TWISTWISE FLOW EQUIVALENCE AND BEYOND ...

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WITH AN APPENDIX ON ALMOST FLOW EQUIVALENCE JOINTLY WITH MIKE BOYLE

Abstract. We survey recent results in classifying embeddings of suspensions of shifts of finite type as hyperbolic basic sets of flows up to twistwise flow equivalence. The problem has been reduced to matrix equivalence over $SL(\mathbb{Z}\mathbb{Z}/2)$. But more is true: G-flow equivalence of G-SFTs, for any finite group G is has a similar algebraic formalization. The appendix proves a new result: all nontrivial faithful irreducible G-SFTs are almost flow equivalent.

1. introduction

Square matrices of nonnegative integers are *flow equivalent* if the suspensions of their corresponding *shifts of finite type* (SFTs) are *topologically equivalent.* (Definitions are in Section 2.) A complete set of easily computed invariants determines flow equivalence of nontrivial irreducible square nonnegative matrices [PS, BF, F2]. When the assumption of irreducibility is dropped the classification of matrices up to flow equivalence becomes harder but has been solved; see [H1, H2, H3] or [H4, BH].

In [Su2] the concept of *twistwise flow equivalence* was introduced to describe the orientability of the stable manifolds of the orbits of a suspended and embedded SFT. The *twist matrices* are square matrices over the semi-group ring

$$\mathbb{Z}_+\mathbb{Z}/2 = \{a + bt \mid a \,\&\, b \text{ are nonnegative integers}\} \quad \text{mod } t^2 = 1.$$

Several computable invariants of twistwise flow equivalence were discovered [Su2, Su3, Su4], but their completeness was unknown and seemed unlikely. In a paper by this author with Mike Boyle [BS] a complete

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algebraic invariant has been found, but it is unknown if it is computable – results in [BS] are more general, hence the "beyond" in our title. This paper surveys these developments. It derives from a series of three lectures given to a graduate student seminar at the University of Maryland in Fall 2002, and again to the Dynamics Seminar at the University of North Texas in Spring 2003. The appendix contains a new result and is joint work with Boyle, who also made many helpful suggestions on a draft of the main body this paper.

2. SYMBOLIC DYNAMICS

A shift of finite type (SFT) is determined by a square matrix over the nonnegative integers $\mathbb{Z}_+$ by way of a directed graph. If M is $n \times n$ matrix over $\mathbb{Z}_+$, we construct a graph $\mathcal{G}_M$ with n vertices and M_{ij} directed edges from vertex i to vertex j. Denote the edges $\mathcal{E}_M = \{e_1, ..., e_k\}$ (k is the sum of entries of M). Let X_M be the set of all bi-infinite sequence from $\mathcal{E}_M$ that can be realized by paths in the graph $\mathcal{G}_M$. The *shift map* $\sigma : X_M \to X_M$ is defined by $\sigma(x)_i = x_{i+1}$. We think of it as taking a step along an path in the graph. A *shift of finite type* is the sequence set with its shift map.

The sequence set X_M is assigned a topology by taking the subset topology of the product space $\mathbb{Z}^{\mathcal{E}_M}$. The shift map is then a homeomorphism.

Example 2.1. Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Label the edges as in Figure 1. Then $x = ...aaaa.bccc...$ is in X_M. Here the dot or "decimal point" tells us that $x_0 = b$. Find all the fixed points of σ. Find all the points of *least period two*, that is the fixed points of $\sigma \circ \sigma$ that are not fixed points of σ.

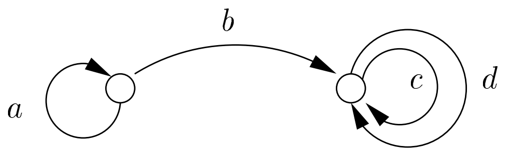

FIGURE 1. Graph for Example 2.1

Definition 2.2. A square matrix M over $\mathbb{Z}$ is *irreducible* if for every i, j which indexes an entry of M there is an n such that $(M^n)_{ij} \neq 0$. An SFT which is generated by an irreducible matrix is also called *irreducible*.

Readers should convince themselves that in the graph of an irreducible matrix over $\mathbb{Z}_+$ there is a path from each vertex to every other vertex. Thus, the matrix in Example 2.1 is reducible (i.e. not irreducible). An SFT is *trivial* if it consists of a finite number of periodic orbits. In this cases any incidence matrix will be a permutation matrix. We will work mostly with nontrivial irreducible SFTs. These can be shown to be Cantor sets.

Definition 2.3 (Topological Conjugacy). Given two SFT (X_i, σ_i), $i = 1, 2$, we say they are *topologically conjugate* if there exists a homeomorphism $h : X_1 \to X_2$ such that $h \circ \sigma_1 = \sigma_2 \circ h$.

It is easy to check that a topological conjugacy takes periodic orbits to periodic orbits, preserving the least period.

Definition 2.4 (Strong Shift Equavalence). Let A and B be square matrices over $\mathbb{Z}_+$. An *SSE-move* from A to B is a dual decomposition $A = RS$, $B = SR$, where R and S are over $\mathbb{Z}_+$, but need not be square. We say A an B are *strong shift equivalent* if there is a finite chain of SSE-moves taking A to B.

It is not yet known if strong shift equivalence is decidable. But many readily computable invariants are unknown. The theorem below is due to R.F. Williams [Wi].

Theorem 2.5 (Fundamental Theorem of SFTs). *Let A and B be square matrices over $\mathbb{Z}_+$. Then X_A is topologically conjugate to X_B if and only if A is strong shift equivalent to B.*

Example 2.6. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = [2]$. Then $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$ while $B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let's construct a topological conjugacy from X_A to X_B. We use the edge and vertex names shown in Figure 2. Define $h : X_A \to X_B$ by letting the i-th coordinate of $y = h(x)$ be e if the edges x_i and x_{i+1} have, respectively, final and initial vertex 1, and be f otherwise. For example:

$$...aabcbcaab.dddddd... \mapsto ...eefefeeef.fffff....$$

The reader should construct the inverse. For the proof of continuity see [LM].

