,0,0,1)$ (middle) and $U^+(0,1,0,0)$ (right)

It follows from a theorem of Franks [6] that the absolute value of the linking number of $a \cup r$ is given by $\mid det(I - S) \mid$ where S is the so-called structure matrix of the saddle set of a simple Smale flow on S^3. The structure matrix S is determined from a Markov partition of the saddle set. For the theorems proved in the next section, the linking number of $a \cup r$ has absolute value 1.

Definition 2.12. A loop in a surface is said to be *inessential* if it bounds a disk in the surface. Otherwise, the loop is called *essential*.

3. Main Results

In this section we prove a theorem concerning simple Smale flows on S^3 whose chaotic saddle set is modeled by a modified version of template U^+. The template U^+ is modified by placing a half-twist in band 1. The resulting template is $U^+(1,0,0,0)$ as in Figure 6. This theorem illustrates that the dynamics is sensitive to half-twists in the bands. Also, we discover for the first time the cinquefoil knot as one component of the attractor-repeller link in a simple Smale flow on S^3. We also state another theorem that results from placing a half twist in band 4 that forms $U^+(0,0,0,1)$ (see Figure 5).

Theorem 3.1. *A simple Smale flow on S^3 with saddle set modeled by $U^+(1,0,0,0)$ is realizable if and only if the link $a \cup r$ is either a Hopf link, a trefoil and meridian, or a $(2,5)$ torus knot (cinquefoil) and meridian. In the latter two cases, the saddle set is modeled by standardly embedded $U^+(1,0,0,0)$. In the Hopf link case, (a), the saddle set is modeled by standardly embedded $U^+(1,0,0,0)$, or (b) one or two bands may form (p,q) torus knots about a or r.*

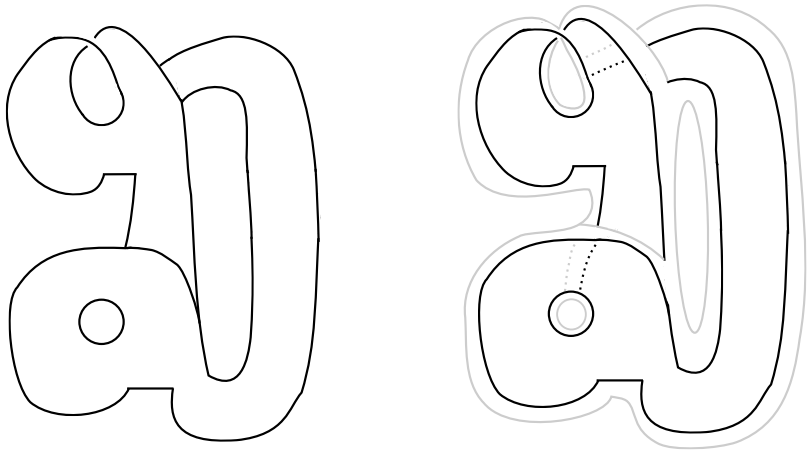


FIGURE 6. Template $U^+(1,0,0,0)$ (left) and Thickened $U^+(1,0,0,0)$ (right)

Proof: Let ϕ_t be a simple Smale flow on S^3 with chaotic saddle set modeled by $U^+(0,0,0,1)$, closed attracting orbit a, and closed repelling orbit r. Let A and R be closed tubular neighborhoods of a and r, respectively. Denote by M the thickened template of $U^+(0,0,0,1)$ which is an isolating neighborhood of the chaotic saddle set. We consider A, M, and R as disjoint manifolds. We observe that A and R are solid tori (possibly knotted) and M is a solid 3-handlebody. We denote the exit set and entrance sets of M by Ex and En, respectively. The closures of the entrance and exit sets of M intersect in four disjoint boundary loops. We recover the simple Smale flow on S^3 by gluing ∂A to ∂M along the the closure of the exit set of M. Then glue ∂R to $\partial(A \cup M)$ forming S^3.

The exit set of M consists of the union of a disc with two smaller disjoint discs removed, a rectangular strip, and an annulus. The core of the exit set of M consists of a single closed loop c_4 and four arcs c_1, c_2, c_3, and l.

The resulting exit set configuration is illustrated in Figure 7. The three arcs c_1, c_2, and c_3 form three closed loops c_1c_2, c_2c_3, and c_1c_3. Note that the arcs c_1 and l must lie on the same side of c_2c_3. Without loss of generality, we assume that c_1 and l lie outside of c_2c_3. We glue ∂A to $cl(Ex)$ by a diffeomorphism $\varphi : cl(Ex) \longrightarrow \partial A$ where $\varphi(c_ic_j) = c_i'c_j'$. For the sake of keeping notation simple, we do not use the prime symbol

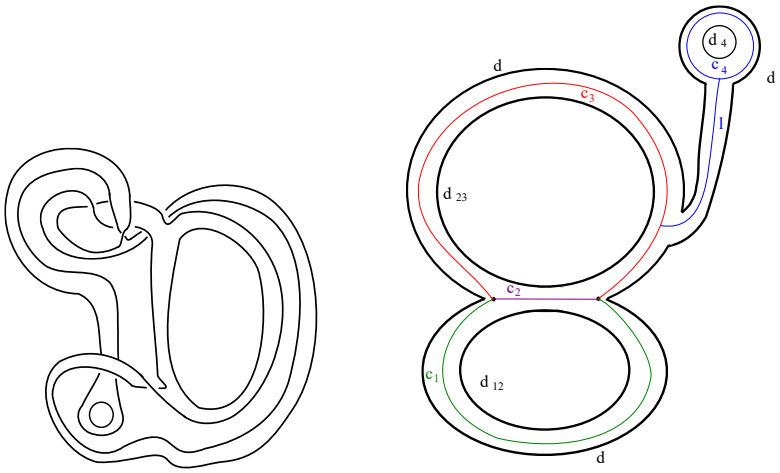


FIGURE 7. Exit Set (left) and Exit Set configuration (right) for Thickened Template $U^+(1,0,0,0)$

when referring to image loops under the diffeomorphism. Instead, we let the context dictate the notation. For example c_ic_j will denote either a loop on M or the corresponding loop $c_i'c_j'$ on ∂A Ultimately the loops are identified during the gluing process. The proof is by cases depending upon which closed loops are essential or inessential in ∂A.

Case 1 All closed loops are inessential in ∂A.

Observe that the loops c_1c_2 and c_2c_3 are inessential if and only of the loop c_1c_3 is inessential. Therefore, there are two possible cases. Either the closed loops c_1c_2, c_2c_3, and c_1c_3 lie inside the disk bounded by c_4 or they lie outside the disk bounded by c_4. The core loops lie on the surface of A as in Figure 8.

