THE PRIME DECOMPOSITION OF KNOTTED PERIODIC ORBITS IN DYNAMICAL SYSTEMS

MICHAEL C. SULLIVAN

Math Dept.
University of Texas
Austin, TX 78712
mike@math.utexas.edu

Abstract

Templates are used to capture the knotting and linking patterns of periodic orbits of positive entropy flows in 3 dimensions. Here, we study the properties of various templates, especially whether or not there is a bound on the number of prime factors of the knot types of the periodic orbits. We will also see that determining whether two templates are different is highly nontrivial.

Keywords: Dynamical systems, flows, knots, templates.

1 Introduction

The periodic orbits of a flow in a 3-manifold may be knotted. These knots and how they are linked have been studied with the aid of *templates* or *knot holders*, which are compact branched 2-manifolds with smooth semi-flows, [5, 6, 11, 14, 15, 16, 28]. Reference [15] is expository.

Theorem 1.1 *Given a flow ϕ_t on a 3-manifold M having a hyperbolic structure on its chain recurrent set, e.g. an Anosov flow, there is a template (T, ϕ'_t), $T \subset M$, such that with a finite number of specified exceptions (usually one or two) the knots and links in (M, ϕ_t) correspond one-to-one with those in (T, ϕ'_t) via an ambient isotopy. (The result can be extended to the pseudo-Anosov case.)*

The proof of this theorem as well as the definitions of "hyperbolic" and "chain recurrent set" can be found in [6]. The idea is that the flow can be collapsed along "strong stable manifolds" [6] onto the template so as to preserve the knot and link types of the periodic orbits. This result is not too dissimilar to the collapsing from a two dimensional diffeomorphism to a one dimensional branched manifold that can be found in [13, Section 5.5].

In this paper we will study the knots in a variety of flows. In particular, we will focus on whether or not there is a bound on the number of prime factors of the knots on a given template. Knots can be factored uniquely into primes, up to order

[19, 7]. This fact makes prime factorization a powerful knot invariant. The relevant definitions and some examples are below.

Definition 1.1 *A knot $k \subset S^3$ is* composite *if there exists a tame sphere S^2 such that $S^2 \cap k$ is just two points, p and q, and if γ is any arc on S^2 joining p to q, then the knots*

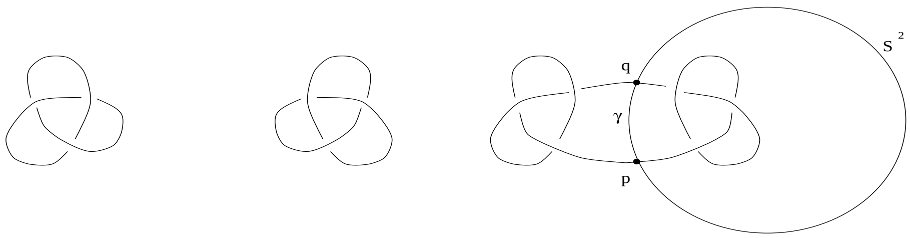
$$k_1 = \gamma \cup (k \cap \text{outside of } S^2) \quad \text{and}$$
$$k_2 = \gamma \cup (k \cap \text{inside of } S^2),$$

are nontrivial, (i.e. not the unknot). We call k_1 and k_2 factors *of k and write*

$$k = k_1 \# k_2.$$

If a knot isn't composite then it is prime.

Examples.



Left-handed Trefoil Prime   Right-handed Trefoil Prime   Square Knot Composite

Theorem 1.2 *Torus knots are prime [7].*

A (p, q)-torus knot is a knot that wraps p times around the longitude of a torus and q times meridionally. The proof that they are prime involves the study of the intersection between the torus and a would-be cutting sphere. By assuming the intersection to be transverse we get a disjoint collection of simple closed curves. We can choose our cutting sphere so that the number of components in the intersection with the torus is minimal. After proving that certain types of simple closed curves can be ruled out one shows that the number of allowed curves can always be reduced. Hence the intersection is empty.

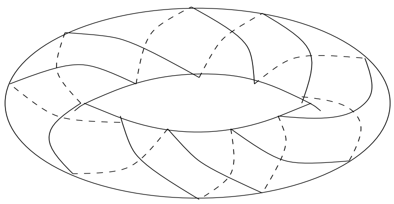


Figure 1.1: A Torus Knot

Theorem 1.3 (Williams [28]) *Lorenz knots are prime.*

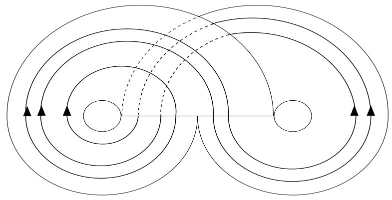
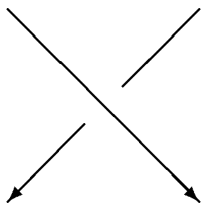


Figure 1.2: The Lorenz Template with the x^2yxy orbit.

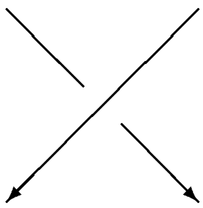
The Lorenz template is a stylized version of the inverse-limit of the Lorenz attractor. The periodic orbits can be represented symbolically by words in two symbols. It was developed by Williams [27] to study the periodic orbits in the solution of the Lorenz differential equations. Its construction is based on computer images of the Lorenz system. The proof that this template is the inverse-limit of the attractor was listed by Smale [20] as one of the ten most important unsolved problems in dynamical systems.

The Lorenz template was studied extensively by Birman and Williams [5]. In [28] Williams showed that only prime knots could be in it. The proof is similar to that of Theorem 1.2 in that one studies the intersection of a sphere and the template, but is much more complex. His techniques will be exploited in section 3 below to find bounds on the number of prime factors that can be possessed by the knots in several templates.

Before we proceed we establish the following sign convention for knot crossings as well as twists in template branches: left-handed is positive, right-handed is negative.



Positive



Negative

1.1 Statement of results

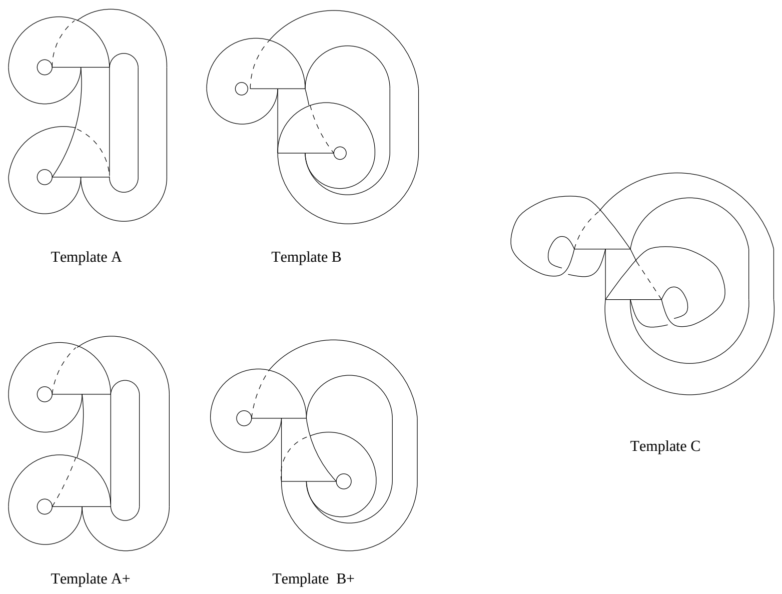


Figure 1.3: Templates

Figure 1.3 shows the principle templates to be discussed in this paper. In addition we will use the notation $L(m,n)$ to denote the Lorenz template with m and n half twists added to the left and right branches respectively, or just L when $n = m = 0$.

In [22] it was shown that,

$$k_1 \in A \text{ and } k_2 \in L \Longrightarrow k_1 \# k_2 \in A.$$

(We have abused notation by considering A and L to represent templates and the set of knots formed by their periodic orbits.) Thus, there is no bound on the number of prime factors knots in A can have. In particular, if k is a knot which factors into Lorenz knots then k is in A. It was also shown in [22] that A was a subtemplate of B and that B was in turn a subtemplate of the Birman-Williams template (see [6, Figure 1.2] or [22, Figure 1]). The Birman-Williams template was shown in [6] to be the knot holder for the gradient flow in S^3 minus the figure-8 knot with Anosov monodromy. (For an expository and very visual treatment of this flow see [9] or [10], however, an understanding of it is not necessary for our purposes.) Birman and Williams had conjectured that it did not contain any knots with more then two prime factors. Theorems A and B below extend the results in [22].

Theorem A: *Template A contains all iterated torus knots.*

Theorem B: $k_1, k_2 \in A \Longrightarrow k_1 \# k_2 \in A$.

Theorem A builds on the proof in [5] that L contains all algebraic knots. The definitions of torus knots and algebraic knots will be given later.

The proof of Theorem B will be done in three steps:

1. $k_1, k_2 \in A \Longrightarrow k_1 \# k_2 \in B$.
2. Template B is a subtemplate of $L(0, -2)$.
3. Template $L(0, -2)$ is a subtemplate of A.

This will give us two corallaries:

Corallary B1: If l is a link then

$$l \in A \iff l \in B \iff l \in L(0, -2).$$

Corallary B2: For $n < 0$ there is no bound on the number of prime factors for the knots in $L(0, n)$.

Corallary B1 and the proof of Theorem B are surprising and lead us to ask, how does one tell if two templates are the same, as well as what should "sameness" mean? Theorems D and E below will, however, allow us to distinguish between templates A^+ and B^+. The essential property that enables us do to so, as the notation suggests, is the uniformity of the possible types of crossings in the later case.

This distinction is reenforced by Corallary B2, since for $n \geq 0$, $L(0, n)$ knots are all prime [28]. This corallary results from and extends work in [21], where it was shown that $L(0, n)$ knots were prime iff $n \geq 0$.

Template C is a double branched cover of the Birman-Williams template and was studied in [6]. (Part of my original thesis problem was to show that knots in it could have at most two prime factors.)

Theorem C: *Template A is a subtemplate of template C.*

In section 3 we will show that there is a bound on the number of prime factors in templates A^+ and B^+.

Theorem D: *Every knot in template A^+ is prime.*

Theorem E: *Every composite knot in template B^+ is the connected sum of just two Lorenz knots.*

In section 4 we return to algebraic and torus knots and answer some questions raised in [21].

Theorem F: $L(0, n)$ *contains all algerbraic knots if and only if*

$$n \in \{2, 0, -1, -2, ...\}.$$

2 Template A

Theorem 2.1 *The template A contains all iterated torus knots.*

Definition 2.1 *If k is any knot, then a (p, q)* cable *about k, where p and q are relatively prime, is defined as follows. Let $N(k)$ be a solid torus neighborhood of k. Let l be a preferred longitude of $\partial N(k)$ for k. Now consider a torus T with a (p, q) knot on it. Let $h : T \to \partial N$ be a homeomorphism that takes a preferred longitude of T to l. The image of (p, q) under this map is said to be a (p, q) cable of k or, $k(p, q)$. This process can be repeated on the new knot. If the original knot was the unknot, then the resulting knots are called* iterated torus knots *and are denoted by $((p_1, q_1), ..., (p_n, q_n))$.*

Before beginning the proof we remark that the set of iterated torus knots is the same as the set of braids with zero entropy [12, 17]. A braid [3], or more correctly a closed braid, is a knot or a link that winds around the interior of a torus in such a way that, for any meridian the standard meridional disk hits the braid n times for some fixed number n. We call n the number of strands. The entropy of a braid is the minimal *topological entropy* [25] of diffeomorphisms of the disk whose mapping torus respect the braid. For example, a trefoil corresponds to a rotation and so has zero entropy.

Proof. The proof will be similar to that of Theorem 6.2 of [5]. Let k be any knot in A and let a and b be relatively prime integers. We may assume a is positive without loss of generality.