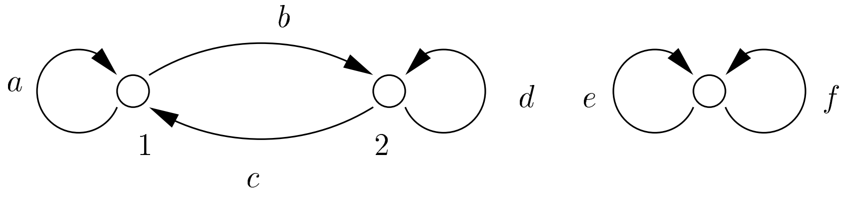

FIGURE 2. These SFTs are toplogically equivalent.

3. Flow Equivalence

Let A be a square matrix over $\mathbb{Z}_+$. Let (X, σ) be the SFT induced by A. Let (F, ϕ_t) be defined by

$$F = X \times \mathbb{R}/(x, t+1) \sim (\sigma(x), t),$$

and

$$\phi_t([x, s]) = [x, s+t].$$

The pair (F, ϕ_t) is called the *mapping torus* or the *suspension flow* of (X, σ). For more details see [LM, §13.6].

Definition 3.1 (Flow Equivalence)**.** Two suspension flows (F_A, ϕ_t) and (F_B, ψ_t) are *topologically equivalent* if there exists a homeomorphism from F_A to F_B taking flow lines to flow lines while preserving the flow direction. We say two SFTs are *flow equivalent* (FE) if their suspensions are topologically equivalent. We also define two square matrices over $\mathbb{Z}_+$ to be FE if their induced SFTs are FE.

Example 3.2. The matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $[1]$ are FE. Their SFTs each consists of a single orbit. Notice these are not SSE.

For permutation matrices FE is determined by just counting the number of closed orbits. Since permutation matrices induce such simple SFTs they are often called *trivial matrices*.

Example 3.3. The matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $[2]$ are FE since they are SSE – think about this. FE is a coarser equivalence relation than SSE.

Example 3.4. The matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ are FE. See Figure 3. Every path that goes through vertex 1 in the graph for A corresponds to a path in the graph for B that goes through $1'$ first.

We define a map from X_A to X_B by replacing each occurrence of a and c in a member if X_A by ae and ce, respectively. Thus,

$$...aaaaa.aaaa.... \mapsto ....aeaeaeae.aeaeaeae....$$

and,

$$....aabddc.bcaabd.... \mapsto ....aeaebddce.bceaeaebd...$$

This is clearly not a topological conjugacy since a fixed point is taken to a point with least period two. The proof that it induces a topological equivalence of F_A and F_B is given in [PS], where flow equivalence was first defined. But, the essential idea can be seen in Figure 4. The matrix A can be recovered from F_A as the incidence matrix for a cross section partitioned into two pieces, 1 & 2. If we add a third partition member $1'$ to this cross section that is parallel to but just before 1, we get the matrix B as an incidence matrix. Thus, heuristically, it seems A and B should be FE. (The partitions are more properly referred to as *Markov partitions*; a precise definition can be found in [PS] or most dynamical systems textbooks.)

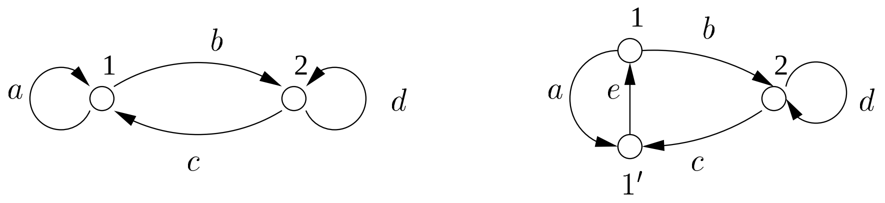

FIGURE 3. FE Graphs

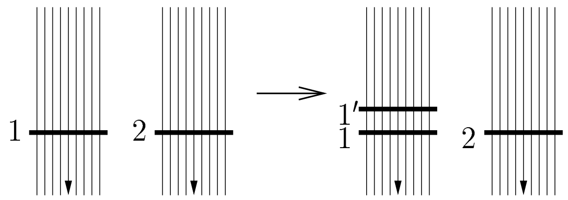

FIGURE 4. Different cross sections, same flows.

Definition 3.5. A PS-move of a matrix A is defined by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 & \cdots \\ a_{11} & 0 & a_{12} & \cdots \\ a_{21} & 0 & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix},$$

or the inverse of this.

Theorem 3.6 (Parry & Sullivan [PS]). *The matrix moves SSE and PS generate FE. That is any FE between matrices can be realized by a finite chain of SEE and PS moves.*

Invariants 3.7. For A an $n \times n$ matrix over $\mathbb{Z}$ define

$$PS(A) = \det(I - A) \qquad \text{(The Parry-Sullivan Number)}$$

and,

$$BF(A) = \frac{\mathbb{Z}^n}{(I-A)\mathbb{Z}^n}. \qquad \text{(The Bowen-Franks Group)}$$

These are invariants of FE; see [PS, BF], respectively.

The Bowen-Franks group of a SFT is a finitely generated Abelian group. Any $n \times n$ integral matrix M defines a finitely generated Abelian group via $\frac{\mathbb{Z}^n}{M\mathbb{Z}^n}$. It can be determined if two such groups are isomorphic by examining their *Smith normal forms.* There is a standard algorithm taking a matrix to its Smith normal form (see any graduate algebra text).

Remark 3.8. $|BF| = |PS|$, unless $PS = 0$, in which case $|BF| = \infty$.

Theorem 3.9 (Franks [F2]). *PS and BF are a complete set of invariants for FE of nontrivial irreducible square matrices over* $\mathbb{Z}_+$.

4. Application to Templates for Smale Flows

A C^1 flow ϕ_t on a compact manifold M is called *structurally stable* if any sufficiently close approximation ψ_t in the C^1 topology is topologically equivalent. Structurally stable C^1 flows have a hyperbolic structure on their chain-recurrent sets [Hu]. We define these concepts next.