Case 1 (a) Assume that the closed loops c_1c_2, c_2c_3, and c_1c_3 lie outside the disk bounded by c_4.

Now c_1c_2, c_2c_3, and c_4 bound disks D_{12}, D_{23}, and D_4, respectively on ∂A. Thicken each of these disks in A to form solid 3-balls B_{12}, B_{23}, and B_4, respectively, such that $B_{12} \cap \partial A = D_{12}$, $B_{23} \cap \partial A = D_{23}$, $B_4 \cap \partial A = D_4$, $B_{12} \cap a = \emptyset$, $B_{23} \cap a = \emptyset$, and $B_4 \cap a = \emptyset$. Cut the balls B_{12}, B_{23}, and B_4 out of A along the surfaces $S_{12} = cl\,(\partial B_{12} - D_{12})$, $S_{23} = cl\,(\partial B_{23} - D_{23})$, and $S_4 = cl\,(\partial B_4 - D_4)$, respectively. After cutting, there are corresponding surfaces left on $A' = cl\,(A - (B_{12} \cup B_{23} \cup B_4))$. Glue B_{12} to M by identifying the annulus bounded by the loops c_{12}

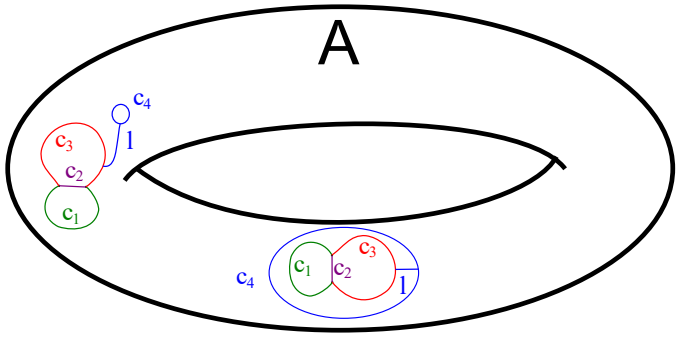


FIGURE 8. Core Loops on A for Case 1 of Theorem 3.2

and d_{12} with the corresponding annulus in ∂B_{12}. Next glue B_{23} to M by identifying the annulus bounded by the loops c_{23} and d_{23} with the corresponding annulus in ∂B_{23}. Finally, glue B_4 to M by identifying the annulus bounded by the loops c_4 and d_4 with the corresponding annulus in ∂B_4. It follows by an Euler chracteristic argument that A' is a solid torus and $M' = M \cup B_{12} \cup B_{23} \cup B_4$ is a solid 3-ball. There is a disk $D \subset \partial M'$ which is bounded by the boundary loop d and contains the surfaces S_{12}, S_{23}, and S_4. We attach A' to M' by identifying the disk D (which contains all the core loops) with the corresponding disk in $\partial A'$. By construction, $A \cup M = A' \cup M'$. It follows that $A \cup M$ is a solid torus with core a. Therefore, R is an unknotted solid torus and $a \cup r$ is a Hopf link.

All cases except case 5 of this theorem are realizable. In fact, all realizations can be obtained by gluing a solid 3-ball B along the closure of the exit set of the thickened template M followed by strategically attaching a 1-handle H to the 3-ball such that the dynamics is respected. $B \cup M$ is a solid 3-ball and $B \cup H$ is a solid torus A whose core is the attractor **a**. The complement of $M \cup B \cup H$ in S^3 is a solid torus R whose core is the repeller **r**. Care must be taken to match the associated vector fields during the gluing.

Figure 9 illustrates a realization of Case 1(a) where $a \cup r$ is a Hopf link. a is the core of the thickened blue ring while r is the core of the thickened red ring. The gray and white screen and the solid light blue regions are part of a solid 3-ball attached to the thickened template M along the exit set.

Case 1 (b) Assume that the closed loops c_1c_2, c_2c_3, and c_1c_3 lie inside the disk bounded by c_4.

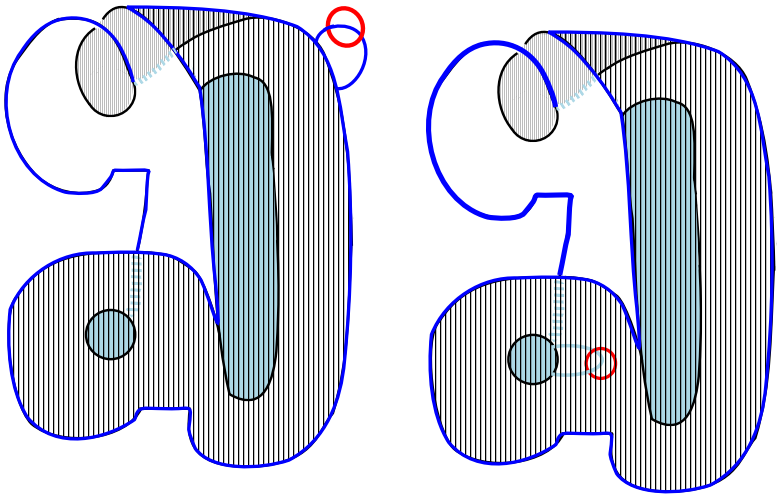


FIGURE 9. Realizations for Case 1(a) (left) and Case 1(b) (right) for Theorem 3.2

Observe that c_1c_2 and c_2c_3 bound disks D_{12} and D_{23}, respectively in ∂A. Thicken the disks D_{12} and D_{23} in A to form solid 3-balls B_{12} and B_{23}, respectively, such that $B_{12} \cap \partial A = D_{12}$, $B_{23} \cap \partial A = D_{23}$, $B_{12} \cap a = \emptyset$ and $B_{23} \cap a = \emptyset$. Cut the balls B_{12} and B_{23} out of A. Then $A' = cl\,(A - (B_{12} \cup B_{23}))$ is a solid torus. Glue B_{12} to M by identifying the annulus An_{12} bounded by the loops c_{12} and d_{12} with the corresponding annulus in ∂B_{12}. Next glue B_{23} to M by identifying the annulus An_{23} bounded by the loops c_{23} and d_{23} with the corresponding annulus in ∂B_{23}. It follows that $M' = M \cup B_{12} \cup B_{23}$ is a solid genus 1 handlebody. Now c_4 bounds a disk D_4 in $\partial A'$ such that l and the surfaces corresponding to S_{12} and S_{23} are contained in D_4. Thicken D_4 in A' to form a solid 3-ball B_4 in A' such that $B_4 \cap \partial A = D_4$ and $B_4 \cap a = \emptyset$. Cut the ball B_4 out of A'. Then $A'' = cl\,(A' - B_4)$ is a solid torus. Denote by An_4 the annulus in $\partial M'$ whose boundary consists of the loops c_4 and and the boundary loop that lies outside D_4. Glue B_4 to M' by identifying the annulus $cl\,(cl\,(Ex) - (An_{13} \cup An_{23} \cup An_4)) \cup S_{13} \cup S_{23} \subset \partial M'$ with the corresponding annulus in $\partial A'$. Here, S_{13} and S_{23} are the cutting surfaces on B_{13} and B_{23}, respectively. It follows that $M'' = M' \cup B_4$ is a solid 3-ball. We attach A'' to M'' by identifying the disk $An_4 \cup S_4$ in $\partial M''$ with the corresponding disk in $\partial A''$ forming a solid torus $A'' \cup M''$. Since