We construct a new knot as follows. Draw a parallel strands to the left of k. If $b > 0$ then add b strands around the top $(+)$ loop so that there are now $a + b$ strands between, α, the left most point of the top branch line and β, the left most point where k meets the top branch line. The b left most of these $a + b$ strands are to wrap around the top loop and land on the b right most of them. The a left most strands coming in from the back branch are now made to land on the a strands closest to α.

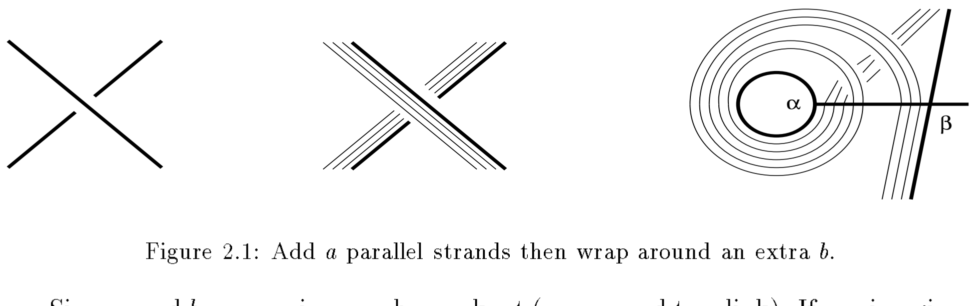


Figure 2.1: Add a parallel strands then wrap around an extra b.

Since a and b are coprime, we have a knot (as opposed to a link). If one imagines k as the core of a knotted torus, then our new knot fits onto this torus. To see this place the strands parallel to k along the bottom of the torus and wrap them meridionally around it $[b/a]$ times and then wrap an additional r strands around, where $[b/a]$ is the integer part of b/a and r is its remainder.

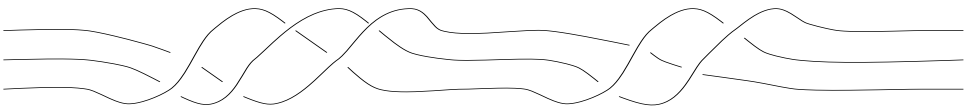


Figure 2.2: We can place the new knot on a torus.

If $b < 0$ then we add the extra b strands to the bottom $(-)$ loop. Our result in either case is an $(a, ac+b)$ cabling of k, where c is the sum of the crossing numbers. Thus, we can get any (p, q) cabling of k by chosing $a = p$ and $b = q - pc$. By starting with an unknotted orbit we can get any iterated torus knot by repeating this procedure. $\square$

Theorem 2.2 $k_1 \& k_2 \in A \Longrightarrow k_1 \# k_2 \in A$.

Proof. The proof will be done in three steps:

STEP 1: $k_1 \& k_2 \in A \Longrightarrow k_1 \# k_2 \in B$. This can be seen by studying Figure 2.3. The shaded regions are copies of template A. See [22] for details.

Figure 2.3: Two copies a A in B.

STEP 2: Template B is a subtemplate of $L(0, -2)$. In [21] it was shown that the template in figure 2.4 (a) is a subtemplate of $L(0, -2)$. The remaining parts of figure 2.4 show that B is also.

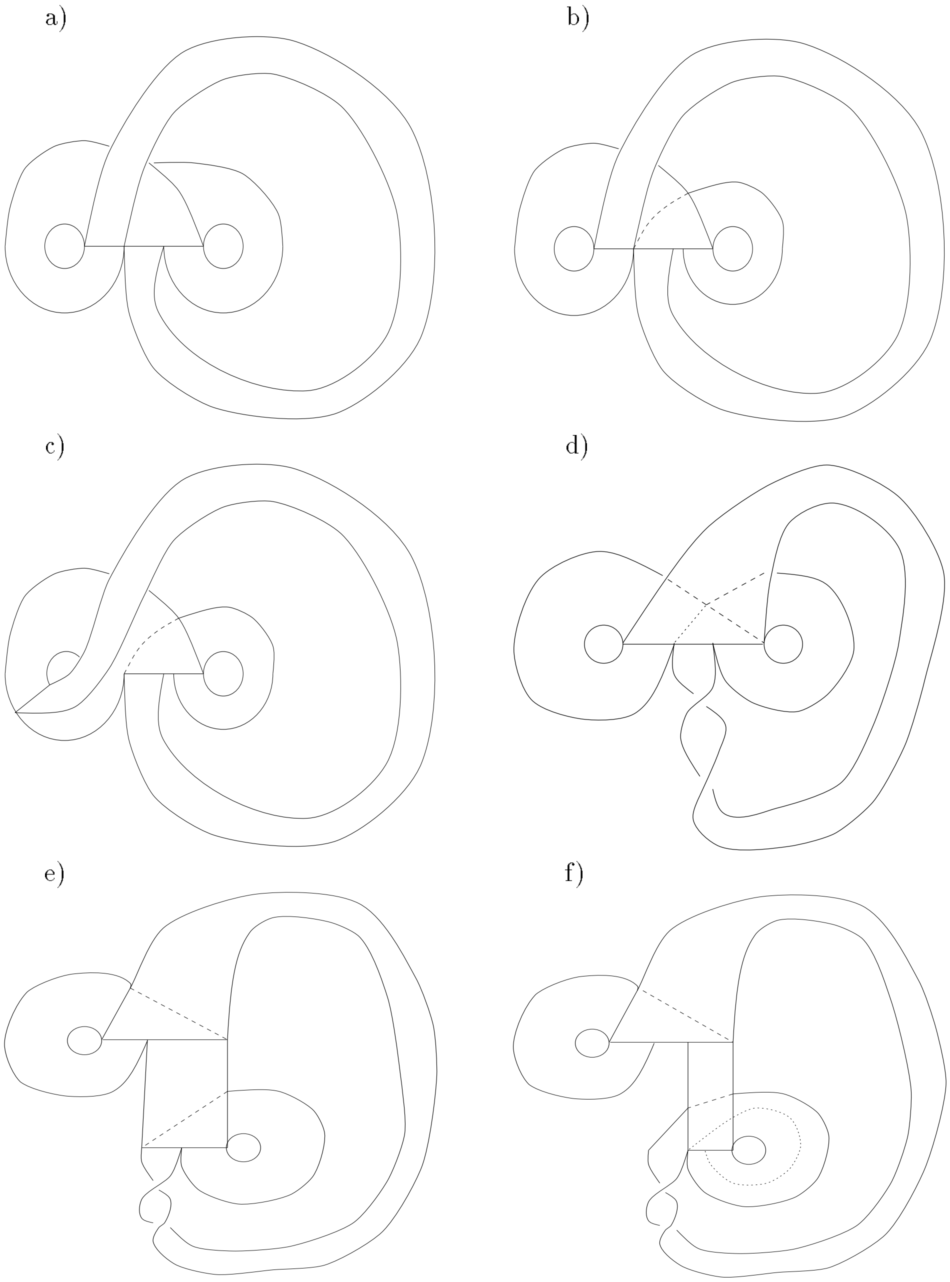


Figure 2.4: B is in L(0,-2). (Continued on next page.)

g) h)

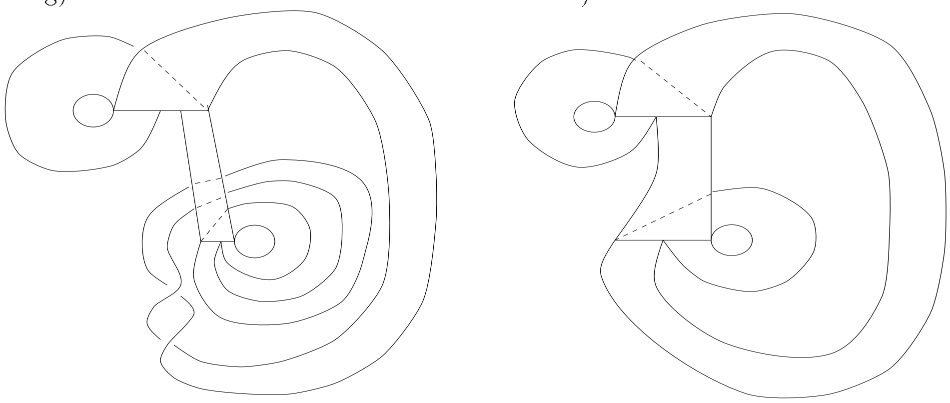


Figure 2.4: B is in L(0,-2).

STEP 3: Template $L(0,-2)$ is a subtemplate of A. See Figure 2.5. $\square$

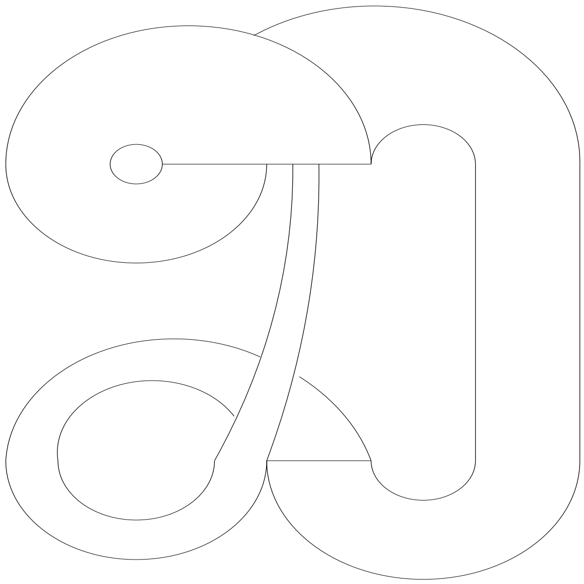


Figure 2.5: L(0,-2) is in A.

Corollary 2.1 *If l is a link then*

$$l \in A \iff l \in B \iff l \in L(0,-2).$$

Corollary 2.2 *For $n < 0$ there is no bound on the number of prime factors for the knots in $L(0,n)$.*

Proof. In [21] it was shown that as set on knots $L(0,n) \subset L(0,n-2)$ and $L(0,-4) \subset L(0,-1)$. $\square$