A point $x \in M$ is *chain-recurrent* for ϕ_t if for every $\epsilon > 0$ and $T > 0$ there exists a chain of points $x = x_0, \dots, x_n = x$ in M, and real numbers $t_0, \dots, t_{n-1}$ all bigger than T such that $d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$ when ever $0 \leq i \leq n-1$. The set of all such points is called the chain-recurrent set $\mathcal{R}$. It is a compact set invariant under the flow.

A compact invariant set K for a flow ϕ_t has a *hyperbolic structure* if the tangent bundle of K is the Whitney sum of three bundles E^s, E^u, and E^c each of which invariant under $D\phi_t$ for all t. Furthermore, the vector field tangent to ϕ_t spans E^c and there exist real numbers $C > 0$ and $\alpha > 0$ such that

$$\|D\phi_t(v)\| \leq Ce^{-\alpha t}\|v\| \text{ for } t \geq 0 \text{ and } v \in E^s,$$

$$\|D\phi_t(v)\| \leq Ce^{\alpha t}\|v\| \text{ for } t \leq 0 \text{ and } v \in E^u.$$

We also define the local stable and unstable manifolds associated to an orbit $\mathbf{O}$. They are respectively,

$$W^s_{\text{loc}}(\mathbf{O}) = \bigcup_{x \in \mathbf{O}} \{y \in M | d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to \infty \text{ and } d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for } t \geq 0\}$$

and

$$W^u_{\text{loc}}(\mathbf{O}) = \bigcup_{x \in \mathbf{O}} \{y \in M | d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to -\infty \text{ and } d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for } t \leq 0\}.$$

The global stable and unstable manifolds are defined by removing the condition that $d(\phi_t(x), \phi_t(y)) \leq \epsilon$.

It was shown by Smale [Sm] that if the chain-recurrent set $\mathcal{R}$ of flow has a hyperbolic structure then $\mathcal{R}$ is the union of a finite collection of disjoint invariant compact sets he dubbed the *basic sets*. Each basic set $\mathcal{B}$ contains an orbit whose closure is $\mathcal{B}$. The periodic orbits of a basic set $\mathcal{B}$ are dense in $\mathcal{B}$.

Definition 4.1. A flow ϕ_t on a manifold M is called a *Smale flow* provided

(a) the chain-recurrent set $\mathcal{R}$ of ϕ_t has a hyperbolic structure,
(b) the basic sets of $\mathcal{R}$ are one-dimensional, and
(c) the stable manifold of any orbit in $\mathcal{R}$ has transversal intersection with the unstable manifold of any other orbit of $\mathcal{R}$.

Some references allow for zero-dimensional basic sets in the definition of Smale flows but we will be working with *nonsingular flows*, flows without rest points. Smale flows on compact manifolds are structurally stable under C^1 perturbations but are not dense in the space of C^1 flows. For dim $M = 3$ a basic set of a Smale flow either consists of a single closed orbit or it is the suspension of an irreducible SFT. A nontrivial basic set is said to be *chaotic*. It is easy to see that each attracting and repelling basic set is a closed orbit. The saddle sets may be chaotic.

For a chaotic saddle set of a Smale flow in a 3-manifold one can construct a neighborhood that is foliated by local stable manifolds of orbits in the flow. Collapsing in the stable direction produces a branched 2-manifold. With a semi-flow induced from the original flow, this branched 2-manifold is known as a *template*. The template models the basic saddle set in that the saddle set itself can be recovered from the template via an inverse limit process and that any knot or link of closed orbits in the saddle set flow is smoothly isotopic to an equivalent knot or link of closed orbits in the template's semi-flow. The proof of this is due to Birman and Williams [BiWi] and can also be found in [GHS, Theorem 2.2.4]. Figure 5 shows two templates, the one on the left is know as the Lorenz template and the one on the right arises for the suspension of the Smale horseshoe map.

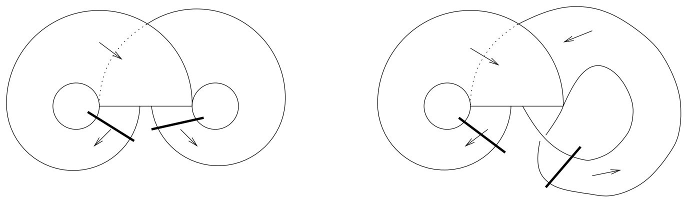

FIGURE 5. Lorenz and Smale Horseshoe Templates

The symbolic dynamics can be recovered from a template by constructing a Markov partition and deducing an incidence matrix of the first return map. For the two templates in Figure 5 an obvious choice for the partition is a pair of line segments where each segment cuts across one of the two bands. The matrix in each case is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. And so the Lorenz and Horseshoe templates are each derived from suspensions of the full 2-shift. Thus, their inverse limits are flow equivalent. But, these two templates look different and I found it bothersome that standard symbolic dynamics did not distinguish them. It was this nagging sense that these two templates *should* be different that lead to the developments described below.

To capture the twisting in the bands we modify the incidence matrix by using the symbol t^p to count the twisting as an orbit goes from partition element i to j. For the Lorenz and Horseshoe templates this produces $\begin{bmatrix} t^2 & t^2 \\ t^2 & t^2 \end{bmatrix}$ and $\begin{bmatrix} t^2 & t^2 \\ t^3 & t^3 \end{bmatrix}$ respectively. Now at least the matrices are different. We use these to define a type of *zeta function.* For a shift map σ the zeta function is

$$\zeta_\sigma(t) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m\right),$$

where N_m is the cardinality of the fixed point set of σ^m, the m-th iterate of σ. If its incidence matrix over $\mathbb{Z}_+$ is A then a standard result [Sh] gives that

$$\zeta_\sigma(t) = \frac{1}{\det(I - tA)}.$$

If we let $A = A(t)$ be *twist matrix* for a template and set

$$\zeta_A(t) = \frac{1}{\det(I - A(t))},$$

we get a zeta-like function that tracks periodic orbits by the amount of twisting. The formal definition of ζ_A is given in [Su1]. There are some important caveats. The definition of twisting is not the standard one used in knot theory, and ζ_A fails to to be well-defined all the crossings in the template are of the same type. And ζ_A is not an invariant of flow equivalence. All these problems are circumvented in the next section by redefining twist matrices mod $t^2 = 1$.