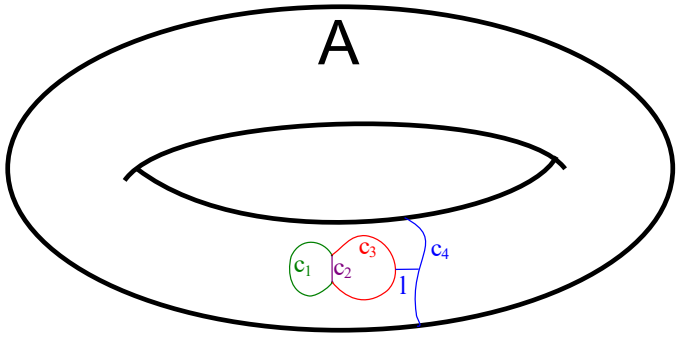


FIGURE 10. Core Loops on A for Case 2 of Theorem 3.2

$A \cup M = A'' \cup M''$, the solid torus R must be unknotted. Therefore, $a \cup r$ is a Hopf link.

Figure 9 illustrates a realization of Case 1(b) where $a \cup r$ is a Hopf link. a is the core of the thickened light blue ring while r is the core of the thickened red ring. The gray and white screen and the solid light blue regions are part of a solid 3-ball attached to the thickened template M along the exit set. There is a crater between solid light blue regions and the black and white screen whose opening is near the twisted band near the top of the template. The crater is more apparent in Figure 12 below.

Case 2 Exactly one core loop is essential.

It must be the case that c_4 is the only essential loop since either c_1c_2 or c_2c_3 is essential if and only if c_1c_3 is essential. Therefore, the loops c_1c_2, c_2c_3, and c_1c_3 are inessential and c_4 is essential. The core loops lie on the surface of A as in Figure 10.

It follows that c_1c_2 and c_2c_3 bound disks $D_{12}$and D_{23}, respectively, in ∂A. Thicken the disks D_{12} and D_{23} in A to form solid 3-balls B_{12} and B_{23}, respectively, such that $B_{12} \cap \partial A = D_{12}$, $B_{23} \cap \partial A = D_{23}$, $B_{12} \cap a = \emptyset$ and $B_{23} \cap a = \emptyset$. Cut the balls B_{12} and B_{23} out of A. Then $A' = cl\,(A - (B_{12} \cup B_{23}))$ is a solid torus. Glue B_{12} to M by identifying the annulus An_{12} bounded by the loops c_{12} and d_{12} with the corresponding annulus in ∂B_{12}. Next glue B_{23} to M by identifying the annulus An_{23} bounded by the loops c_{23} and d_{23} with the corresponding annulus in ∂B_{23}. It follows that $M' = M \cup B_{12} \cup B_{23}$ is a solid 1-handlebody. Finally, we glue A' to M' by identifying the annulus $S = cl\,(cl\,(Ex) - (An_{12} \cup An_{23})) \cup S_{12} \cup S_{23} \subset \partial M'$ with the corresponding annulus in $\partial A'$. Again, S_{12} and S_{23} are cutting surfaces. Fixing a basepoint in $S \subset M$, we compute the fundamental group of $A \cup M$ using the Seifert-Van Kampen Theorem. See Figure 11 for generators. Call a the generator for A, α is the generator

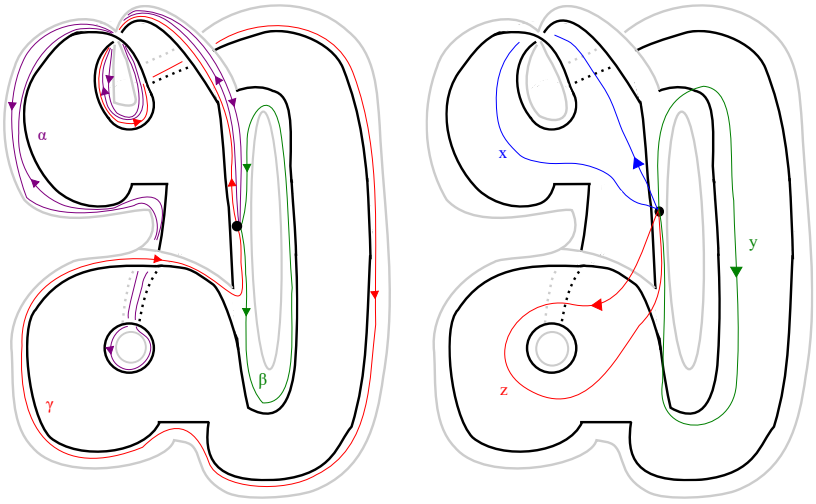


FIGURE 11. Generators for Fundamental Group of Exit Set (left) and Thickened Template (right) for Theorem 3.2

of S, and note that M' has generators x, y, and z. Since α is ambient isotopic in to c_4 which is a (p, q) curve in ∂A, we have

$$\begin{aligned}
\pi_1(A \cup M) &= \langle a, x, y, z \mid a^p = x^2 z x^{-2},\ y = 1,\ xyz^{-1} = 1 \rangle \\
\pi_1(A \cup M) &= \langle a, x, z \mid a^p = z = x \rangle \\
\pi_1(A \cup M) &= \langle a \rangle .
\end{aligned}$$

It follows that R is a solid unnotted torus. Thus, $a \cup r$ is a Hopf link.

Figure 12 illustrates a realization of Case 2 where $a \cup r$ is a Hopf link. a is the core of the thickened blue and aqua blue loop, while r is the core of the thickened red ring. Again, there is a crater between the solid light blue regions and the gray and white screen whose opening is near the twisted band near the top of the template. Part of A runs through this crater.

Case 3 Exactly two core loops are essential.

Since either c_1c_2 or c_2c_3 is essential if and only if c_1c_3 is essential, c_4 must be inessential. Therefore, three possible subcases exist. The core loops lie on the surface of A as in Figure 13, Figure 14, and Figure 15.