Theorem 2.3 *Template A is a subtemplate of template C.*

Proof. The proof is again pictorial. See figure 2.6 (a-g). □

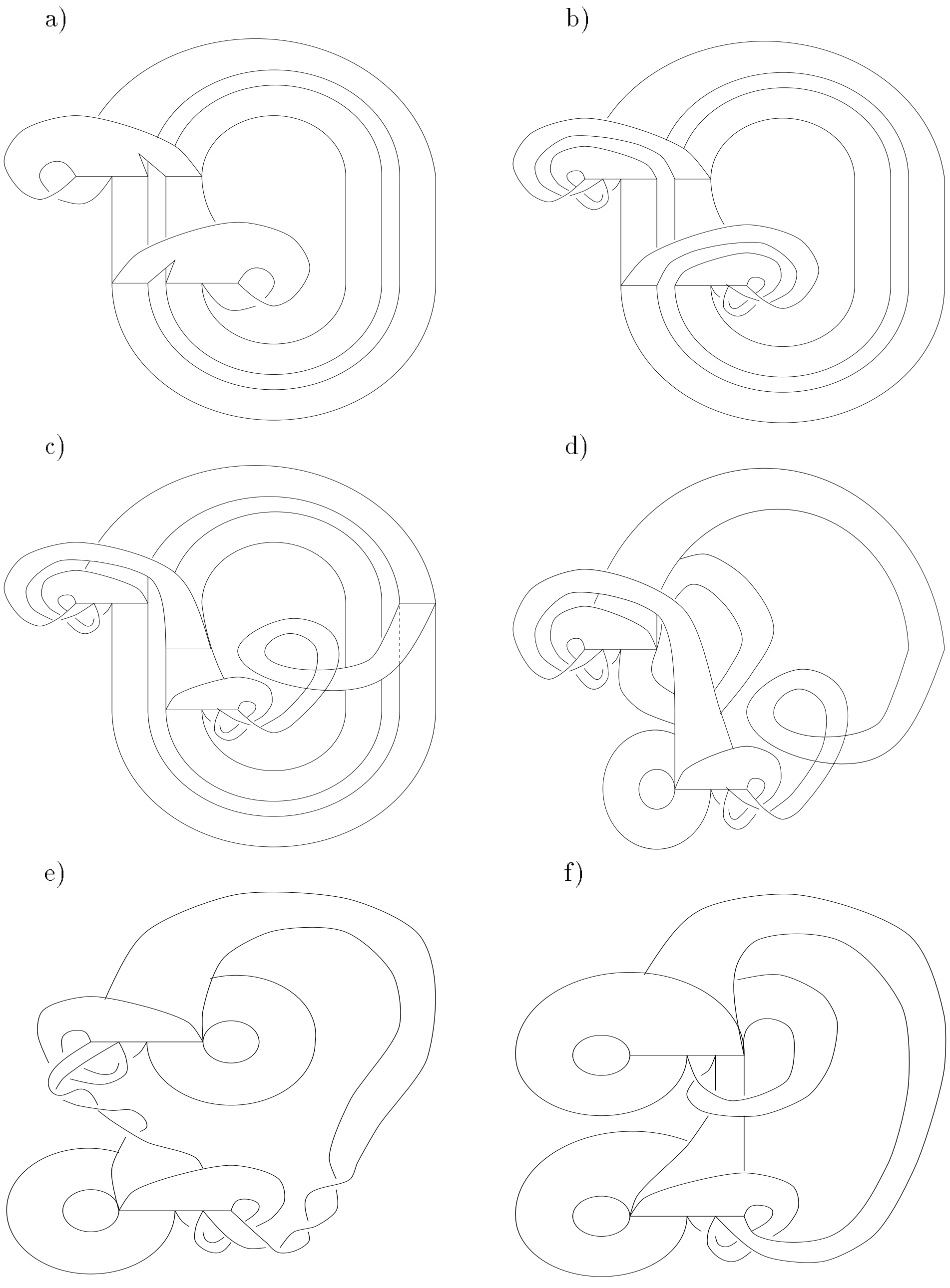


Figure 2.6: L(0,-2) is in A. (Continued on next page.)

g)

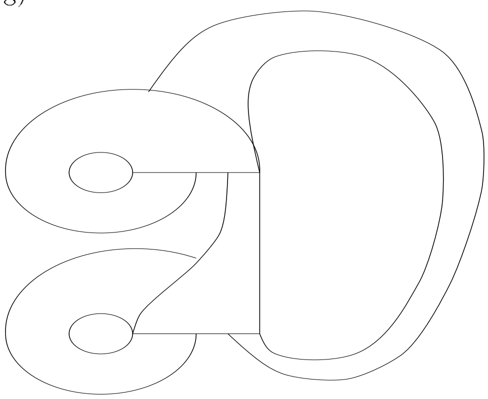


Figure 2.6: L(0,-2) is in A.

Corollary 2.3 *Template C contains the figure-8 knot the Whitehead link and the Borromean rings.*

Proof. These can be drawn fairly easily on the version of C in Figure 2.6 (f). See figure 2.7 (a-c). It is worth noting that despite extensive searching, these links were not found on the original version of C and still have not been found on templates A or B. $\square$

a) b)

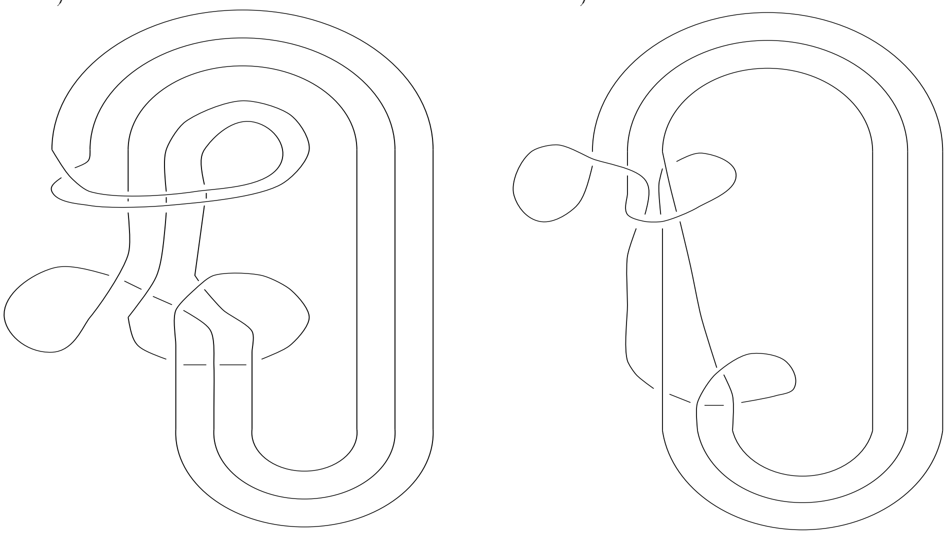


Figure 2.7: a) The figure-8 knot. b) The Whitehead link. (Continued on next page.)

c)

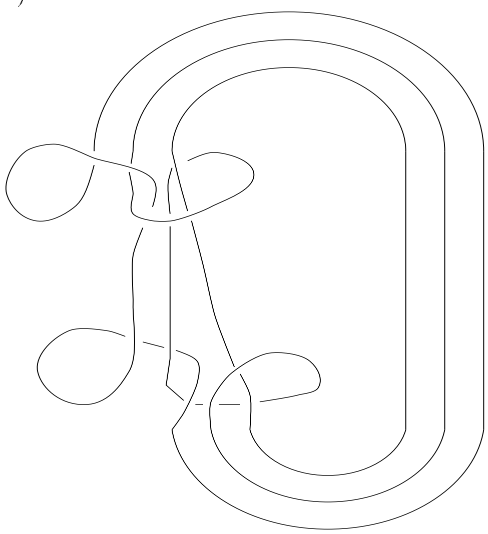


Figure 2.7: c) The Borromean rings.

3 Positive Templates

For the templates studied in this chapter a bound on the number of prime factors has been found. Their distinguishing feature is that all of the of the crossings in them are positive. The possible implications of this will be discussed in Section 5.

The template A^+ is similar to A except that the bottom loop now crosses in front, making all of the crossings positive. Here we shall prove that all the knots are prime. Template B^+ is modeled after template B. Again the lower loop has been changed so as to make all the crossings positive. On template B^+ all the composite knots are the connected sum of two Lorenz knots, which are known to be prime. Thus, two is the upper bound for the number of prime factors. We will also explore some variants of these templates.

(It is worth noting that the author spent a great deal of time trying to prove analogous theorems for templates B and C. It was the failure of these attempts that produced the previous section.)

The strategy of the proofs is as follows: Suppose k is a composite knot in the template which violates the hypothesis of the theorem and let S be a cutting sphere of k. We assume that k is "minimal" in the sense that it is to have the smallest number of symbols in its *word* of all knots with the necessary properties. (The *word* of an orbit can be thought of as its word in the fundamental group.) This means that there are no "redundant" loops as in the figure 3.1 below.

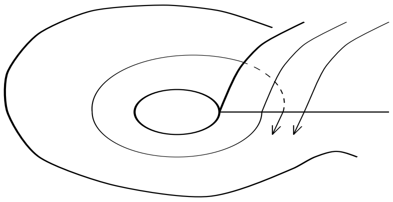


Figure 3.1: No redundant loops allowed.

If we assume S is normal to the semi-flow then its intersection with the template is a one dimensional branched manifold, or, if you like, a graph. Its branch points lie in the branch lines of the template. We partion the intersection set into segments whose end points are the branch points and any terminal points where the intersection meets an edge of the template.

A contradiction is produced by developing an algorithm for tracing out a path in the graph which has infinitely many distinct segments.

The next section contains several lemmas which are independent of the template in question. Except for the last of them, they were, in one form or another, first developed by Williams in [28] for his proof that Lorenz knots are prime.

3.1 Generic Lemmas

Let T be an arbitrary template with semi-flow ϕ_t. Let β denote the set of branch points in T. Let k be a minimal composite knot in T and S be a cutting sphere normal to ϕ_t. It will be helpful to think of the template and hence the knot, as being rigid and the sphere as being flexible.

We start by requiring S to be a sphere whose intersection with T has the smallest number of segments of all cutting spheres for k. This leads to our first lemma:

Lemma 3.1 *$S \cap T$ contains no trivial loops*

Proof. The components of $S \cap T$ that do not hit $\partial T \cup \beta$, are called trivial loops. Let γ be an inner most trivial loop in $S \cap T$. The knot k cannot (transversely) intersect S in such a loop least one of the factors of k be the unknot. The loop bounds a disk in T and two in S. These form two spheres. Only of them can contain any portion of the knot since the knot cannot pass through the disk γ bounds in T. Hence, the empty half of S can be homotopied onto this disk which is then pushed off T. Thus, we have a new cutting sphere for which $S \cap T$ has fewer segments. $\square$

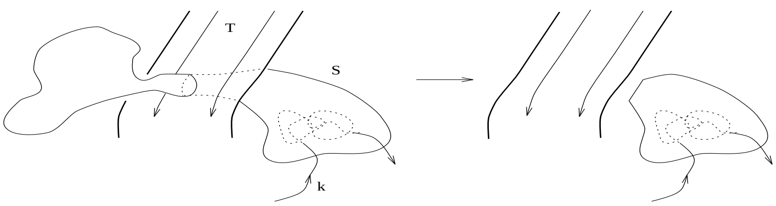


Figure 3.2: Trivial loops can be removed.

The next lemma shows that we can "assume away" three additional types of segments by just pushing the sphere around a bit.

Lemma 3.2 *Segments that join two points on the same branch line from below, a point on ∂T and a point on a branch line from below or, that join two points on the same segment of ∂T can be removed by homotopies of S that reduce the number of segments is $S \cap T$.*

Proof. As Figure 3.3 shows these segments can be pushed away without creating or eliminating any intersection points of k with S or creating any additional segments. $\square$

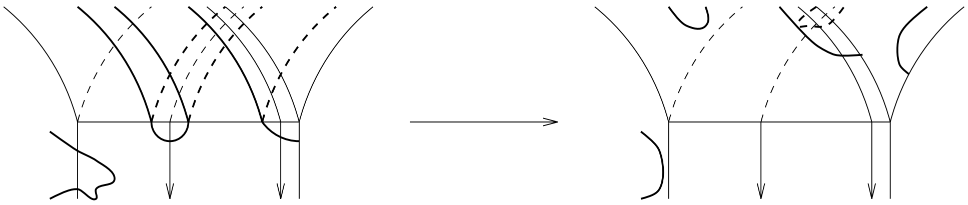
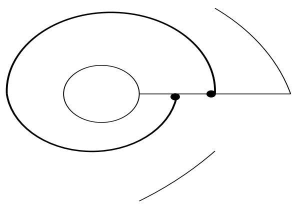


Figure 3.3: Push up or out to get rid of segments.

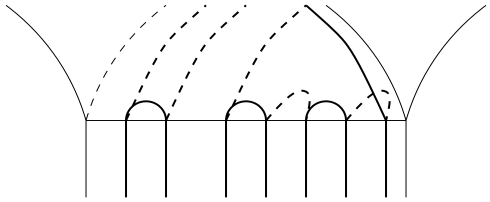
With these segments out of the way we can now smooth out the intersection to be with the flow except for just above β. To see this imagine placing a comb on each branch line and then combing downward, with the flow, straightening out $S \cap T$ as you go. Pull the comb around until you arrive just above a branch line.