5. Twistwise Flow Equivalence

Let $G = \langle\, t \,|\, t^2 = 1 \,\rangle \cong \mathbb{Z}/2$. Given a matrix $A(t)$ over $\mathbb{Z}_+G$ (a twist matrix) we define the *ribbon set* R of $A(t)$ to be a certain fiber bundle over the suspension flow (F, ϕ) of $A(1)$. The fiber will be the interval $(-1, 1)$. Without loss of generality we can assume $A(t)$ has only ones, tees, and zeros, since $A(t)$ is SSE to such a matrix [F1]. Then place an oriented Markov partition $\{d_1, \ldots, d_k\}$ on a cross section of F which induces $A(1)$ as its incidence matrix. For y in any d_i let $\tau(y)$ be the first return time for y. Let

$$F_{ij} = \{x \in F | x \in \phi_t(y), \text{where } y \text{ is such that } y \in d_i \;\& \\ \phi_{\tau(y)}(y) \in d_j, \text{ and } 0 \le t \le \tau(y)\}.$$

In words, F_{ij} is the union of segments of flow lines from d_i to d_j. Some F_{ij} may be empty. Let $R_{ij} = F_{ij} \times (-1, 1)$. Attach the R_{ij}'s so that the core is F and the gluings of the end fibers are identity maps if $A_{ij} = 1$ and multiplications by -1 if $A_{ij} = t$. Call this set R. We can place a flow on R that agrees with F at is core and has flow lines converging to the core elsewhere, as in Figure 6. This is the ribbon set for $A(t)$; it can be shown to be independent of the choice of Markov partition.

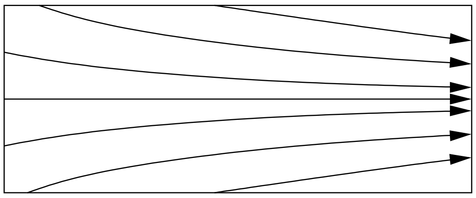

Figure 6. Flow on a chart of the ribbon set.

For a given chaotic saddle set of a Smale flow on a 3-manifold, the ribbon set is topologically equivalent to the stable portion of the tangent bundle. (In [Su4] it was mistakenly confounded with a local stable manifold. But, ribbon sets can be thought of as *infinitesimal stable manifolds.*)

Definition 5.1. Two twist matrices are *twistwise flow equivalent* if they have topologically equivalent ribbon sets.

Notation: Let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If $A(t)$ is $n \times n$, let $A(T)$ be the $2n \times 2n$ matrix over $\mathbb{Z}_+$ formed by replacing each entry $a_{ij} + b_{ij}t$ of $A(t)$ with $\begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{bmatrix}$. Then $A(T)$ is the incidence matrix for the SFT defined by placing a flow on the boundary of the ribbon set of $F_{A(t)}$ and using the same Markov partition. The flow $F_{A(T)}$ is a double cover of $F_{A(1)}$ that records the "twisting" give by $A(t)$.

Invariants 5.2. The following are invariants of twistwise flow equivalence.

- $PS^{\pm}(A(t)) = PS(A(\pm 1))$.
- $BF^{\pm}(A(t)) = BF(A(\pm 1))$.
- $BF^{\partial}(A(t)) = BF(A(T))$.
- $O(A(t))$ equals "orientable" if $\operatorname{tr}(A^k(t))$ has no tees for all k, and equals "nonorientable" otherwise.

These where established in [Su2, Su3, Su4]. It is easy to show that $O(A(t))$ can be found by checking the trace of $A(t)^n$ for t's for only a finite number of powers. For example $O(\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}) =$ orientable. A more sophisticated view of $O(A)$ will be given in Section 6.

Example 5.3. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} t & 1 \\ 1 & 1 \end{bmatrix}$. Then for both matrices $PS^{\pm} = -1$ and all the Bowen-Franks groups are trivial. But they are distinguished by orientability.

Example 5.4. The matrices $\begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ are not distinguished by the invariants above. Are they twistwise flow equivalent? I asked this question in 1997 [Su4]. The answer was found in 2002 and will appear in a joint paper with Mike Boyle [BS]. We begin our discussion of these ideas in the next section.

6. Enter K-theory

There is a new approach to symbolic dynamics. It is being developed by a number of researches largely in response to the difficulties that arose around attempts to settle the Williams Conjecture (that Strong Shift Equivalence could be reduced to a weaker and computable relation called Shift Equivalence). The Williams Conjuncture is now known to be false [KR].

The new approach exploits tools from algebraic K-theory. I initially found the prospect of having to learn K-theory rather daunting. Fortunately much help is available. The expository articles on K-theory and symbolic dynamics [B2], [BW] and [Wa] were most helpful. For a beginners look at K-theory itself I recommend [Si], and [R] for a more advanced treatment. Fortunately, few details of K-theory are needed to understand its application in symbolic dynamics. The central feature of the new approach is that the awkward matrix moves for SSE and PS are replaced with the more natural row and column operations, but these act on infinite matrices. This paper confines itself to how this new approach was applied to settle the twistwise flow equivalence problem.

6.1. **Positive Equivalence.** In this subsection we restrict ourselves to the case where $G = \langle 1 \rangle$, the trivial group. Given an $n \times n$ matrix A define A_∞ to be the infinite matrix, one indexed by i, j in $\mathbb{N} = \{1, 2, 3, ...\}$, whose upper right corner agrees with A and is zero elsewhere. We let $I - A_\infty$ be the infinite identity matrix minus A_∞. Let $SL(\mathbb{N}, \mathbb{Z})$ be the set of infinite matrices indexed by $\mathbb{N}$ with entries in $\mathbb{Z}$ and determinant equal to one. For U and V in $SL(\mathbb{N}, \mathbb{Z})$ let $(U, V)(A) = I - U(I - A)V = B$. That is, B is determined by $U(I - A)V = I - B$. [1]

Let for $i \neq j$ let E_{ij} be the infinite elementary matrix with 1 as its ij-entry and equal to the identity matrix elsewhere.