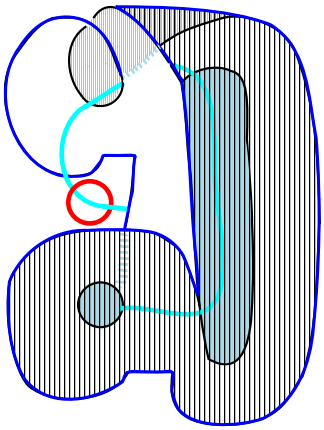


FIGURE 12. Realization for Case 2 of Theorem 3.2

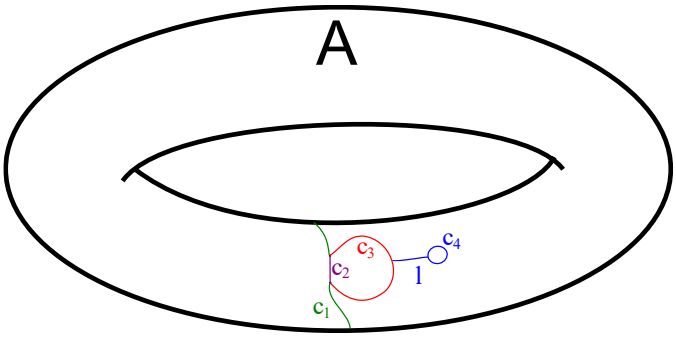


FIGURE 13. Core Loops on A for Case 3(a) of Theorem 3.2

Case 3 (a) c_1c_2 is essential which implies c_1c_3 is essential. c_2c_3 and c_4 are inessential.

The loops c_2c_3 and c_4 bound disks D_{23} and D_4, respectively, in ∂A. Thicken the disks D_{23} and D_4 slightly in A to form solid 3-balls B_{23} and B_4, respectively, such that $B_{23} \cap \partial A = D_{23}$, $B_4 \cap \partial A = D_4$, $B_{23} \cap a = \emptyset$ and $B_4 \cap a = \emptyset$. Cut the balls B_{23} and B_4 out of A. Then $A' = cl\,(A - (B_{23} \cup B_4))$ is a solid torus. Glue B_{23} to M by identifying the annulus An_{23} bounded by the loops c_{23} and d_{23} with the corresponding annulus in ∂B_{23}. Next glue B_4 to M by identifying the annulus An_4 bounded by the loops c_4 and d_4 with the corresponding annulus in ∂B_4. Then M' is a solid 1-handlebody or torus. Finally, we glue A' to M' by identifying the annulus $S = cl\,(cl\,(Ex) - (An_{23} \cup An_4)) \cup S_{23} \cup S_4 \subset \partial M'$

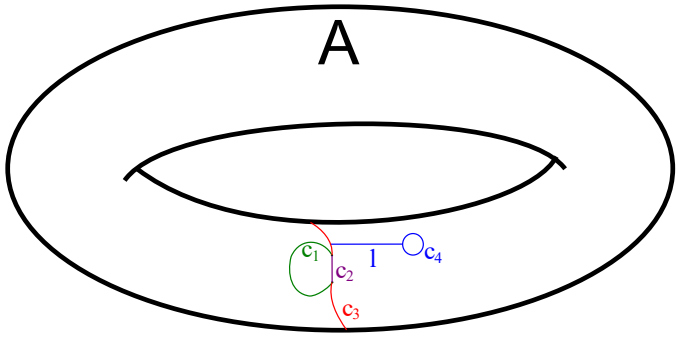


FIGURE 14. Core Loops on A for Case 3(b) of Theorem 3.2

with the corresponding annulus in $\partial A'$. Fixing a basepoint in $S \subset M$, we compute the fundamental group of $A \cup M$ using the Seifert-Van Kampen Theorem. Refer to Figure 11 for the appropriate generators. We note that A has generator a, α is the generator of S, and M' has generators x, y, and z. Since β and δ are ambient isotopic to c_1c_2 and c_1c_3, respectively, which are (p, q) curves in ∂A, we have

$$\begin{aligned} \pi_1(A \cup M) &= \langle a, x, y, z \mid a^p = y^{-1},\ a^p = xyz^{-1}y^{-1},\ z = 1 \rangle \\ \pi_1(A \cup M) &= \langle a, x, y \mid a^p = y^{-1} = x \rangle \\ \pi_1(A \cup M) &= \langle a \rangle . \end{aligned}$$

It follows that R is a solid unknotted torus. Thus, $a \cup r$ is a Hopf link.

Figure 16 illustrates a realization of Case 3(a) where $a \cup r$ is a Hopf link. a is the core of the thickened aqua blue ring while r is the core of the thickened red ring.

Case 3 (b) c_2c_3 is essential which implies c_1c_3 is essential. c_1c_2 and c_4 are inessential.

The loops c_1c_2 and c_4 bound disks D_{12} and D_4, respectively, in ∂A. Thicken the disks D_{12} and D_4 slightly in A to form solid 3-balls B_{12} and B_4, respectively, such that $B_{12} \cap \partial A = D_{12}$, $B_4 \cap \partial A = D_4$, $B_{12} \cap a = \emptyset$ and $B_4 \cap a = \emptyset$. Cut the balls B_{12} and B_4 out of A. Then $A' = cl(A - (B_{12} \cup B_4))$ is a solid torus. Glue B_{12} to M by identifying the annulus An_{12} bounded by the loops c_{12} and d_{12} with the corresponding annulus in ∂B_{12}. Next glue B_4 to M by identifying the annulus An_4 bounded by the loops c_4 and d_4 with the corresponding annulus in ∂B_4. Then M' is a solid 1-handlebody or torus. Finally, we glue A' to M' by identifying the annulus $S = cl(cl(Ex) - (An_{12} \cup An_4)) \cup S_{12} \cup S_4 \subset \partial M'$ with the corresponding annulus in $\partial A'$. Fixing a basepoint in $S \subset M$, we

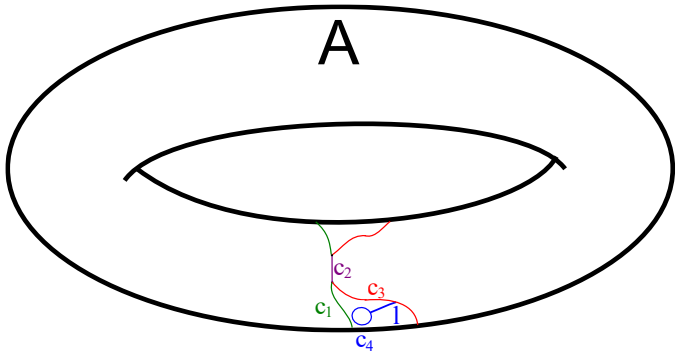


FIGURE 15. Core Loops on A for Case 3 (c) of Theorem 3.2

compute the fundamental group of $A \cup M$ using the Seifert-Van Kampen Theorem. See Figure 11 for the appropriate generators. We have a is the generator for A, γ is the generator of S, and M' has generators x, y, and z. Since γ is ambient isotopic in to c_2c_3 which is a (p, q) curve in ∂A, we have

$$\begin{aligned} \pi_1(A \cup M) &= \left\langle a, x, y, z \mid a^p = xyz^{-1},\, a^p = xyz^{-1}y^{-1},\, y = 1,\, z = 1 \right\rangle \\ \pi_1(A \cup M) &= \left\langle a, x \mid a^p = x \right\rangle \\ \pi_1(A \cup M) &= \left\langle a \right\rangle. \end{aligned}$$

It follows that R is an unknotted solid torus. Thus $a \cup r$ is a Hopf link.