We now classify the possible segments in $S \cap T$ as follows:

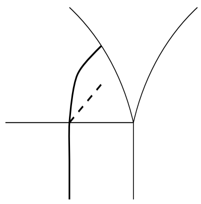
- Those which connect two points of β by traveling around a branch of T. They are with the flow except possibly for a short diagonal stretch which is combed to be just above a branch line.



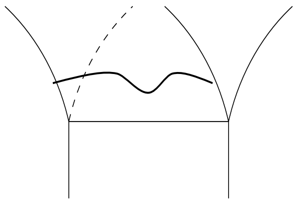
- Those that lie just above a single branch line and connect two of its points will be called U-joints. A connected collection of U-joints will be called a U-string.



- Those that lie just above a single branch line but connect a point of it to ∂T are called edge-joints.



- Those that connect two opposite sides of ∂T are double edge-joints. They closely parallel a branch line.



The next lemma requires an additional minimality assumption: the number of branch points in $S \cap T$ must be minimal, relative to those spheres which satisfy the earlier conditions.

Lemma 3.3 *Every U-joint and edge-joint must be "guarded" by an arc of the knot on the branch opposite from it.*

Proof. See figure 3.4 below. If we push a guarded segment through the branch line we will create new intersection points between the knot and the sphere. But if there is no arc of the knot then pushing the segment through reduces the number of segments just as it did in Lemma 3.2 or if the U-joint gets split into two edge-joints we have a reduction it the number of branch points. Thus unguarded U-joints and edge-joints cannot exist because of the minimality requirements. $\square$

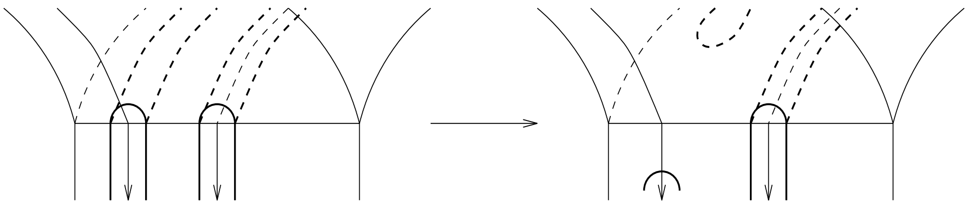


Figure 3.4: Unguarded U-joints can be pushed away.

Lemma 3.4 *U-strings do not double back.*

Remark. The lemma means that situations like that the one below cannot occur. We do not give the proof here. It can be found in [28].

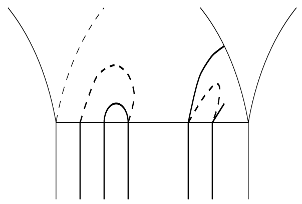


Figure 3.5: A U-string cannot double back like the ones above.

Lemma 3.5 *The normal bundle of any loop in $S \cap T$ cannot be twisted since it lies on the sphere S.*

Proof. Think of the normal bundle as a closed ribbon perpendicular to T, attached to it along the loop. The ribbon fits around the sphere since the sphere is normal to the semi-flow. Thus, it cannot be twisted. $\square$

Lemma 3.6 *A U-joint contributes $\pm\frac{1}{2}$ to the twist of normal bundle of a loop in $S \cap T$. A U-string with an even number of U-joints does not add any twist to a loop's bundle while one with an odd number gives only $\pm\frac{1}{2}$.*

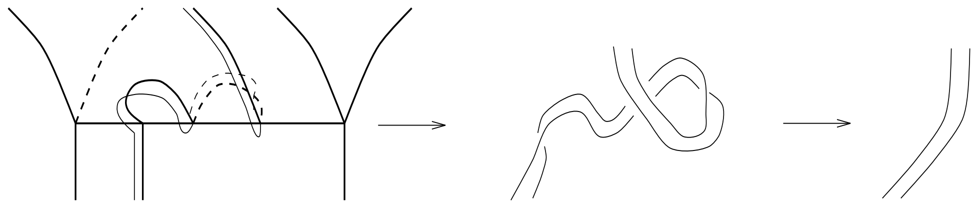


Figure 3.6: U-strings alternate $\pm\frac{1}{2}$ twist at each cusp.

Proof. The proof is in Figure 3.6. $\square$

In the sections that follow, a twist calculus for closed loops is developed for the templates in question. To this end we establish some notation. Segments that travel around a branch are labeled with the letter that denotes the branch. U-strings with an even number of U-joints are designated with an N, for neutral. U-strings with an odd number of U-joints are labeled with an L or an R depending on which direction, left or right, you would travel if you encountered one while tracing up from just below a branch line. As an example, the loop below on the Lorenz template would be denoted $LxNy$. Notice that it has twist $+1/2$.

Since any loop with nonzero twist cannot be in the intersection of a sphere and a template, Lemma 3.6 will give us a great deal of control over the possible forms of $S \cap T$. It is here that the positiveness of the templates is exploited. Since a positive crossing in the loop produces a full positive twist in the normal bundle and a U-string can at most cancel out half a twist, we will see that there will only be a finite number of possible loops. This fails in the templates of Chapter 2.

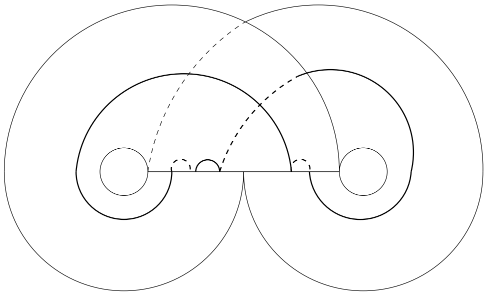


Figure 3.7: The loop $LxNy$ in the Lorenz template.

We now show how to rule out certain twist zero loops on a branch that bounds a disk and contains only a single branch line.

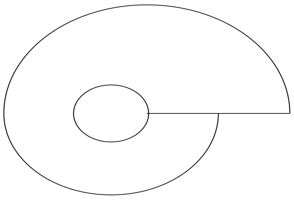


Figure 3.8: A branch that bounds a disk with only one branch line.

Lemma 3.7 *Suppose T contains a branch,* a, *of the type shown in Figure 3.8. Then the following loops cannot appear in $S \cap T$ if S is a cutting sphere for a composite knot* k*: a single* a *segment, an* a *segment with a neutral, leftward U-string,*Na*, or one with two* a *segments and two odd, leftward U-strings,* LaLa.

Proof. We do each case separately.

a: If the loop below misses the knot k then it can be pushed off T into the small disk as shown. But this reduces the number of segments of $S \cap T$. Thus k must hit our loop. This can only happen once. Call this point q.

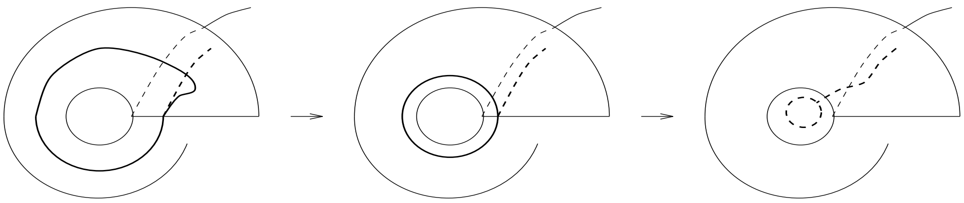


Figure 3.9: The a loop slides off into the disk.

Tracing up on k we wrap around and land at p on β. We claim p must be the point of $k \cap \beta$ closest to the disk. For if it is not one of two things happen. Either k wraps around some more, producing trivial i.e., nonminimal arcs of k, or another arc of k comes into β in between the disk and p. But in this case the knot exits (or enters) the cutting sphere twice, which is impossible. This fact referred to in [28] as the *no double entry lemma*.

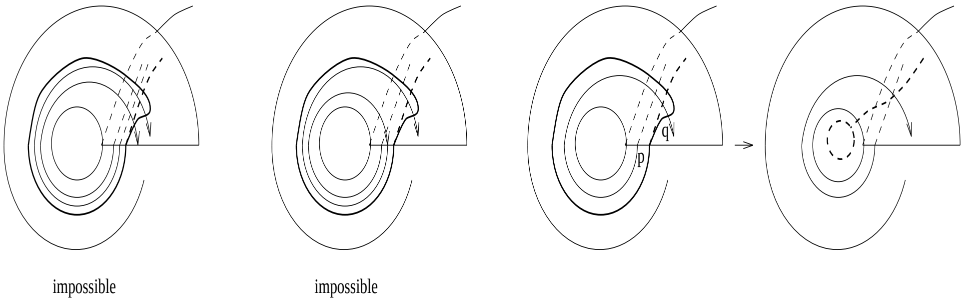


Figure 3.10: Can't guard a.

However, we now see that S can be deformed so that k misses the loop a altogether. Thus it can be pushed off into the small disk as before.

Na: The requirement that the U-joints of the U-string be guarded force k to pierce the a segment twice from the same side since the flow is expanding.

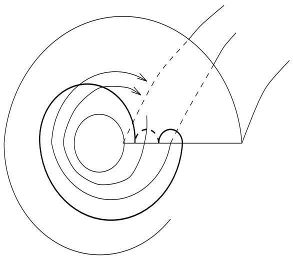


Figure 3.11: Can't guard U-joints.

$LaLa$: Suppose we have such a loop and let p be the right most point of its intersection with the branch line. Let q be the left end of the U-string that starts at p. The a segment coming from q must land between p and q, else the next a segment could not get back to p. See below:

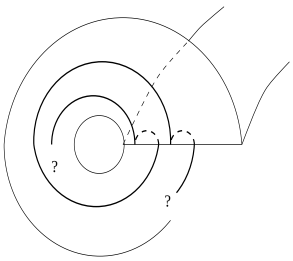


Figure 3.12: You can't get there from here.

Thus we are left with the situation in Figure 3.13. However, the need to guard the inner U-joint causes the knot to intersect the sphere four times. This is impossible. □

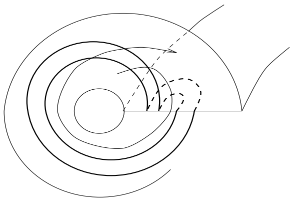


Figure 3.13: Once again we can't guard the U-joints.

The next lemma deals with edge-joints.

Lemma 3.8 *Edge-joints on the disk side of a branch that contains only one branch line cannot be in $S \cap T$.*

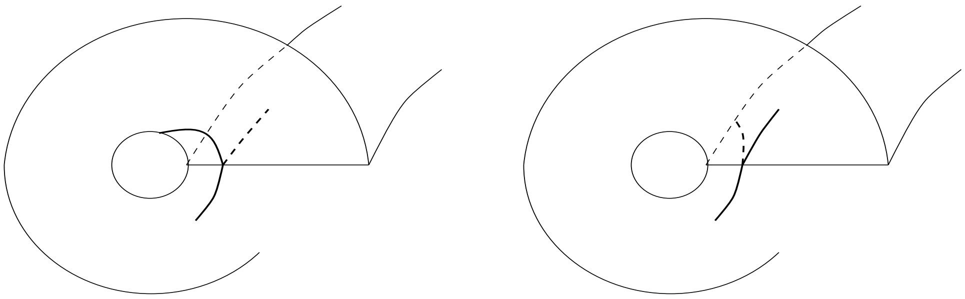


Figure 3.14: These edge-joints can be ruled out.

Remark. The proof is in Section 6 of [28]. We remark only that an edge-joint on the disk itself violates the minimality of the number of segments while one on the other branch implies the knot must have a redundant loop.

Lemma 3.9 *Let γ be a simple closed curve on S that does not meet k. Then the linking number between γ and k is at most one.*

Proof. Isotope S minus the two points where it meets k to a planar annulus. Push the "knotted parts" of k far below and far above the plane of the annulus. Now the linking number of the image of γ in the annulus with the knot is just its winding number with the center of the annulus. Since γ is a simple closed curve this must be zero or one, up to sign. □

The strategy of the proofs then proceeds as follows. Pick a point on $S \cap T$ below the branch line. Trace "up", that is against the flow. When we arrive at the branch line we must choose which branch to continue tracing on. This freedom of choice is the key. Using these lemmas and by being clever in the choices we make at the branch line we show how to trace out a path that neither goes off an edge nor forms a loop. But then it must wind on forever. That is impossible.