Definition 6.1. Let A and B be a square matrices over $\mathbb{Z}_+$ (not necessarily of the same size), and assume the ij-entry of A is positive. Then there is a *basic positive equivalence* (BPE) from A to B if (I, E_{ij}) or (E_{ij}, I) takes A_∞ to B_∞. Because we want to define an equivalence relation, we will say there is a BPE from B to A, whenever there is one from A to B. If there is a sequence of basic positive equivalences from A to B we say there is a *positive equivalence* (PE) from A to B, and write $A \overset{+}{\sim} B$. Now PE is an equivalence relation.

Definition 6.2. A matrix M over $\mathbb{Z}^+$ is *essentially irreducible* if it has a unique principal submatrix that is irreducible and that is contained

[1] In [BS] $(U, V)(M)$ was defined to be UMV and one worked directly with I minus the incidence matrix.

in no larger irreducible principal submatrix; such a submatrix is called the *irreducible core* of M.

Example 6.3. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and apply $(I, E_{32}(1))$. We get $A \overset{+}{\sim} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The corresponding irreducible core is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Theorem 6.4. *PE and FE are the same.*

Sketch of Proof. That PE implies FE can be observed in Figure 7; it shows how a BPE effects a graph (ignore the labels for now). This was first shown explicitly by Boyle [B3] but was implicit in Franks paper [F2]. The other direction is harder. It is well known the any SSE can be broken down into basic splitting and their inverses (amalgamations). One shows that these can be factored into BPEs. The PS move can also be factored into BPEs. This direction is due to Boyle [B3]. $\square$

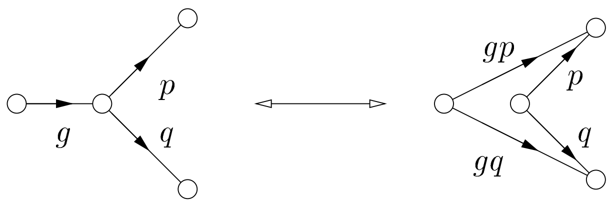

FIGURE 7. BPE gives a FE

Example 6.5. (a)Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $(E, I)(A) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. We see in Figure 8 that one edge from vertex 1 to 2 is deleted, but an edge is added for each length 2 path that started with the removed edge.

(b) Next observe that $(I, E)(B) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. We see in Figure 9 that he same edge is deleted but now we add an edge for each length 2 path that ended with the deleted edge.

(In Figures 8 & 9, the edge to be deleted in the left graph is dashed and the edges added to form the right graph are dashed.)

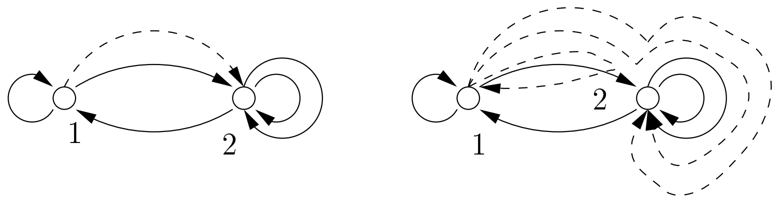

FIGURE 8. Graphs for Example 6.5a

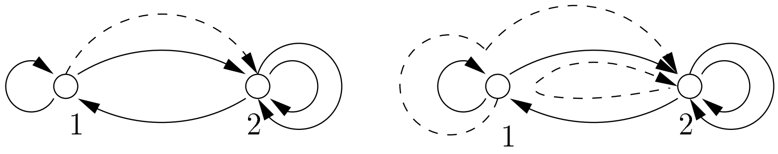

FIGURE 9. Graphs for Example 6.5b

We have traded one problem for another. The awkward matrix moves of SSE and PS have been replaced by row and column operations. However, we must now tread very carefully lest our new matrix fail to be nonnegative. The next result eliminates this difficulty. It was proved by Boyle [B3, Theorem 3.3] in greater generality than we give here; specifically the matrices were allowed to be reducible and the statement of the theorem included special notation for tracking irreducible components.

Theorem 6.6. *Let A and B be essentially irreducible square matrices over $\mathbb{Z}_+$. Suppose U and V are in $SL(\mathbb{N}, \mathbb{Z})$ and $(U, V)(A) = B$. Then (U, V) can be factored into BPEs.*

The proof of the Theorem 6.6 involves an intricate and clever series of matrix moves.

6.2. **Back to twistwise flow equivalence.** We return to the setting $G = \langle t \mid t^2 = 1\rangle$, but stress that many of the results discussed hold for any finite group. In particular there is a notion of *G-flow equivalence*, which is defined algebraically, that generalizes twistwise flow equivalence. The idea of BPE still works in this setting. The elementary matrices $E_{ij}(g)$ have ij-entry $g \in G$, $i \neq j$. We can act on a matrix A over $\mathbb{Z}_+G$ with $(E_{ij}(g), I)$ and $(I, E_{ij}(g))$, provided the ij-entry of A has g as a summand. See Figure 7, but now pay close attention to the labels. Theorems 6.4 and 6.6 were generalized to the case where G is a finite group in [BS]. For the former this was straight forward, even the finiteness of G was not required. For the generalization of Theorem 6.6

more needs to be said. Both the finiteness of G and the irreducibility assumption will be required.

Suppose A is a matrix over $\mathbb{Z}_+G$. We associate to A a labeled graph $\mathcal{G}_A$ such the there is an edge from vertex i to j with label g for each occurrence of g in the ij-entry of A. For example, if $A(i,j) = 2+3g+12h$ there would be two edges with label 1, the group identity element, three with label g and 12 with label h. The *weight* of an allowed path $e_1e_2\dots e_k$ is the group product of the labels in order. (For finite G [BS, §2] shows that G labeled SFTs can be viewed as SFTs with a free right group action. Then a G-FE is a flow equivalence that the commutes with the group action. We will only need this point of view in the appendix.)

Definition 6.7. Suppose G is a finite group, A is an essentially irreducible matrix over $\mathbb{Z}_+G$ and i is a vertex indexing a row of the irreducible core of A. Then $W_i(A)$ is the subgroup of G which is the set of weights of paths from i to i, and the weight class of A, $W(A)$, is the conjugacy class of $W_i(A)$ in G.