Figure 16 illustrates a realization of Case 3(b) where $a \cup r$ is a Hopf link. a is the core of the thickened blue ring while r is the core of the thickened red ring.

Case 3 (c) c_1c_2, c_2c_3 are essential and c_1c_3, c_4 are inessential.

Now c_4 bounds a disk D_4 in ∂A. Thicken the disk D_4 slightly in A to form a solid 3-ball B_4 in A such that $B_4 \cap \partial A = D_4$ and $B_4 \cap a = \emptyset$. Cut B_4 out of A. Then $A' = cl(A - B_4)$. Glue B_4 to M by identifying the annulus An_4 bounded by the loops c_4 and d_4 with the corresponding annulus in ∂B_4. It follows that $M' = M \cup B_4$ is a solid 2-handlebody. Next, we glue A' to M' by identifying the set $S = cl(cl(Ex) - An_4) \cup S_4 \subset \partial M'$ with the corresponding set in $\partial A'$. By construction $A \cup M = A' \cup M'$. Fixing a basepoint in $S \subset M$, we compute the fundamental group of $A \cup M$ using the Seifert-Van Kampen Theorem. See Figure 11 for the appropriate generators. Let a is the generator for A. Note that β and γ are generators of S, and M' has generators x, y, and z. Since β and γ is ambient isotopic to c_1c_2 and c_2c_3, respectively, which are (p, q) curves in

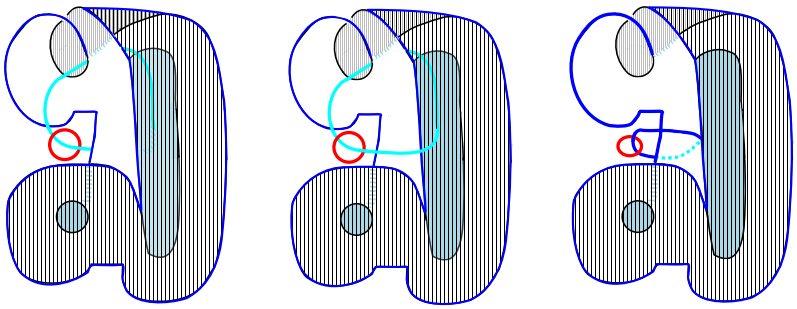


FIGURE 16. Realizations for Case 3(a) (left), Case 3(b) (middle) and Case 3(c) (right) for Theorem 3.2

∂A, we have

$$\begin{aligned}\pi_1(A \cup M) &= \langle a, x, y, z \mid a^p = y^{-1},\ a^p = xyz^{-1},\ z = 1\rangle \\ \pi_1(A \cup M) &= \langle a, x, y \mid a^p = y^{-1},\ a^p = x\rangle \\ \pi_1(A \cup M) &= \langle a\rangle .\end{aligned}$$

It follows that R is an unknotted solid torus. Thus $a \cup r$ is a Hopf link. We have the additional relation $xyz^{-1}y^{-1} = 1$ since c_1c_3 is nulhomotopic. Therefore, $p = 0$ and the embedding is standard.

Figure 16 illustrates a realization of Case 3(c) where $a \cup r$ is a Hopf link. a is the core of the thickened blue ring while r is the core of the thickened red ring.

Case 4 Exactly three core loops are essential.

Since either c_1c_2 or c_2c_3 is essential if and only if c_1c_3 is essential, there are two cases to consider.

Case 4 (a) c_1c_2 is essential which implies c_1c_3 is essential. c_4 is essential and c_2c_3 is inessential.

The core loops lie on the surface of A as in Figure 17.

The loop c_2c_3 bounds a disk D_{23} in ∂A. Thicken the disk D_{23} slightly in A to form a solid 3-ball B_{23} in A such that $B_{23} \cap \partial A = D_{23}$ and $B_{23} \cap a = \emptyset$. Cut B_{23} out of A. Then $A' = cl(A - B_{23})$ is a solid torus. Glue B_{23} to M by identifying the annulus An_{23} bounded by the loops c_2c_3 and d_{23} with the corresponding annulus in ∂B_{23}. Then $M' = M \cup B_{23}$ is a solid 2-handlebody. Next, we glue A' to M' by identifying the annulus $S = cl(cl(Ex) - An_{23}) \cup S_{23} \subset \partial M'$ with the corresponding annulus in $\partial A'$. By construction $A \cup M = A' \cup M'$. Fixing a basepoint in $S \subset M$, we compute the fundamental group of $A \cup M$ using the Seifert-Van

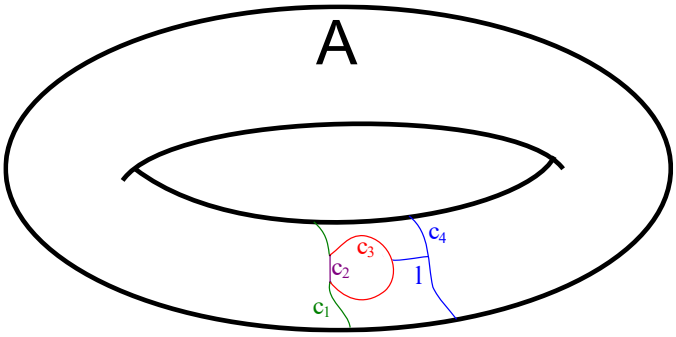


FIGURE 17. Core Loops on A for Case 4(a) of Theorem 3.2

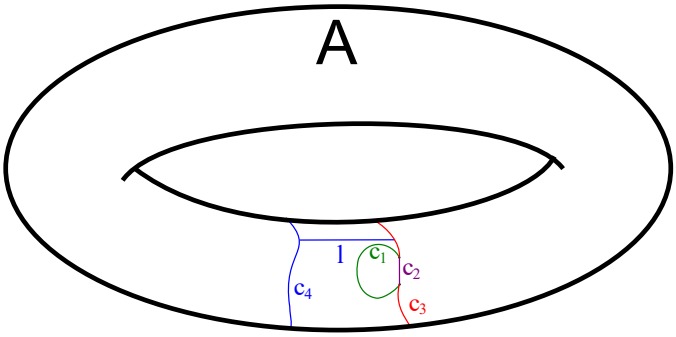


FIGURE 18. Core Loops on A for Case 4(b) of Theorem 3.2

Kampen Theorem. See Figure 11 for the appropriate generators. Let a be the generator for A. Note that α and β are generators of S and M' has generators x, y, and z. Now α and β are ambient isotopic to c_4 and c_1c_2, respectively, which are (p,q) curves in ∂A. Since c_1c_3 is also essential, we have