3.2 Template A^+

Theorem 3.1 *All the knots on template A^+ are prime.*

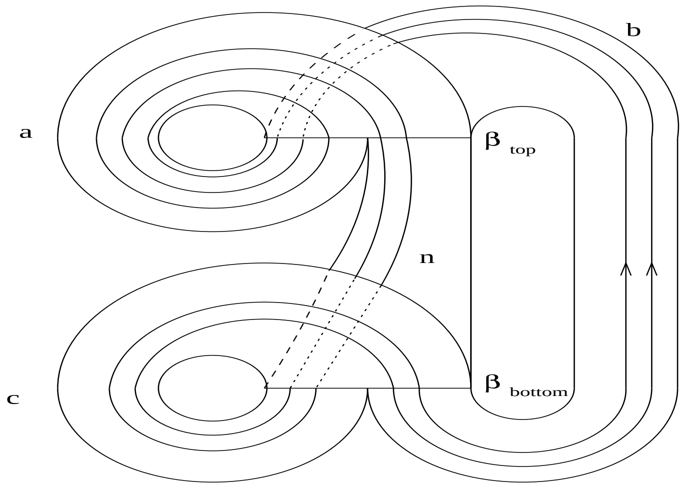


Figure 3.15: The A^+ template with a knot.

Proof. Each branch of A^+ is denoted by the letter used in the Figure 3.15. In describing a knot by a symbolic word we will ignore the n. Thus, the knot shown corresponds to the word a^2cbacb. It is a (2,5)-torus knot. The two branch lines will be referred to as β_{top} and β_{bottom}, as indicated in Figure 3.15 also.

Assume that k is a composite knot on A^+ and let S be a cutting sphere that is normal to A^+. We further assume the following minimality conditions:

- The number of symbols in the word for k is the smallest of any composite knot in A^+.
- The number of segments in $S \cap A^+$ is the smallest possible relative to the knot k.
- The number of branch points in $S \cap A^+$ is the smallest possible relative to the first two assumptions.

Pick a point in $S \cap A^+$ just below a branch line and begin tracing up, against the flow and towards the branch line. Let Γ be the path we form.

By Lemma 3.8, $S \cap A^+$ cannot have any leftward edge-joints. Thus, as we trace out our path, whenever we encounter a branch line we will choose to go straight, or if we must follow a U-string, take the left-hand turn. We wish to show that Γ contains no loops and hence must end at an leftward edge-joint, which is impossible.

If γ is a loop in $S \cap A^+$ then the twist of its normal bundle is

$$T(\gamma) = a + b + c - 1 + \tfrac{1}{2}U \qquad \text{where,}$$
$$U = Ra - La - Rb + Lb + Rc - Lc - Rn + Ln.$$

Here, the symbols refer to the number of each type of segment. That is a, means the number of a segments in γ, Rb the number of Rb type U-strings, etc.

The total number of possible U-strings in γ is $a + 2b + c$. For the twist to be zero there must be at least $2(a+b+c-1)$ negative U-strings. Therefore, $a+c \leq 2$. We consider the following cases.

CASE: $a = 2, c = 0$ ($a = 0, c = 2$ is similar): If $b = 0$ then the only possible loop is $LaLa$. But this has been ruled out be Lemma 3.7. Thus $b > 0$. For the twist to be zero there must be at least $2 + 2b$ negative U-strings. Thus, there must be a negative U-joint each time we meet β. In particular, as we pass from b to n we must go through a Rn. But this turn is not allowed.

CASE: $a = 1, c = 0$ ($a = 0, c = 1$ is similar): If $b = 0$ then the only possible loops are a or Na. But these have been ruled out by Lemma 3.7. (If the Na contained rightward U-joints, then we would always avoid it.) Thus, $b > 0$. For the twist to be zero we need at least $2b$ negative U-strings. There are three places they could go: From a to b: But Lb is positive. From b to n: But Ln is positive. From b to a: La is negative, but this not enough.

CASE: $a = 1, c = 1$: Clearly, $b = 0$ is impossible. If $b > 0$ we get the same contradiction as before.

CASE: $a = 0, c = 0$: If $b > 1$ then there must be a negative U-string and it can only be Rb or Rn, but these are right turns. Thus, we suppose $b = 1$. There are three subcases to consider.

SUBCASE: The loop nb: Any strand of k that passes over this type of loop must pierce it on its way back. Thus, by the *no double entry lemma* only one strand on k does so. But then we can push the loop off the template as shown. Thus such a loop in $S \cap A^+$ violates the minimality conditions. This is similar the the first case of Lemma 3.7.

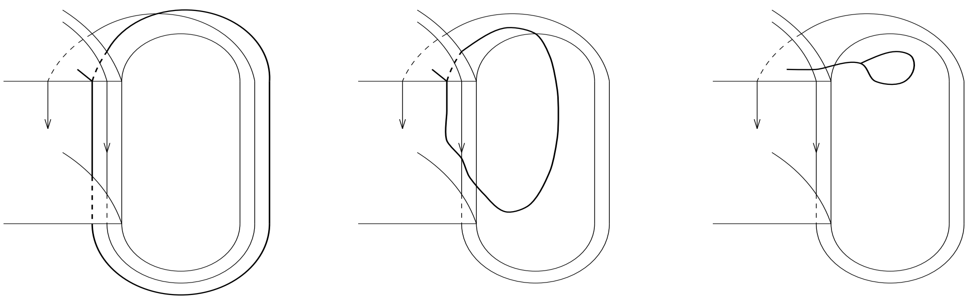


Figure 3.16: Push the bn loop off into the disk.

SUBCASE: The loop $NnNb$: Let γ be a loop in $S \cap A^+$ of type $NbNn$, with both U-strings going to the left. S can be deformed so that $k \cap \gamma = \phi$. The deformations are shown below.

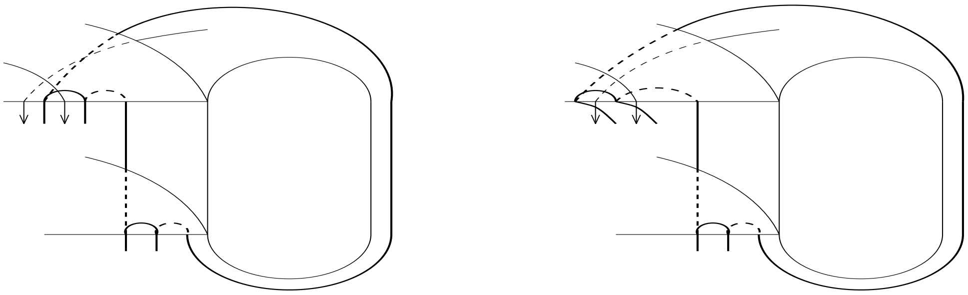


Figure 3.17: Deform loop to miss the knot.

The requirement that the forward U-joints be guarded forces the linking number of the knot and the loop to be at least two. This contradicts Lemma 3.9. See figure 3.18.

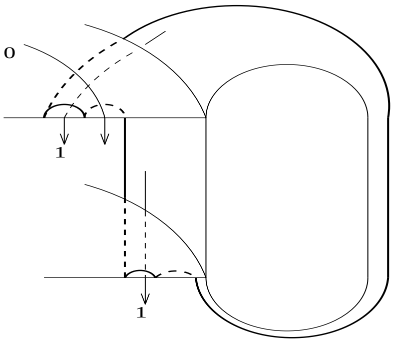


Figure 3.18: Linking number is at least two.

SUBCASE: nNb (Nnb is similar): Suppose we have encountered an nNb loop. As before we can assume $k \cap nNb = \phi$. The strand that guards the left most U-joint (and there can only be one such strand, least the linking number exceed one) must originate from the c branch. Otherwise we would have a redundant loop in k. This contributes $+1$ to the linking number between k and nNb.

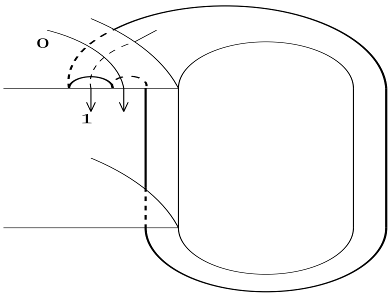


Figure 3.19: Linking number is at least one.

Let p be the right most point of the U-string. We will show that the segment of $S \cap A^+$ in the a-branch that contains p must be a type a segment. Assume not. Since U-strings don't double back we have either a rightward edge-joint or a rightward U-joint. In both cases it must be guarded by a strand of the knot. There are two ways this can be done. In one we have a redundant loop in k, in the other we force the knot to link the loop at least twice. Both are impossible.

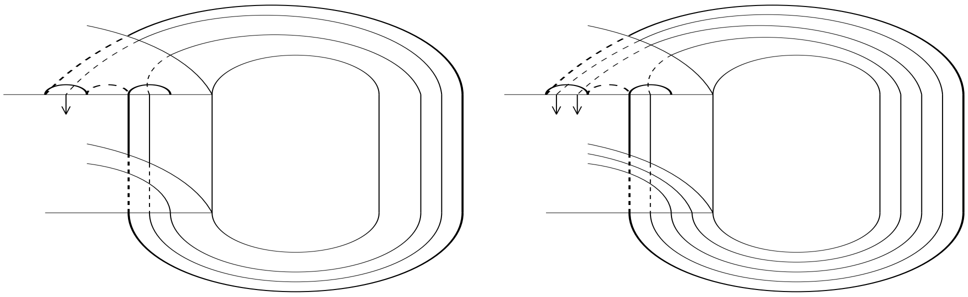


Figure 3.20: Can't guard a rightward U-joint.

Therefore, we will have to go along the a segment and avoid this type of loop. We now see that Γ never ends and never forms a loop. Thus it contains infinitely many segments. This concludes the proof. $\square$

If one adds a negative half twist to any one branch of A^+ then there are composite knots as such a template will contain $L(0, n)$ as a subtemplate for some $n < 0$. However, we conjecture that if one adds any number of positive twists to a or c then all the knots are still prime. Note that one looses the prohibition on leftward edge-joints, and the conjecture, if true, may be more difficult to prove. However, if one adds positive half twists to the n branch then it is an easy corallary that the knots are still prime. We call such templates $A^+(n)$ where there are n half twists on the n branch.

We record the following applications.

Corollary 3.1 *Every composite knot in B that is factored by the sphere S below is the connected sum of a knot in A^+ with the mirror image of a knot in A^+. Thus, these knots have just two prime factors and they are fibered knots.*

Proof. Figure 3.21 shows that, although the part of the factor outside of S may not "fit" onto the Lorenz template, it can be (ambiently) isotoped to a periodic orbit of A^+. Thus, it is prime. Similarly the inner factor can be placed on the mirror image of A^+. Both factors are fibered knots because any knot whose crossings are all the same sign is fibered [5, Theorem 5.2]. Further, the connected sum of two fibered knots is a fibered knot [7]. $\square$

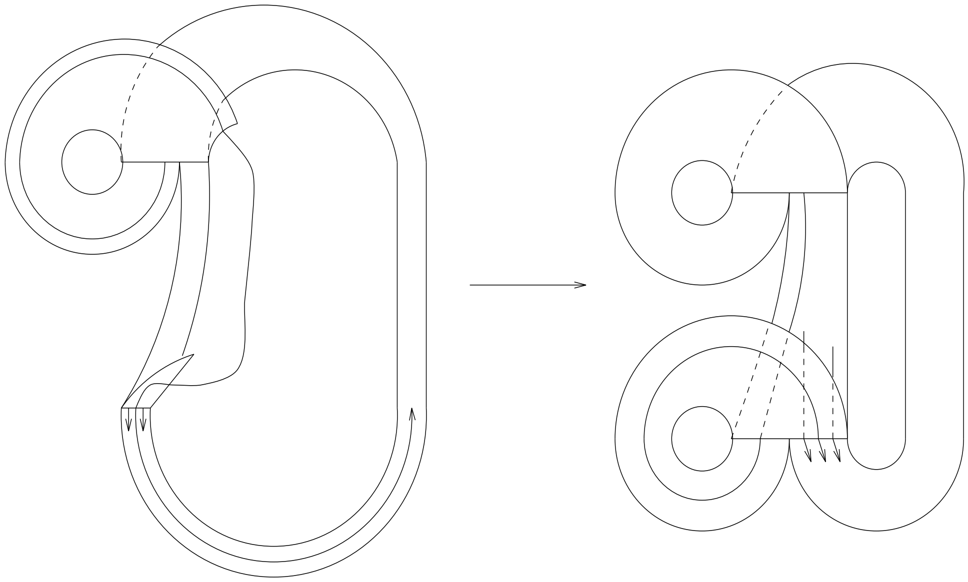


Figure 3.21: The outside factor can be placed on A^+.

