POSITIVE KNOTS AND ROBINSON'S ATTRACTOR

MICHAEL C. SULLIVAN

Abstract. We study knotted periodic orbits which are realized in an attractor of a certain ODE. These knots can be presented so as to have all positive crossings, but may not be restricted to positive braids.

1. Introduction

In [8] Clark Robinson analyzes the flow arising from the system ordinary differential equations below.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y + yz \\ \dot{z} &= -\gamma z + \delta x^2 \end{aligned} \tag{1}$$

He shows that there is a transitive attractor similar to that of the geometric model for the attractor of the Lorenz equations developed by Williams [10, 11] for parameter values, $\alpha = -0.71$, $\beta = 1.8690262$, $\delta = 0.1$, and $\gamma = 0.6$. This means that the periodic orbits of the attractor are smoothly isotopic to closed orbits that arise in a semi-flow on a branched 2-manifold [1]. Such a branched 2-manifold is called a *template* for the system. Robinson shows that the template for (1) has two bands each with a half twist. Using Mathematica the author has confirmed that the orientation of the twists are as depicted in Figure 1a. For convenience we isotopically embed the interior of this template into the template show in Figure 1b. The later template is denoted by $L(-1,-1)$ in the notation of [12]. All of the periodic orbits in the attractor of (1) are in $L(-1,-1)$, but not vice versa.

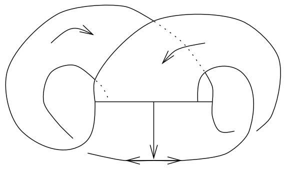
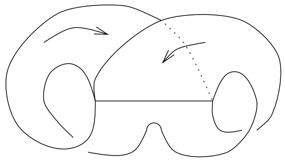
a.  b. 

Figure 1. a) Template model of attractor, b) $L(-1,-1)$

In order to organize our study of the knots in $L(-1,-1)$ we shall use words in the symbols x and y for each closed orbit. Every time an orbit passes through the left half of the branch line we record an x and for each

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pass through the right half we record a y. For example, x^2y^2 corresponds to the orbit show in Figure 2. Because the semi-flow is expanding it is known that each word corresponds to just one closed orbit. This correspondence is one-to-one modulo cyclic permutations of the words [1].

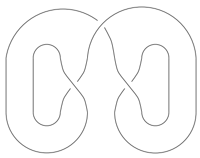

FIGURE 2. A closed orbit with word x^2y^2

2. Positive Knots

Definition 2.1. A knot is *positive* if it has a presentation where the crossings are all of the same type. A knot is a *positive braid* if it has a braided presentation with all crossings of the same type.

Theorem 2.2. *All the knots in $L(-1,-1)$ are positive, but they are not all positive braids.*

Proof. We perform a type of surgery on $L(-1,-1)$. We cut open along an exiting orbit and then isotope the template as shown in Figure 3. The details of this now standard procedure can be found in [3]. The point is that the knot types of the closed orbits have not been altered. From Figure 3 it is clear that each closed orbit can be presented so as to have only one type of crossing.

Now consider the orbit with word $xy^4x^2yx^4y^2$. It can be presented as the following braid on four strands, $(32233232221\bar{4})^2$. A calculation shows that its Conway polynomial has leading coefficient 3. Hence it is not a positive braid by [9]; see also [2]. ☐

Corollary 2.3. *The $L(-1,-1)$ template does not contain all knots.*

Proof. It is known that nonpositive knots exist. The figure-8 knot is an example. See [2]. ☐

Remark 1. A template is said to be *universal* if it contains all knots and all links. Rob Ghrist has proved the existence of universal templates and studied many examples [4]. He has also shown that $L(-1,-1)$ does not contain all links, by computing a bound of the linking number [3, page 93]. Until now all known examples of templates not containing all knots were either positive braids (i.e. all the knots were positive braids) or all the knots were satellites of a nontrivial knot. Thus, the positivity of $L(-1,-1)$ represents a new type of obstruction to universality for templates.