$$\begin{aligned}
\pi_1(A \cup M) &= \left\langle a, x, y, z \mid a^p = x^2 z x^{-2},\ a^p = y^{-1},\ a^p = xyz^{-1}y^{-1} \right\rangle \\
\pi_1(A \cup M) &= \left\langle x, y, z \mid y^{-1} = x^2 z x^{-2},\ y^{-1} = xyz^{-1}y^{-1} \right\rangle \\
\pi_1(A \cup M) &= \left\langle x, y, z \mid y^{-1} = x^2 z x^{-2},\ x = zy^{-1} \right\rangle \\
\pi_1(A \cup M) &= \left\langle x, z \mid z^{-1}x = x^2 z x^{-2} \right\rangle \\
\pi_1(A \cup M) &= \left\langle x, w \mid x^5 = w^2 \right\rangle
\end{aligned}$$

where we introduced the superfluous generator w, the superfluous relation $w = xzx$, and applied Teitze's theorem. But this is the knot group of the $(2, 5)$ torus knot. Therefore, the knot group of r is the knot group of the $(2, 5)$ torus knot or cinquefoil knot.

Computing $\pi_1(R \cup M')$ gives the same results. We glue R to $M' = M \cup B_{23}$ along the closure of the entrance set of M. Fixing a basepoint in $cl\,(Ex)$ and noting the generators in Figure 19, we apply the Seifert-Van Kampen Theorem to compute $\pi_1(R \cup M')$. Let e_4, e_{12}, and e_{13} denote core loops of the entrance set of M. As e_4, e_{12}, and e_{13} are parallel (p_1, q_1) curves in ∂R and xyz^{-1} is null-homotopic in $\partial M'$ we have $r^{p_1} = x^{-1}yzy^{-1}x$, $r^{p_1} = zy^{-1}z^{-1}$, $r^{p_1} = xyz^{-1}y^{-1}$, $xyz^{-1} = 1$. So

$$\begin{aligned}
\pi_1(R \cup M') &= \left\langle x, y, z \mid x^{-1}yzy^{-1}x = zy^{-1}z^{-1},\ y = x^{-1}z \right\rangle \\
\pi_1(R \cup M') &= \left\langle x, z \mid x^{-1}x^{-1}zzz^{-1}xx = zz^{-1}xz^{-1} \right\rangle \\
\pi_1(R \cup M') &= \left\langle x, y, z \mid x^{-2}zx^2 = xz^{-1} \right\rangle \\
\pi_1(R \cup M') &= \left\langle x, w \mid x^{-2}x^{-1}wx^{-1}x^2 = xxw^{-1}x \right\rangle \\
\pi_1(R \cup M') &= \left\langle x, w \mid w^2 = x^5 \right\rangle
\end{aligned}$$

where we have introduced the superfluous generator w, the superfluous relation $w = xzx$, and applied Teitze's theorem. Therefore, the knot group of a is the knot group of a $(2, 5)$ torus knot.

We claim that the template must be standardly embedded. We note that the boundary loop d bounds a disk D in ∂A. Consider a loop ρ that is parallel to both d and the union $c_1c_3 \cup l \cup c_4$. Then ρ bounds a disk D_ρ in ∂A. Thicken this disk to form a solid 3-ball B_ρ in A such that $B_\rho \cap a = \emptyset$ and $B_\rho \cap \partial A = D_\rho$. Cut the ball B_ρ out of A. Then $A^\rho = cl\,(A - B_\rho)$ is a solid torus with core a. Glue B_ρ to M' along an annulus in M' corresponding to the annulus $An_{\rho d}$ in D_ρ. Then $M^\rho = M' \cup B_\rho$ is a solid torus. The boundary loops d_{12} and d_4 divide ∂A^ρ into two annuli A_1 and A_2 such that say A_2 contains no core loops of the exit set. Now A_2 has core u which is a (p, q) torus curve. Similary, the boundary loops d_{12} and d_4 divide the torus ∂R into two annuli R_1 and R_2. Then one of these annuli, say R_2, is diffeomorphic to A_2 by a map ψ and has core v which is a (p_1, q_1) torus curve. But then $A^\rho \cup_\psi R = cl\,((M^\rho)^c)$ is an unknotted solid torus. Now $\pi(A^\rho) = \langle s \rangle$ and $\pi(R) = \langle t \rangle$ for some generators s and t. Then $\pi(A^\rho \cup R) = \langle s, t \mid s^p = t^{p_1} \rangle$ where $p = 1$ or $p_1 = 1$. Without loss of generality suppose $p = 1$. Then r is a (2.5) torus knot. If $p_1 = 0$, then a is an unknot and we are done. If $p_1 \neq 1$, then $\pi(A^\rho \cup R) = \langle t \rangle$ and t is simultaneously the generator of the fundamental group of the unknotted solid torus $A^\rho \cup R$ and the knotted solid torus R which is a contradiction. Therefore, we must have $p_1 = 0$ whenever $p = 1$. A similar argument shows that if $p_1 = 1$, the we must have $p = 0$. We conclude

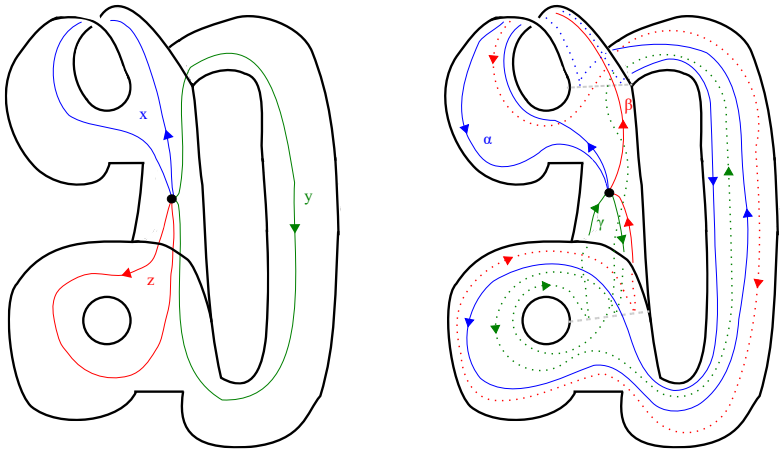


FIGURE 19. Generators for Fundamental Group of Thickened Template (left) and Entrance Set (right) for Theorem 3.2

that the template must be standardly embedded.Thus the link $a \cup r$ is a cinquefoil knot and meridian.