That the weight class is well defined is shown in [BS] – the finiteness of G and the irreducibility of A are used. In the case that G is Abelian all of the $W_i(A)$ are the same and we may talk about the *weight group* of A. If $G \cong \mathbb{Z}/2$ then $W(A)$ is either G or trivial. It is equivalent to the orientation invariant $O(A)$.

The promised generalization of Theorem 6.6 is given by Theorem 6.3 of [BS]. We restate it below for the case $G \cong \mathbb{Z}/2$. First note that if A and B have trivial weight groups then they are twistwise flow equivalent if and only if the PS^+ and BF^+ invariants are equal. (It is not hard to show that if $W(A)$ is trivial, A is twistwise flow equivalent to a matrix over Z_+.)

Theorem 6.8. *Let $G = \langle t \mid t^2 = 1\rangle$. Let A and B be essentially irreducible matrices over $\mathbb{Z}_+G$ and assume both have weight group G. Then A and B are twistwise flow equivalent if and only if there is a $SL(\mathbb{N},\mathbb{Z}G)$ equivalence from $I - A_\infty$ to $I - B_\infty$.*

Example 6.9. Let $A = \begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $E(I-A) = I - B$, A and B are twistwise flow equivalent. This settles the question raised in Example 5.4. Notice E does not give a basic positive equivalence. However, following the philosophy of the proofs in [BS], we let $Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Then (I, Q_1), (I, Q_2),

(E, I), (I, Q_2^{-1}), (I, Q_1^{-1}) is a sequence of basic positive equivalences taking A to B.

To fully exploit Theorem 6.8 we would like to have an algorithm that determines when two matrices are $SL(n, \mathbb{Z}G)$ equivalent. If the ring $\mathbb{Z}G$ was a PID then we could put two such matrices into their Smith normal forms and compare them. (See any graduate algebra text for this result.) But even for $G \cong \mathbb{Z}/2$ is not a PID. There are zero divisors: $(1-t)(1+t) = 0$. To the best of our knowledge the general problem of deciding $SL(n, \mathbb{Z}G)$ has not been explicitly addressed in the literature. The problem may be quite difficult. There are matrices over $\mathbb{Z}\mathbb{Z}/2$ that are not equivalent to a triangular matrix or to their own transpose [BS, §8].

However, there is Smith normal form for a special case [BS, §8].

Theorem 6.10. *Let $G = \mathbb{Z}/2$. Let M be an $n \times n$ matrix over $\mathbb{Z}G$. Write $M = A + Bt$ with A and B $n \times n$ matrices over $\mathbb{Z}$. If $\det(A+B)$ is is not divisible by four, then M is $SL(n, \mathbb{Z}G)$-equivalent to a Smith normal form. This is the form corresponding to (C, D), where C and D are the Smith normal forms for $A + B$ and $A - B$.*

6.3. Open questions.

- Can these results be extended to infinite groups? The group $\mathbb{Z}^n$ is of special interest in ergodic theory. The weights are probabilities which generate of copy of $\mathbb{Z}^n$ embedded as a multiplicative subgroup of the positive reals.
- Can these results be extended to reducible matrices?
- Is there an an algorithm to classify matrices over $SL(n, \mathbb{Z}\mathbb{Z}/2)$?

Appendix A. Almost Flow Equivalence by Mike Boyle and Michael C. Sullivan

For this appendix, we switch to joint authorship and prove a new result (Theorem A.1).

Theorem A.1. *Let G be a finite group. Then all nontrivial faithful irreducible G-SFTs are almost flow equivalent.*

We begin with some definitions. Let (X_i, σ_i) (or just X_i) denote an irreducible SFT and let $(F_i, (\phi^i)_t)$ (or just F_i) denote its standard suspension flow (Definition 3.1). An irreducible SFT is *trivial* if it contains only one orbit; equivalently, the (mapping torus) domain of its suspension flow is a topological circle. A *semiequivalence* of flows $f : F_i \to F_j$ is a continuous surjection whose restriction to any orbit in the domain is an orientation preserving local homeomorphism onto some orbit in the range. A *semiconjugacy* of flows is a semiequivalence $f : F_i \to F_j$ such that, in addition, $(\phi^j)_t f = f(\phi^i)_t$.

Irreducible SFTs X_1, X_2 are *almost topologically conjugate* if there is a third irreducible SFT X_3 such that for $i = 1, 2$ there is a continuous shift-commuting surjection $f_i : X_3 \to X_i$ which is uniformly finite to one (i.e. there is a uniform finite bound on the number of preimages of any point) and one-to-one almost everywhere (i.e. any point of X_i in a bilaterally transitive orbit has a unique preimage). (Here X_3 is an *almost conjugate extension* of X_i.) Note, such a map f_i induces a semiconjugacy of flows $F_3 \to F_i$. We have then the following natural flow equivalence analogue of almost topological conjugacy. Irreducible SFTs $(X_1, \sigma_1), (X_2, \sigma_2)$ are *almost flow equivalent* if there is a third irreducible SFT (X_3, σ_3) such that for $i = 1, 2$ there is a semiequivalence of flows $F_3 \to F_i$ which is uniformly finite to one and one-to-one almost everywhere (i.e. any point on a bilaterally transitive flow line has a unique preimage).

Almost topological conjugacy is a weakening of conjugacy which is useful in particular for studying the SFTs with respect to certain invariant measures. One of the basic results in symbolic dynamics is the Adler-Marcus Theorem : two irreducible SFTs are almost topologically conjugate if and only if they have the same topological entropy and period (see [AM] or [LM, Theorem 9.3.2]). The flow equivalence analogue of the Adler-Marcus Theorem is the following fact [B4]: all nontrivial irreducible SFTs are almost flow equivalent. This is the result which is generalized to G-SFTs by Theorem A.1.