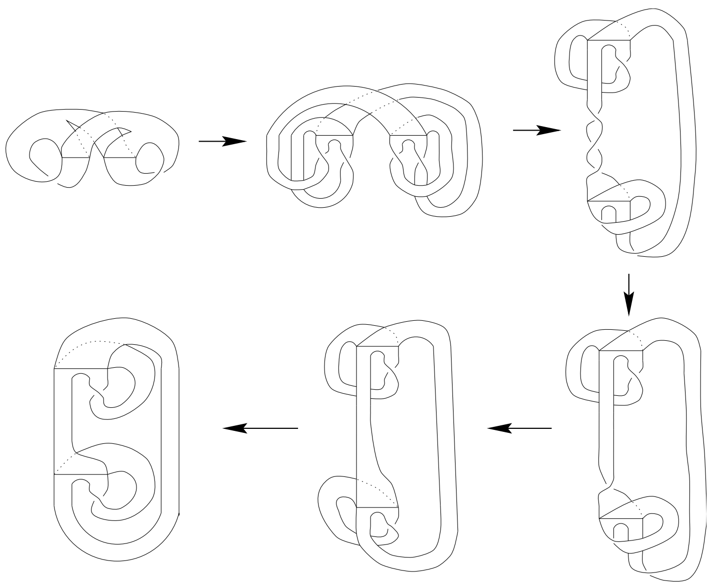

FIGURE 3. Surgery on $L(-1,-1)$

3. Composite knots

The $L(-1,-1)$ template seems strange in another way. It contains composite knots. Figure 4 gives an example. The knot is shown on the final template of Figure 3. Its word on $L(-1,-1)$ is x^4y^4. We conjecture, however, that the range of composite knots on $L(-1,-1)$ is rather narrow. To explain what we mean by this a couple of definitions are needed.

A word of a closed orbit on $L(p,q)$ is always a concatenation of *syllables* of the form $x^m y^n$. The *trip number* of a closed orbit is the minimum number of syllables of words in the permutation class of words representing the closed orbit. Thus, the trip number is the number of times an orbit pierces the xz-plane in the canonical coordinate system. We conjecture that all composite knots in $L(-1,-1)$ have trip number one.

The template $L(-1,-1)$ is a branched double cover of $L(0,1)$. See Figure 8. The $L(0,1)$ template has been studied extensively as it arises in the study of a suspension flow of Smale's horseshoe map [7, 6, 5]. The knots on $L(0,1)$ are often called *horseshoe knots*. It is known that they are prime positive braids [12]. For trip one horseshoe knots we can also report the following.

Lemma 3.1. *Trip one horseshoe knots are torus knots.*

Proof. First, notice that the knot type of $x^m y^n$ (on $L(0,1)$) is independent of m. Henceforth, we will only work with words of the form xy^n. Let γ be such a closed orbit. Let $\hat{y}$ be the point where the orbit with word y meets the branch line. Let $y_1, ..., y_i, y_{i+1}, ..., y_n$ be the n points where γ intersects

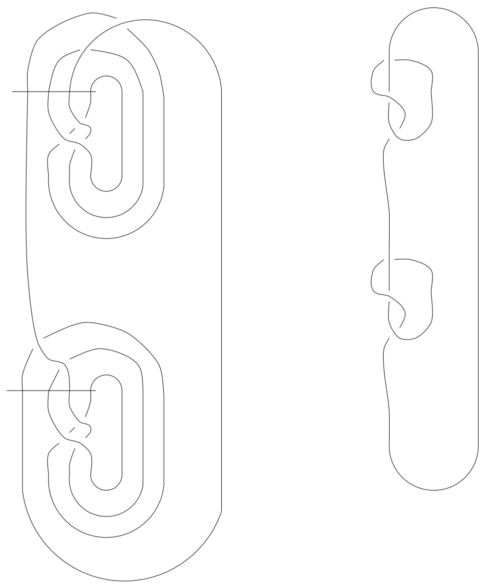

FIGURE 4. The connected sum of two trefoils

the right half on the branch line. Assume the ordering is consecutive with y_1 left most, y_n right most, and $y_i < \hat{y} < y_{i+1}$. From y_n the orbit γ heads back to the left half on the branch line. It is also clear that in order to have a complete circuit, γ must land on y_i or y_{i+1} when coming around the left band to the right half of the branch line.

To aid in visualization we have split the template open along the y orbit. See Figure 5.

We will divide the proof into two cases, n even and n odd. Suppose n is even. Then as γ comes around on the left band it must land on y_i. In Figure 6 we have deleted parts of the template that are not used by γ. Figure 6 also shows how to place (isotopically embed) this portion of $L(0,1)$, which is not branched, into a torus. Hence, γ is a torus knot.