Figure 20 illustrates a realization of Case 4(a) where $a \cup r$ is a cinquefoil knot and meridian. a is the core of the thickened blue ring while r is the core of the thickened red ring. The knot can be visualized easier when removed from the template in a knot projection and following $a \to b \to c \to d$ in Figure 21.

Case 4 (b) c_2c_3 is essential which implies c_1c_3 is essential. c_4 is essential and c_1c_2 is inessential.

Figure 35 is an illustration of how the core loops lie on the surface of A.

Now c_1c_2 bounds a disk D_{12} in ∂A. Thicken the disk D_{12} slightly in A to form a solid 3-ball B_{12} in A such that $B_{12} \cap \partial A = D_{12}$ and $B_{12} \cap a = \emptyset$. Cut B_{12} out of A. Then $A' = cl\,(A - B_{12})$ is a solid torus. Glue B_{12} to M by identifying the annulus An_{12} bounded by the loops c_1c_2 and d_{12} with the corresponding annulus in ∂B_{23}. Then $M' = M \cup B_{12}$ is a solid 2-handlebody. Glue A' to M' by identifying the set

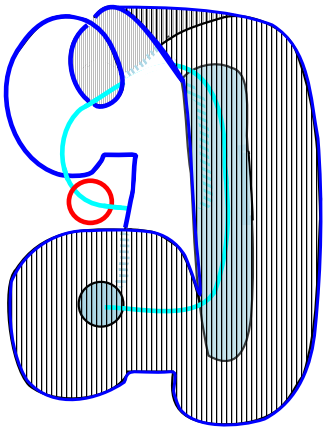


FIGURE 20. Realization for Case 4(a) of Theorem 3.2

$S = cl\left(cl\left(Ex\right) - An_{12}\right) \cup S_{12} \subset \partial M'$ with the corresponding set in $\partial A'$. By construction $A \cup M = A' \cup M'$. Fixing a basepoint in $S \subset M$, we compute the fundamental group of $A \cup M$ using the Seifert-Van Kampen Theorem. See Figure 11 for the appropriate generators. Let a denote the generator for A. Note that α and γ are generators of S and M' has generators x, y, and z. Since α and γ are ambient isotopic to c_4 and c_2c_3, respectively, which are (p,q) curves in ∂A, we have

$$\begin{aligned}
\pi_1\left(A \cup M\right) &= \left\langle a,x,y,z \mid a^p = x^2zx^{-2},\, a^p = xyz^{-1},\, y = 1\right\rangle \\
\pi_1\left(A \cup M\right) &= \left\langle a,x,z \mid a^p = x^2zx^{-2},\, a^p = xz^{-1}\right\rangle \\
\pi_1\left(A \cup M\right) &= \left\langle a,x,z \mid a^p = x^2zx^{-2},\, x = a^pz\right\rangle \\
\pi_1\left(A \cup M\right) &= \left\langle a,z \mid a^p = a^pza^pzz\left(a^pz\right)^{-1}\left(a^pz\right)^{-1}\right\rangle \\
\pi_1\left(A \cup M\right) &= \left\langle a,z \mid a^pa^pza^pz = a^pza^pzz\right\rangle \\
\pi_1\left(A \cup M\right) &= \left\langle a,z \mid a^pza^p = za^pz\right\rangle .
\end{aligned}$$

But this is the knot group of the unknot for $p = 0$ and the knot group of the $(2,3)$ torus knot (trefoil) if $p \neq 0$. It follows that R is an unknotted solid torus or a tubular neighborhood of the trefoil.

Next we compute $\pi_1\left(R \cup M'\right)$ where $M' = M \cup B_{12}$. We glue R to $M' = M \cup B_{12}$ along the closure of the entrance set of M. Fixing a basepoint in $cl\left(Ex\right)$ and noting the generators in Figure 19, we apply

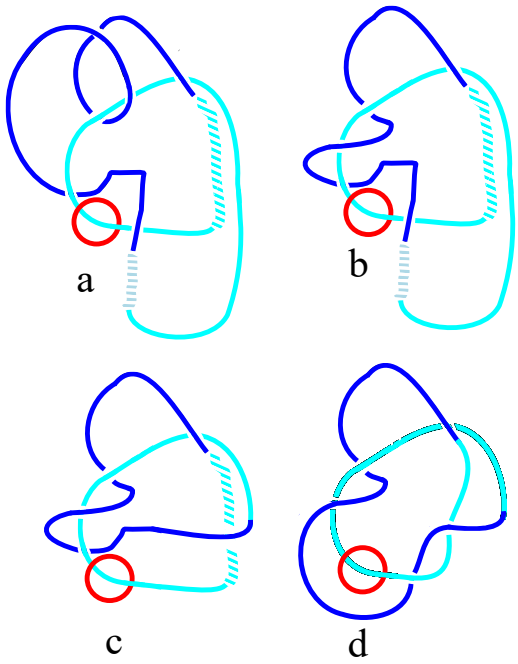


FIGURE 21. Link $a \cup r$ for Case 4(a) of Theorem 3.2

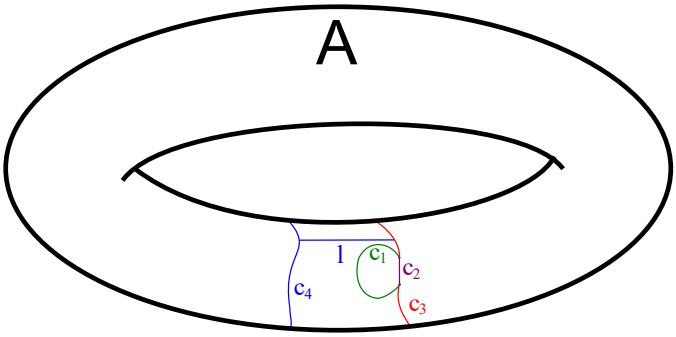


FIGURE 22. Core Loops on A for Case 4(b) of Theorem 3.2

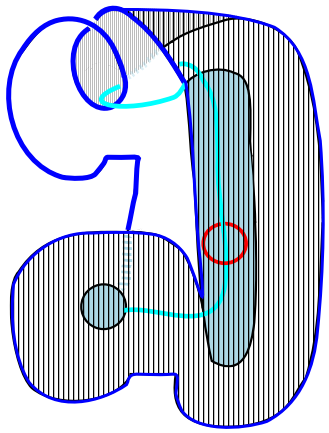


FIGURE 23. Realization of Case 4(b) for Theorem 3.2

the Seifert-Van Kampen Theorem to compute $\pi_1(R \cup M')$. Let e_4, e_{12}, and e_{13} denote core loops of the entrance set of M. As e_4, e_{12}, and e_{13} are parallel (p_1, q_1) curves in ∂R and $zy^{-1}z^{-1}$ is null-homotopic in $\partial M'$ or equivalently $y = 1$, we have $r^{p_1} = x^{-1}yzy^{-1}x$, $r^{p_1} = xyz^{-1}$, $r^{p_1} = xyz^{-1}y^{-1}$, $y = 1$. So

$$\begin{aligned}
\pi_1(R \cup M') &= \left\langle r, x, z \mid r^{p_1} = x^{-1}zx,\, x = r^{p_1}z \right\rangle \\
\pi_1(R \cup M') &= \left\langle r, z \mid r^{p_1} = \left(r^{p_1}z\right)^{-1} z r^{p_1} z \right\rangle \\
\pi_1(R \cup M') &= \left\langle r, z \mid r^{p_1} z r^{p_1} = z r^{p_1} z \right\rangle .
\end{aligned}$$