Let G be a group. A G-SFT is an SFT together with a continuous right G action which commutes with the shift (i.e., for all x, g we have $(\sigma x)g = \sigma(xg)$). We will only consider finite groups. A G-SFT is irreducible and nontrivial if the underlying SFT is. The G action is *faithful* if no element other than the identity in G acts by the identity map. A *faithful G-SFT* is a G-SFT for which the G action is faithful. The G action on a G-SFT X_i induces in an obvious way a G action on the suspension flow $(F_i, (\phi^i)_t)$ such that $(\phi^i)_t g = g(\phi^i)_t$ for all g in G. With this action we call F_i a G-flow. We say irreducible G-SFTs X_1, X_2 are *almost flow equivalent* (as G-SFTs) if there are semiequivalences of flows $F_3 \to F_1$, $F_3 \to F_2$ as above for which in addition each semiequivalence $F_3 \to F_i$ is equivariant with respect to the G-action. The relation of being almost flow equivalent is indeed an equivalence relation, by a standard type of pullback argument (compare [AM, Theorem 2.17]).

A G-SFT is *free* if the G action is free, i.e., if $g \in G$ and there exists x in the SFT such that $gx = x$, then g must be the identity element of

G. We will summarize some facts reviewed in detail in [BS, Section 2]. Suppose that A is a square matrix over $\mathbb{Z}_+G$. Then A gives rise to a G-labeled directed graph, where the adjacency matrix of the unlabeled graph is denoted $|A|$ (it is the image of A under entrywise application of the augmentation map $\mathbb{Z}G \to \mathbb{Z}$). This graph defines an SFT $X_{|A|}$ with a continuous map into G, from which a skew product $\mathcal{S}_A$ may be constructed. This skew product is an SFT which carries a natural G-action with which it is a free G-SFT. Conversely, any free G-SFT is conjugate to one induced by such a matrix A. (A conjugacy of G-SFTs is simply a G-equivariant topological conjugacy of SFTs.)

For the proof of Theorem A.1, we will use three more facts, which follow from the adjacent citations.

Fact A.2. [B4, Lemma 2.4] Every irreducible nontrivial SFT is flow equivalent to a mixing SFT with entropy $\log 2$.

Fact A.3. [AKM, Theorem 3] Let G be a finite group. Then any irreducible faithful G-SFT has an almost conjugate extension to an irreducible free G-SFT.

Fact A.4. [AKM, Theorem 4] Let G be a finite group. Then two faithful mixing G-SFTs are almost topologically conjugate if and only if they have the same entropy.

Remark A.5. Fact A.4 is a generalization of the Adler-Marcus Theorem to G-SFTs. For the irreducible case and more general actions, also see [AKM]. For a different proof see [P]. For analogous generalizations of *right closing almost topological conjugacy* to G-SFTs, and some clarification of the [AKM] invariants for irreducible G-SFTs (a special case in [AKM]), see [D].

We can now prove Theorem A.1. Suppose G is a finite group and X_1, X_2 are irreducible nontrivial faithful G-SFTs. By Fact A.3, each X_i has an almost conjugate extension to an irreducible free G-SFT. Thus without loss of generality we may assume that X_i is a skew product over an SFT $X_{|A(i)|}$ defined by an irreducible matrix $A(i)$ over $\mathbb{Z}_+G$, with weights class G. By Fact A.2, the SFT $X_{|A(i)|}$ is flow equivalent to a mixing SFT of entropy $\log 2$. This flow equivalence naturally lifts to the skew product. So without loss of generality, we may assume that each $X_{|A(i)|}$ is mixing with entropy $\log 2$. By the Adler-Marcus Theorem, there is a common mixing almost conjugate extension of $X_{|A(1)|}$ and $X_{|A(2)|}$ to some X_C. This can be done by one block codes [AM], under which the G-labelings (defined from the $A(i)$) on the graphs with adjacency matrices $|A(i)|$ lift to G-labelings on the graph with adjacency matrix C. Thus without loss of generality, we may assume that each $|A(1)| = |A(2)| = C$ where X_C is a mixing SFT of entropy $\log 2$.

Now the only barrier to citing Fact A.4 is the possibility that one or both of the skew product SFTs S_i defined from $A(i)$ is not mixing. (These skew products remain irreducible SFTs through all the constructions.) Let $\mathcal{G}_i$ be the labeled graph defined by A_i. Let $\mathcal{G}$ denote the underlying unlabeled graph, the same for $\mathcal{G}_1$ and $\mathcal{G}_2$. The period of the irreducible SFT S_i is the g.c.d. of the lengths of those loops in $\mathcal{G}_i$ which have weight e (where e denotes the identity element in G). If this g.c.d. is not 1 for the $\mathcal{G}_i$, then we will pass to new labeled graphs $\mathcal{G}'_i$, with the same underlying unlabeled graph $\mathcal{G}'$, as follows.

By positive entropy, there are distinct (not necessarily simple) loops ℓ_1, ℓ'_1 in $\mathcal{G}_1$ of equal length with weight e. Likewise there are loops ℓ_2, ℓ'_2 of equal length, which are distinct from each other and from ℓ_1, ℓ'_1, and which have weight e in $\mathcal{G}_2$. After passing to the same higher block presentation of C (pulling along the G-labelings), we can assume without loss of generality that there is an edge e_1 traversed exactly once by ℓ_1 but not at all by ℓ'_1, ℓ_2 or ℓ'_2; and there is an edge e_2 traversed exactly once by ℓ_2 but not at all by ℓ'_2, ℓ_1 or ℓ'_1. For $i = 1, 2$, construct $\mathcal{G}'_i$ from $\mathcal{G}_i$ by making the following changes to $\mathcal{G}_i$

- Delete the labeled edges e_1 and e_2.
- For $j = 1, 2$, add a new vertex v_j; add a new edge e'_j from the initial vertex of e_j to v_j; and add a new edge e''_j from v_j to the terminal vertex of e_j.
- Label e''_1 and e''_2 with the identity element of G.
- Label e'_1 and e'_2 respectively with the labels of e_1 and e_2 in $\mathcal{G}_i$.

We have $\mathbb{Z}_+G$ matrices B_1, B_2 describing the new labeled graphs, and their induced skew products are clearly G-flow equivalent respectively to S_1 and S_2. Moreover, these skew products must be mixing. Finally, because $|B_1| = |B_2|$, they also have equal entropy. By Fact A.2, they are almost flow equivalent. This concludes the proof of Theorem A.1.