Now, suppose n is odd. Then as γ comes around on the left band it must land on y_{i+1}. In Figure 7 we have deleted parts of the template that are not used by γ. Next, since only one arc of γ passes around the left band on $L(0,1)$, we can replace the loop indicted in Figure 7 with a loop of different twist without changing the knot type of γ. This too is shown in Figure 7. The new surface, which is unbranched and supports γ can then be placed into a torus as is shown in the last part of Figure 7. Hence, γ is a torus knot. $\square$

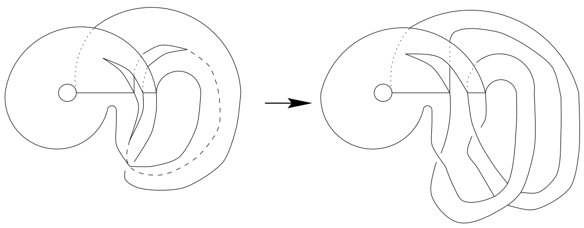

FIGURE 5. $L(0,1)$ split along y

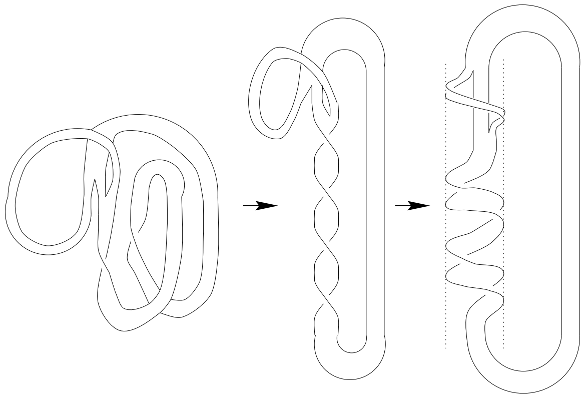

FIGURE 6. Surface fits around a torus

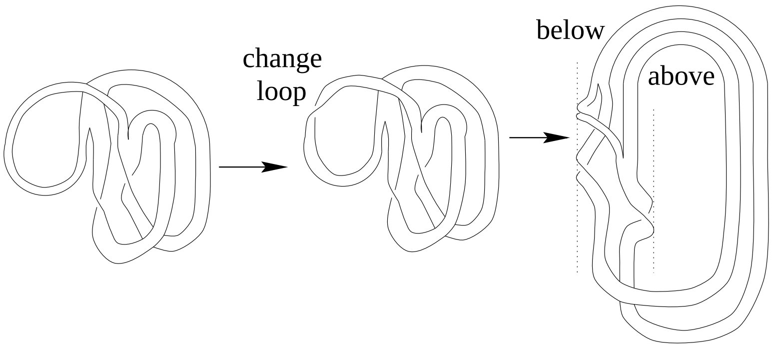

FIGURE 7. Modified surface fits around a torus

The trip one knots on $L(-1,-1)$ are connected sums of trip one horseshoe knots of the form xy^n. Thus, we conjecture that the composite knots on $L(-1,-1)$ have only two prime factors, each of which is a torus knot. If so, then the same is true of the knotted orbits in Robinson's attractor.

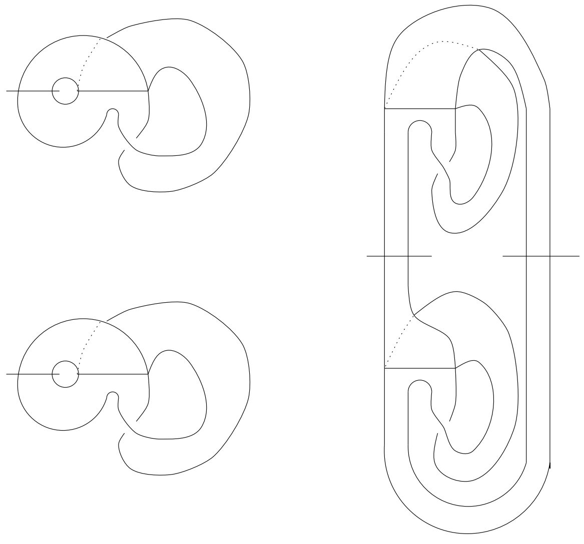

FIGURE 8. Template on the right, $L(-1,-1)$, is seen to be a double branched cover of $L(0,1)$.

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Department of Mathematics (4408), Southern Illinois University, Carbondale, IL 62901, USA, msulliva@math.siu.edu, http://nkrs465.math.siu.edu/~mike