But this is the knot group of the trefoil if $p_1 \neq 0$ and the unknot if $p_1 = 0$. The template must be standardly embedded by the same arguments as in Case 4(a) above. Thus $a \cup r$ is a trefoil and meridian.

Figure 23 illustrates a realization of Case 4(b) where $a \cup r$ is a trefoil knot and meridian. a is the core of the thickened blue ring while r is the core of the thickened red ring. The link $a \cup r$ removed from the template is depicted in Figure 24.

Case 5 All of the core loops are essential.

We glue A to M by identifying the closure of the exit set of with the corresponding set in ∂A. Fixing a basepoint in $cl(Ex) \subset M$, we compute the fundamental group of $A \cup M$ using the Seifert-Van Kampen Theorem. See Figure 11 for the appropriate generators. Let a be the generator for A. Note that α, β, and γ are generators of $cl(Ex)$ and M has generators x, y, and z. Since α, β, and γ are ambient isotopic to c_4, c_1c_2 and c_2c_3,

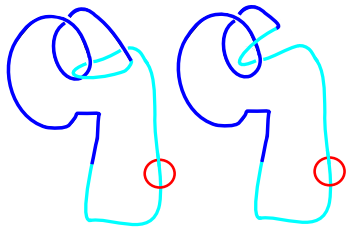


FIGURE 24. Link $a \cup r$ for Case 4(b) for Theorem 3.2

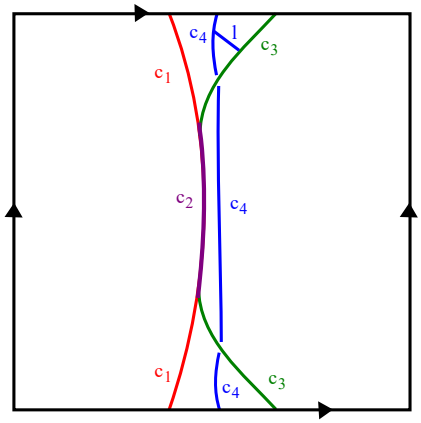


FIGURE 25. Core Loops on Flat Torus Case 5 of Theorem 3.2

respectively, which are parallel (p, q) curves in ∂A, we have

$$\pi_1(A \cup M) = \left\langle a, x, y, z \mid a^p = x^2 z x^{-2},\ a^p = y^{-1},\ a^p = xyz^{-1} \right\rangle.$$

We have the additional relation $a^p = xyz^{-1}y^{-1}$ since $c_1 c_3$ is essential in ∂A which gives $y = 1$ and forces $p = 0$. Therefore,

$$\begin{aligned} \pi_1(A \cup M) &= \langle a, x, y, z \mid x = y = z = 1 \rangle \\ \pi_1(A \cup M) &= \langle a \rangle. \end{aligned}$$

It follows that R is an unknotted solid torus. Thus $a \cup r$ is a Hopf link.

This case is not realizable. To see this, we note that since $p = 0$, the core loops must all be meridians or parallel $(0, 1)$ torus knots on the surface of the unknotted solid torus A. However, when the core loops are placed on the flat torus (see Figure 25), the loop c_4 must intersect either

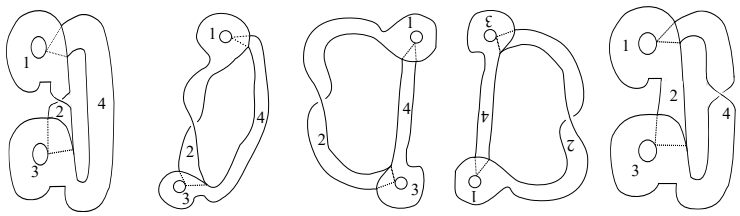


FIGURE 26. $U^+(0,0,0,1)$ (far left) and $U^+(0,1,0,0)$ (far right) are Equivalent

c_1c_2 or c_2c_3. Alternatively, it can be observed that if both c_1c_2 and c_2c_3 are essential in ∂A, the c_1c_3 bounds a disk in ∂A.

By the Gordon-Leucke Theorem [9], the knot complement in S^3 is a complete knot invariant.

□

If a half-twist is placed in band three of template U^+, the new template $U^+(0,0,1,0)$ is formed. However, this template is equivalent to $U^+(1,0,0,0)$. This can be seen by an obvious modification of Figure 26. So we may replace $U^+(1,0,0,0)$ with $U^+(0,0,1,0)$ in Theorem 3.1.

Theorem 3.2. *A simple Smale flow on S^3 with saddle set modeled by $U^+(0,0,0,1)$ is realizable if and only if the link $a \cup r$ is either a Hopf link or a figure eight knot and meridian. In the later case, the saddle set is modeled by standardly embedded template $U^+(0,0,0,1)$ where the bands are untwisted, unlinked, and unknotted. In the Hopf link case, (a) the saddle set is standardly embedded $U^+(0,0,0,1)$ or (b) one or two bands may form (p,q) torus knots about a or r. However, the two looped bands on the left of Figure 5 cannot both be knotted, twisted, and linked.*

Proof: A proof of this theorem is given in [13]. We remark that the proof can be done without the use of the Fox calculus.

□

Consider placing a half twist in band 2 of U^+ which yields the template $U^+(0,1,0,0)$. It is clear from observing Figure 26 that placing a half-twist in band 2 of template U^+ yields a template that is equivalent to $U^+(0,0,0,1)$. So, we may replace $U^+(0,0,0,1)$ with $U^+(0,1,0,0)$ in Theorem 3.2 above.

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School of Mathemaical and Statistical Sciences; Southern Illinois Uni④versity; Carbondale, Illinois 62901
Email address: anthony.sloan@siu.edu

School of Mathematical and Statistical Sciences; Southern Illinois University; Carbindale, Illinois 62901
Email address: mikesullivan@math.siu.edu