Corollary 3.2 *Any composite knot on the C template which is factored by the sphere S below is the connected sum of a knot on* $A^+(1)$ *and a knot on the mirror image of* $A^+(1)$.

Proof. See Figure 3.22 (a-j). The strand of the outside factor that enters the sphere is the left most strand on the twisted branch. We pull it off of this branch and follow its course along the sphere (not shown) down into the untwisted branch and then back to the upper branch line (by disregarding the lower flap, this is now the only branch line). We then (Figure 3.22 d) push the strand into the template, along the flow in the twisted branch, but from the back. We see that the strand off the template now comes from the far left end and goes into the branch line somewhere in the twisted branch. By adding a new branch we create a template onto which we can isotop our knot, making it a periodic orbit on the new template. This template is readily seen to be $A^+(1)$. $\square$

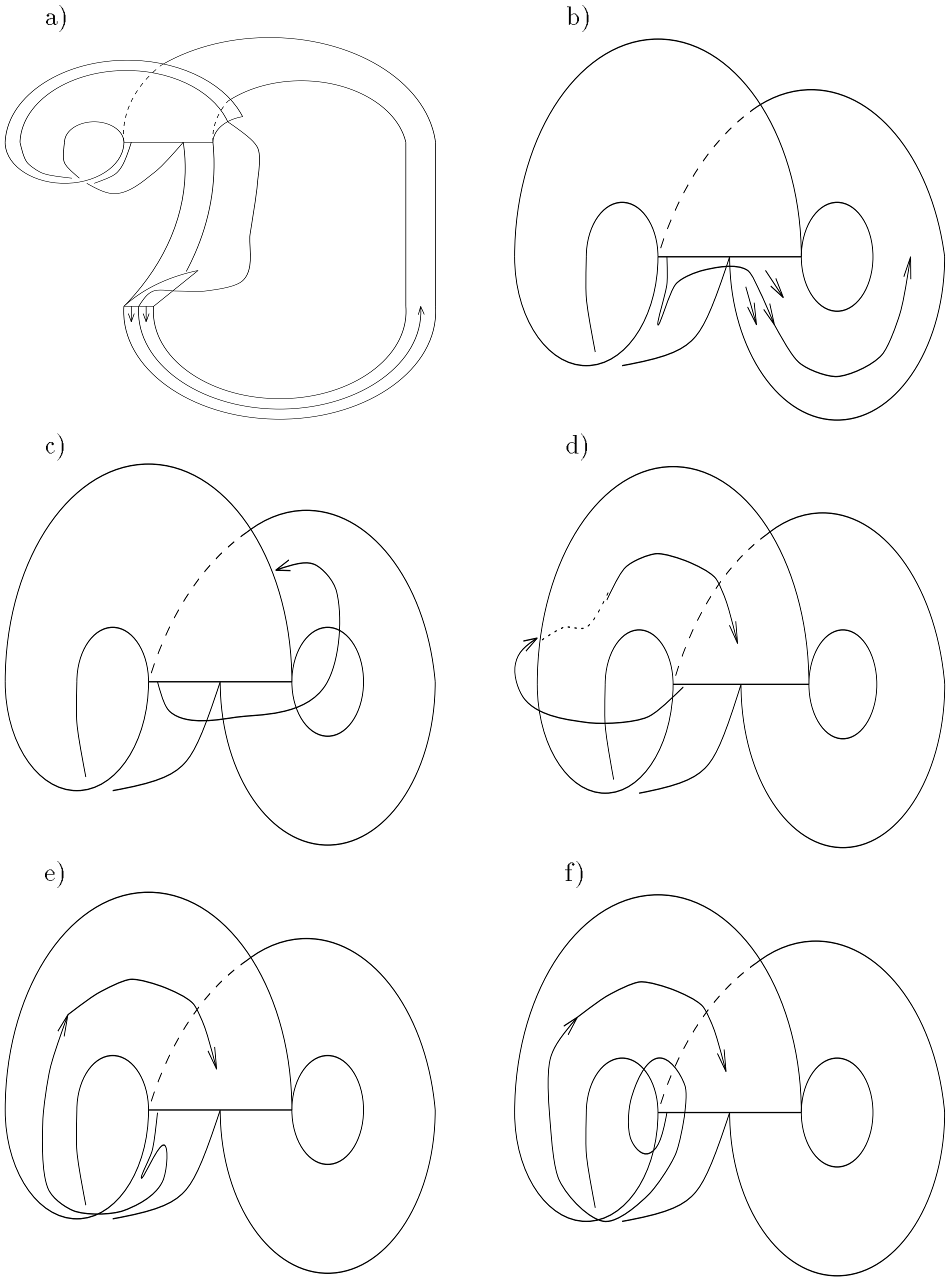


Figure 3.22: (a-f), (Continued on next page.).

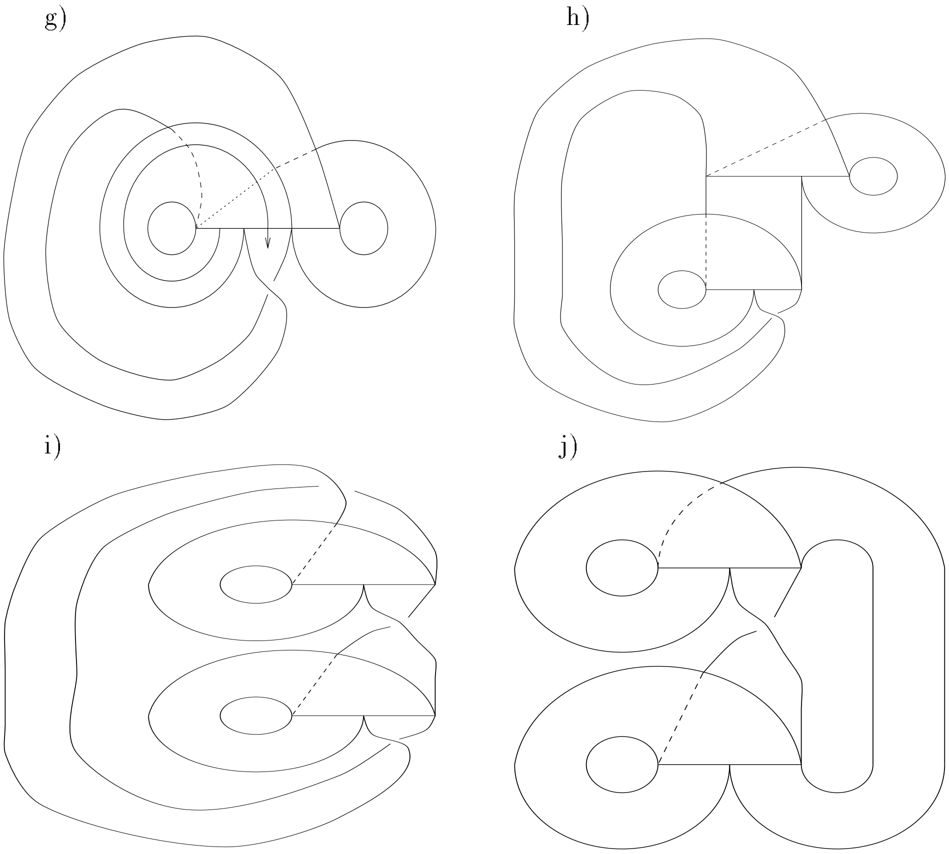


Figure 3.22: (g-j), (Continued from previous page.)

3.3 Template B^+

Theorem 3.2 *All the composite knots on template B^+ are the connected sum of two Lorenz knots.*

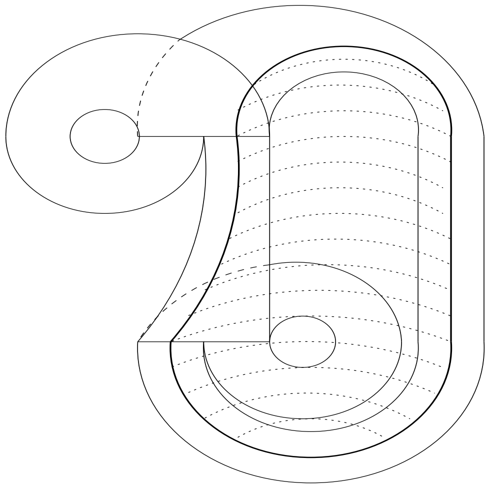


Figure 3.23: B^+ with the sphere S.

Lemma 3.10 *Let k be a knot in B^+ that is factored by the sphere S shown above. Then the factors of k are two Lorenz knots.*

Proof. Let k be a knot in B^+ such that $k \cap S$ is just two points. Let k_1 and k_2 be the factors of k when it is cut by S. Let k_1 be the factor outside of S. From the Figure 3.23 we see that S divides T into two pieces that are "almost" Lorenz; k_1 can be pushed onto a Lorenz flow in this outer piece but for an *exceptional* arc. Let this arc have end points p and q as shown in Figure 3.24.

The next step is easier to see from behind, so we have rotated the outer piece 180^o. Now we slide q up along a flow line on the b-branch until we get back to the branch line. Since we are going against the flow the new point q is closer to the end point, labeled e. After a finite number of such moves there will be no arcs of k_1 hitting the branch line between p and q. (One could think of this as backing in towards the repeller.) Now the exceptional arc can be pushed into the template and then combed onto a periodic orbit, i.e. a Lorenz knot.