Finally we remark that Araújo [A] studies almost flow equivalence of *stochastic systems*. These can be viewed as SFTs with a skew product over a group which is a copy of $\mathbb{Z}^n$ embedded in the multiplicative group of positive real numbers [P]. Araújo shows that if the group is infinite cyclic, then the group is the only invariant of almost flow equivalence, and he shows that this is not true for more general groups.

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References

[AM] R. Adler and B. Marcus. Topological entropy and equivalence of dynamical systems. *Mem. Amer. Math. Soc.* **20** (1979), no. 219, iv+84 pp.

[AKM] R. Adler, B. Kitchens and B. Marcus. Finite group actions on shifts of finite type. *Ergodic Theory Dynam. Systems* **5** (1985), no. 1, 1–25.

[A] P. Araùjo. A stochastic analogue of a theorem of Boyle's on almost flow equivalence. *Ergodic Theory Dynam. Systems* **13** (1993), no. 3, 417–444.

[BiWi] J. Birman and R. Williams, Knotted periodic orbits in dynamical systems II: knot holders for fibered knots. *Cont. Math.*, 20, (1983) 1–60.

[B1] M. Boyle. Symbolic dynamics and matrices. In *Combinatorial and graph-theoretical problems in linear algebra (Minneapolis, MN, 1991)*, volume 50 of *IMA Vol. Math. Appl.*, pages 1–38. Springer, New York, 1993.

[B2] M. Boyle. Positive K-theory and symbolic dynamics. In *Dynamics and Randomness*, editors A. Maass, S. Martinez and J. San Martin, Kluwer, pp. 31–52, 2002.

[B3] M. Boyle. Flow equivalence of shift of finite type via positive factorizations. *Pacific J. of Math.*, 204 (2), 2002, 273 – 317.

[B4] M. Boyle. Almost flow equivalence for hyperbolic basic sets. *Topology* **31** (1992), no. 4, 857–864.

[BH] M. Boyle and D. Huang. Poset block equivalence of integral matrices. *Trans. Amer. Math. Soc.* **355** (2003), 3861-3886.

[BS] M. Boyle and M. Sullivan. Equivariant Flow Equivalence for Shifts of Finite Type. Preprint (2004): `http://www.math.siu.edu/sullivan/`

[BW] M. Boyle and J. Wagoner. Positive algebraic K-theory and shifts of finite type. *Modern Dynamical Systems and Applications*, Cambridge University Press (2004), 45-66. Preprint: `http://www.math.umd.edu/~mmb/papers/index.html`

[BF] R. Bowen and J. Franks. Homology for zero-dimensional basic sets. *Annals of Math.* **106** (1977), 73–92.

[D] A. Dykstra. Right closing almost conjugacy for G-shifts of finite type. Preprint (2004): `http://www.math.umd.edu/~dykstraa/`

[F1] J. Franks. Knots, links and symbolic dynamics. *Ann. of Math.* (2) 113 (1981), no. 3, 529–552.

[F2] J. Franks. Flow equivalence of subshifts of finite type. *Ergod. Th. & Dynam. Sys.* **4** (1984), 53–66.

[GHS] R. Ghrist, P. Holmes, and M. Sullivan. *Knots and links in three-dimensional flows.* Springer-Verlag, New York, NY, 1997.

[Hu] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. of Math.* (2), 145(1) (1997) 81–137.

[H1] D. Huang. Flow equivalence of reducible shifts of finite type. *Ergod. Th. & Dynam. Sys.* **14** (1994), 695–720.

[H2] D. Huang. Flow equivalence of reducible shifts of finite type and Cuntz-Krieger algebra. *J. Reine. Angew. Math.* **462** (1995), 185–217.

[H3] D. Huang. The classification of two-component Cuntz-Krieger algebras. *Proc. Amer. Math. Soc.* **124**(2) (1996), 505–512.

[H4] D. Huang. Automorphisms of Bowen-Franks groups for shifts of finite type. *Ergod. Th. & Dynam. Sys.* **21** (2001), 1113–1137.

[KR] K. H. Kim and F. Roush. The Williams conjecture is false for irreducible subshifts. *Ann. of Math.* (2) 149 (1999), no. 2, 545–558.

[LM] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press (1995).

[P] W. Parry. Notes on coding problems for finite state processes. *Bull. London Math. Soc.* 23 (1991), no. 1, 1–33.

[PS] W. Parry and D. Sullivan. A topological invariant for flows on one-dimensional spaces. *Topology* **14**(1975), 297–299.

[R] J. Rosenberg. *Algebraic K-theory and Its Applications*, Springer-Verlag (1994).

[Sh] M. Shub. *Global Stability of Dynamical Systems*, (with the collaboration of Albert Fathi and Remi Langevin, translated by Joseph Christy) Springer-Verlag (1987).

[Si] J. Silvester. *Introduction to algebraic K-theory*, Chapman & Hall (1981).

[Sm] S. Smale. Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.* **73** (1967) 797–817.

[Su1] M. Sullivan. A zeta function for flows with positive templates. *Topology & Its Applications*, 66 (1995) 199–213.

[Su2] M. Sullivan. An invariant of basic sets of Smale flows. *Ergod. Th. & Dynam. Sys.* **17** (1997), 1437–1448; Errata 18 (1998), no. 4, 1047.

[Su3] M. Sullivan Invariants of twist-wise flow equivalence. *Electron. Res. Announc. Amer. Math. Soc.* **3** (1997), 126–130.

[Su4] M. Sullivan. Invariants of twist-wise flow equivalence. *Discrete Contin. Dynam. Systems* **4** (1998), no. 3, 475–484.

[Wa] J. Wagoner. Strong shift equivalence theory. *Symbolic Dynamics and its Applications: AMS Short Course, January 4-5, 2002, San Diego, Proceedings of Symposia in Applied Mathematics*, Vol. 60, AMS, Providence RI, 2004.

[Wi] R. Williams. Classification of subshift of finite type. *Ann. Math.* **98** (1973), 120–153; Errata **99** (1974), 380–381.

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