The proof for k_2 is similar. Also, it is trivial is see that if either factor was the unknot then k was Lorenz to begin with. $\square$

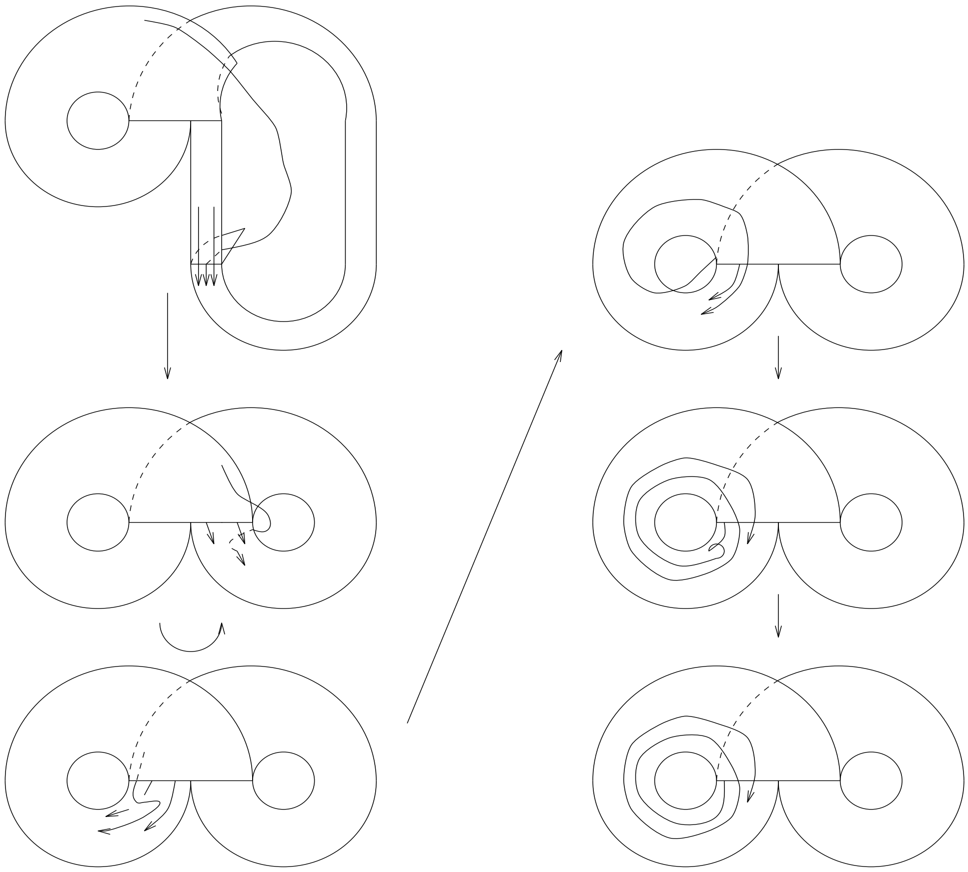


Figure 3.24: S creates Lorenz factors.

We now assume that k is a composite knot on B^+ that is not factored by S but rather by some other cutting sphere O. Like before in order to get a contradiction we make the following minimality assumptions:

- The number of symbols in k is the smallest of any composite knot in B^+ that is not factored by S.
- The number of segments in $O \cap B^+$ is the smallest possible relative to the knot k.
- The number of branch points in $O \cap B^+$ is the smallest possible relative to the two assumptions above.

We again trace out a path in $O \cap B^+$ by going against the flow, however, this time we if we must choose a U-joint we go to the right if we are on β_{bottom} and left on β_{top}. Lemma 3.8 insures that we will not encounter an edge-joint in our travels. Thus, it only remains to rule out loops.

A loop γ in $O \cap B^+$ has a twist number given by

$$T(\gamma) = a + b + c - 1 + \tfrac{1}{2}U \qquad \text{where,}$$
$$U = Ra - La - Rb + Lb - Rc + Lc + Rn - Ln.$$

Just as before $a + c \leq 2$ and we have six cases to check. However, except for the case, $a = c = 0$, the proof is virtually the same as for Theorem 3.1.

Suppose $a = c = 0$. Then $b = 1$, and we have four possibilities: nb, Nnb, nNb and $NnNb$, where the neutral U-strings go to the left or right depending or whether they are on β_{top} or β_{bottom} respectively. They can be deformed so as to insure that they do not intersect the knot. Figure 3.25 illustrates this for the $NnNb$ loop. The others are similar.

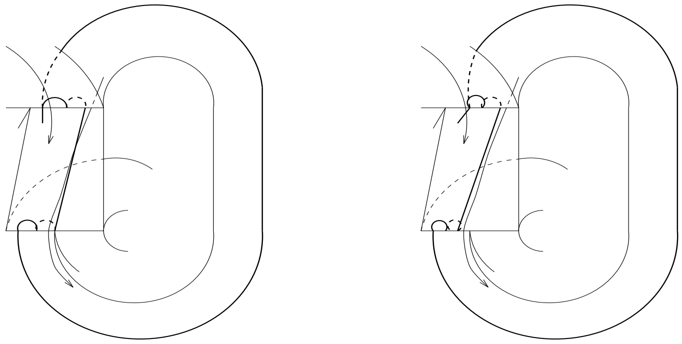


Figure 3.25: Can deform O so that the loop misses k.

In each case we exploit the fact that since k is not factored by S, at least two strands of k must go from the a-branch over to the c-branch (perhaps wrapping around b one or more times first) before passing back to a. This causes k to link any of these loops at least twice, i.e., they have linking number greater than one, which contradicts Lemma 3.9. Figure 3.26 illustrate this.

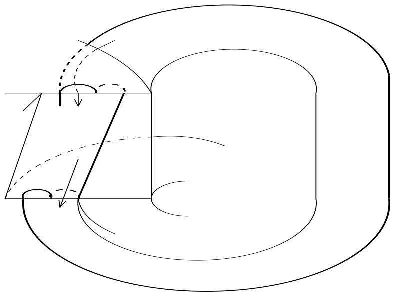


Figure 3.26: The linking number must be at least two.

The proof is now complete. $\square$

Corollary 3.3 *The braid index of a composite knot k in B^+ is given by the number of "syllables" in the word for k of the form $a^m b^n$ plus those of the form $b^p c^q$ minus one.*

Proof. The braid index of a knot is the minimum number of strands needed to present it as a closed braid. For the trefoil it is two, for the figure-8 knot three. The braid index is a knot invariant.

The braid index of a knot in the Lorenz template has been shown to be the number of $x^m y^n$ syllables [16, page 132]. Since Birman and Menasco [4] have shown that the braid index is additive minus one under connected sums, we have our result. $\square$

Corollary 3.4 *If we add any number of positive half twists to the n branch of B^+ then all the knots are prime.*

Proof. Let $B^+(p)$ be B^+ with p half twists on the n branch. The twist equation is just the equation for B^+ plus $pn/2$. For $p > 0$ one easily checks that $a = c = 0$. The only way a loop without a and c segments could have twist zero would be to have a right U-string along β_{top} and a left one along β_{bottom}. But of course we choose just the opposite. $\square$

We now know that A^+ and B^+ are different since only the latter contains composite knots. However, Figure 3.27 shows us that $A^+(1)$ and $B^+(1)$ are the same. The reader can check that

$$A^+(n) = B^+(n)$$

for n odd. For non-zero even n the question is open. In particular, it is known that $B^+(2)$ is the Lorenz template [5, ?]. Thus, we wonder if $A^+(2)$ is also.

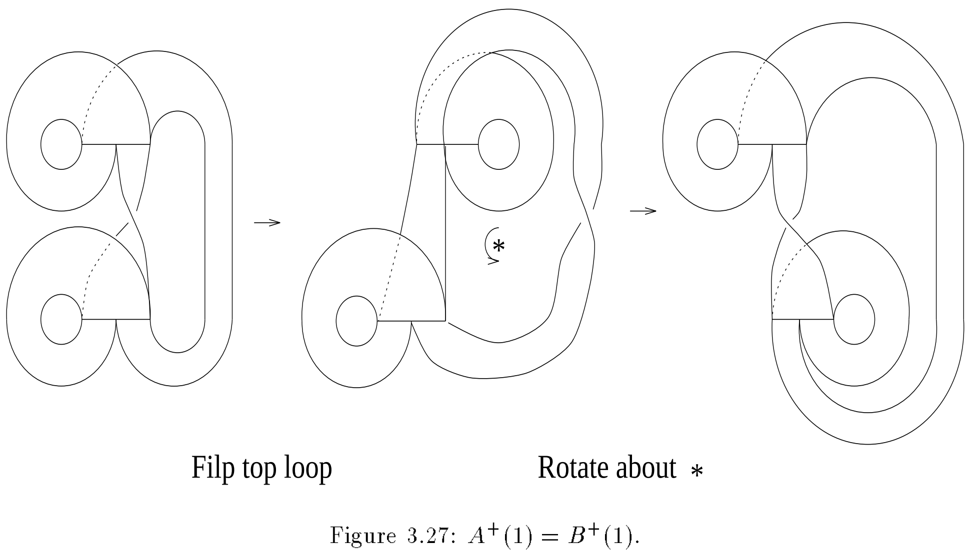


Figure 3.27: $A^+(1) = B^+(1)$.

4 Torus and algebraic knots

The goal of this section is to prove the following two theorems. They extend results arrived at in [21].

Theorem 4.1 *For every nonnegative integer n there exists a knot k, such that $k \in L(0, 2n)$ but $k \notin L(0, 2n+2)$.*

Theorem 4.2 *$L(0, n)$ contains all algebraic knots if and only if*

$$n \in \{2, 0, -1, -2, ...\}.$$

The definition of algebraic knots will be given later. The next three lemmas prove Theorem 4.1.

Lemma 4.1 *For knots in $L(0, 2n)$, $n \geq 1$, $2g \geq nb(b-1)$, where g is the genus and b is the braid index of the knot.*

Proof. The *genus* of a knot is the minimum genus of oriented surfaces whose boundary is the knot. It is a standard knot invariant [7]. The *braid index* is the minimum number of strands needed to display a knot as a braid [3]. For knots in $L(0, 2n)$, $n \geq 1$, the braid index is just the number of strands that go down the twisted branch. See Theorem 6.1.3 and the accompanying note in [16]. From Theorem 5.2 of [5] we know that, for a braid whose crossings are all the same sign

$$2g = c - b + 1,$$

where c is the number of crossings when the knot is placed on b strands. We can see that c is bounded below:

$$c \geq 2nb(b-1)/2 + b - 1.$$

Thus,

$$c - b + 1 \geq nb(b-1),$$

and we have our result. □

The next lemma was also discovered by David Armstrong [1].

Lemma 4.2 *A (p, q)-torus knot, $q > p$, is in $L(0, 2n)$ if and only if $q > np$.*

Proof. There is no loss of generality in assuming $q > p$ since $(p, q) = (q, p)$.
($\Rightarrow$): We will use the fact that the genus of (p, q) is known to be $(p-1)(q-1)/2$ and that the braid index is p. Thus we have

$$(p-1)(q-1) > np(p-1) \Rightarrow q \geq np + 1.$$

($\Leftarrow$): Let $q = mp + r$, where $m \geq 0$ and $0 < r < p$. Figure 4.1 shows us that $(p, q) \in L(0, 2m)$. (It is this step that is difficult to extend to the odd cases.)

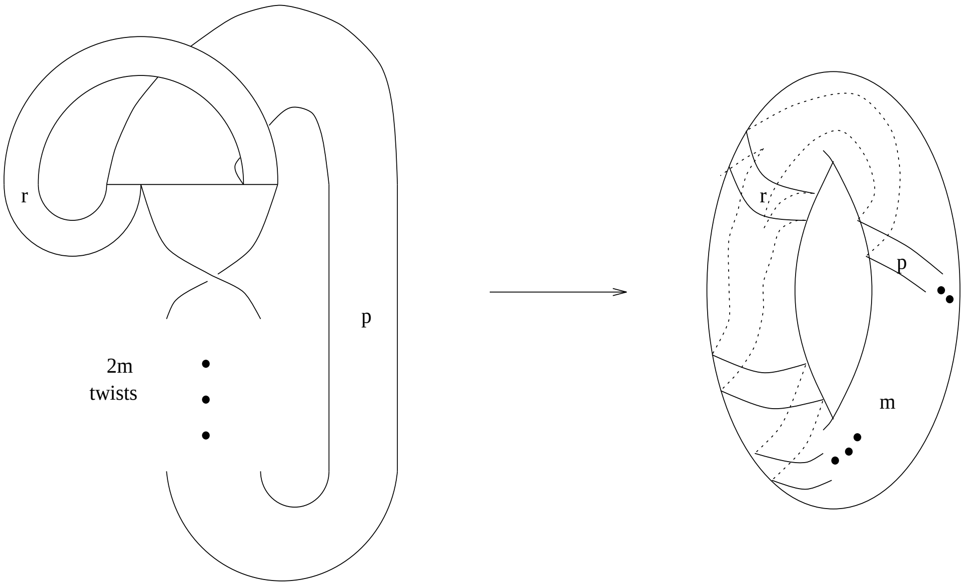


Figure 4.1: Torus knot "fits" onto the template.

Now, if $q > np$ then there exist an integer $n' \geq 0$ such that $q = (n + n')p + r$. Thus,

$$(p, q) \in L(0, 2(n' + n)) \subset L(0, 2n),$$

and the proof is done. $\Box$

This shows us exactly which torus knots are in $L(0, 2n)$ for every n. To complete the proof of Theorem 4.1 we need only find a knot which is on $L(0, 0)$ that is not on $L(0, 2)$; both contain all torus knots. The next lemma does this.

Lemma 4.3 *There exits a Lorenz knot which is not on $L(0, 2)$.*

Braid Notation. Let B_n be the braid group for braids on n strands. If $b \in B_n$ let $\hat{b}$, the *closure* of b, be the link formed by identifying the top and bottom of b as show in Figure 4.2.

Each braid $b \in B_n$ can be defined by a word in the integers set $\{1, 2, ..., n-1\}$ according to its crossings. The word for the braid in Figure 4.2 is $1^2 2^{-1}$. Let Δ_n be the positive half twist on the n strands of B_n. All this is standard. See [3]. For convenience we will let $\Delta = \Delta_3$.

Proof of lemma 4.3. Let $b = 1^3 2^3$ and $k = b\Delta^2$ in B_3. To see that $\hat{k}$ is a Lorenz knot we view it on the *braid index* form of the Lorenz template developed in [5] and shown in Figure 4.3.

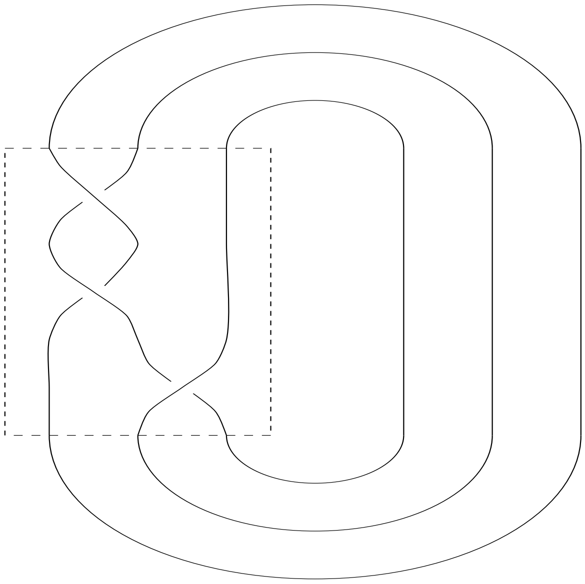


Figure 4.2: Close up braid to form a link.

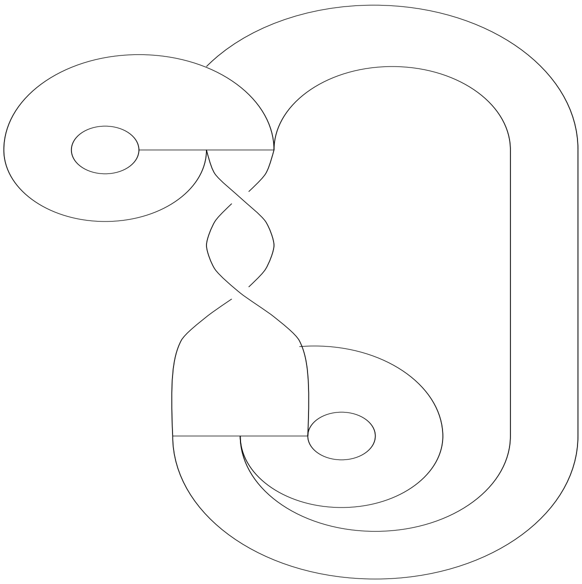


Figure 4.3: $\hat{k}$ on $L(0,0)$

Now suppose $\hat{k}$ is on $L(0,2)$. Then it is the closure of a braid of the form $l\Delta_n^2$, where $\hat{l}$ is a Lorenz knot. See Figure 4.4. If $\hat{l}$ were a link then $\widehat{l\Delta_n^2}$ would also be a link and hence not equivalent to $\hat{k}$.

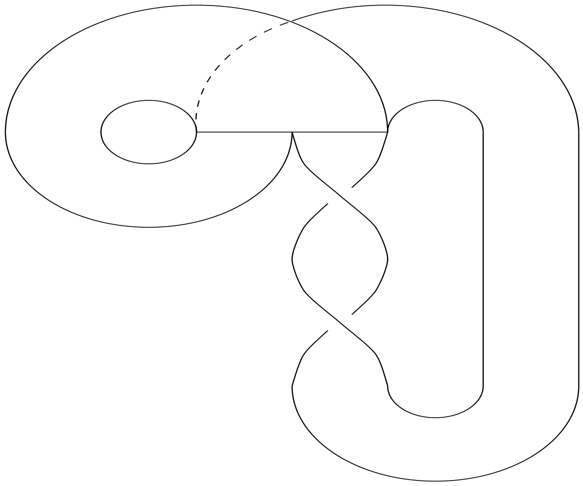


Figure 4.4: Is $\hat{k}$ on $L(0,2)$?

To get a contradiction, we first show that $n = 3$. But it is clear from a theorem in [11] that the braid index of $\hat{k}$ is 3 while the braid index of $\widehat{l\Delta_n^2}$ is n. Hence $n = 3$.

Notice that l cannot equal b since $\hat{b}$ is composite. Now, there are infinitely many words in B_3 whose closures are Lorenz knots. But, because the geneses of $\widehat{b\Delta^2}$ and $\widehat{k\Delta^2}$ are the same and since these are positive braids, they must have the same crossing number. Hence l has letter length six. One can list all of these and check (I used the two variable Jones polynomial via a computer program) that for no such l is $\widehat{l\Delta^2}$ the same as $\hat{k}$. $\square$

We now move on to algebraic knots. Algebraic knots form an important class of iterated torus knots, that contains all torus knots. That is

$$\text{torus knots} \subset \text{algebraic knots} \subset \text{iterated torus knots}.$$

Birman and Williams [5] showed that all algebraic knots are in $L(0,0)$. We will show that this is also true for $L(0,2)$. This is clearly not true for $L(0,2n)$, $n > 1$, since they do not contain all torus knots.

Definition 4.1 *An* algebraic knot *is an iterated torus knot,*

$$((p_1,q_1),...,(p_n,q_n))$$

for which,

$$\begin{array}{rcll} p_i & > & 0 & \text{for } i = 1,...,n, \\ q_1 & > & 0 & \text{and,} \\ q_i & > & p_i p_{i-1} q_{i-1} & \text{for } i = 2,...,n. \end{array}$$

Algebraic knots arose from the study of the solutions to polynomial equations of two complex variables at an isolated singular point. The above definition is really a theorem [7].

Lemma 4.4 *$L(0,2)$ has all algebraic knots.*

Proof. The proof follows the same lines as that of Theorem 6.3 of [5]. Let $N_r = ((p_1, q_1), ..., (p_r, q_r))$ be an algebraic knot. Below we not only see that $N_1 = (p_1, q_1)$ is on $L(0,2)$, but that it can have crossing number $p_1 q_1$. As an inductive hypothesis we assume $N_{r-1} = ((p_1, q_1), ..., (p_{r-1}, q_{r-1}))$ is on $L(0,2)$ with crossing number is $c = p_{r-1} q_{r-1}$.

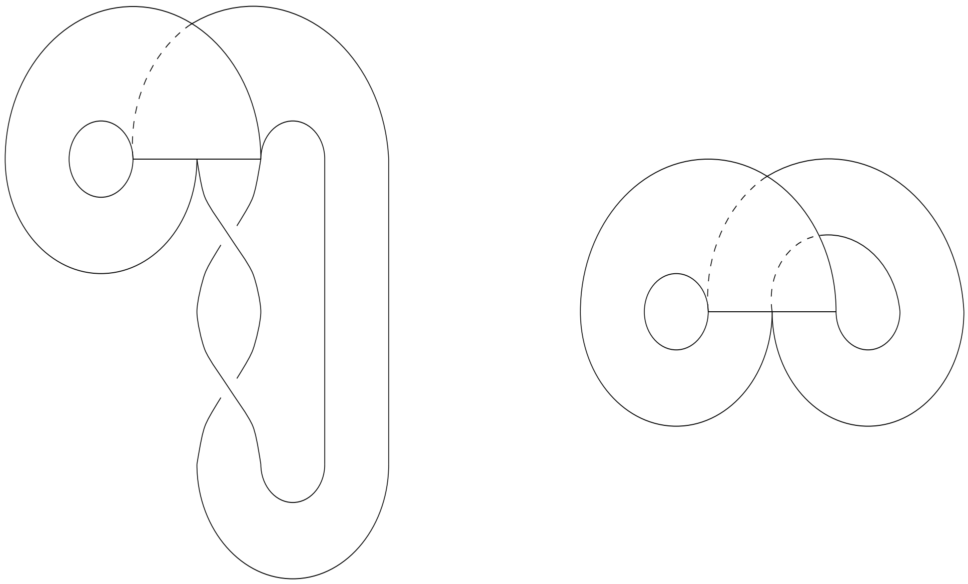


Figure 4.5

From the proof of Theorem 2.1 we know we can get all cablings of the form $N_{r-1}(a, ac+b)$, where a and b any coprime integers, but now both must be positive. Thus, in order to get N_r we need $a = p_r$ and $b = q_r - p_r c$. But the definition of an algebraic knot tells us that b will be positive and so this cabling can be done. Furthermore, it is clear from the construction in Theorem 2.1 that

$$c(N_r) = c(N_{r-1})a^2 + ab = p_r q_r.$$

This establishes the inductive hypothesis. □

It was shown in [16] that $L(0,1)$ does not contain all torus knots. Hence the same conclusion holds for any number of odd positive twists. The negative cases all contain all Lorenz knots. This proves Theorem 4.2.

5 Conclusion

An obvious pattern has emerged. A bound on the number of prime factors for the periodic orbits in a template has been found only is those cases where all the crossings are the same sign. The counter-examples to the Birman-Williams conjecture occurred on templates with mixed crossings. This leads to

Conjecture 5.1 *For all positive templates there is a bound on the number of prime factors. More precisely, if there is a presentation of a template so that all the crossings are of the same sign then there is a bound on the number of prime factors of the periodic orbits.*

As a special case we make

Conjecture 5.2 *For all positive integers, m and n, the knots in $L(m, n)$ have at most two prime factors.*

The converse of Conjecture 5.1 is most likely false. It may well be possible to construct a template whose branches are knotted in such a way as to rule out composite knots, even if both positive and negative crossing occur.

If a knot or link has a diagram with all positive or all negative crossings then it is known to be fibered. Whitten has shown [26] that determining the primeness of a fibered knot can be reduced to an algebraic condition. The number of prime factors of a fibered knot is equal to the rank of a certain subgroup of the commutator subgroup of the fundamental group of the knot's complement in S^3. This leads us to wonder if the theorems in Chapter 3, as well as Williams' result for Lorenz knots and perhaps Conjecture 5.1, can't be proved by purely algebraic means as is the case with Theorem 1.2, that torus knots are prime [7].

It is now known that flows with entropy zero can only possess periodic orbits whose knot types are iterated torus knots or their connected sums. In the case where the flow is a suspension of a diffeomorphism of a punctured disk, then entropy zero is equivalent to the braid type of the of the diffeomorphism being that of an iterated torus knot and the periodic orbits can only be iterated torus knots. See [12, 17]

For positive entropy flows we may never have as clear a picture as in the zero entropy case. However, for flows which possess positive templates, for example the pseudo-Anosov parts of the suspensions of diffeomorphisms (in the sense of Thurston's classification theorem of diffeomorphisms [24]) whose braid types are positive braids, some head way has now been made. Aside from the work here, a Zeta function for positive templates has been discovered by the author and will be the subject of a future paper [23].

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