REALIZING FULL N-SHIFTS IN SIMPLE SMALE FLOWS

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ABSTRACT. Smale flows on 3-manifolds can have invariant saddle sets that are suspensions of shifts of finite type. We look at Smale flows with chain recurrent sets consisting of an attracting closed orbit a, a repelling closed orbit r and a saddle set that is a suspension of a full n-shift and draw some conclusions about the knotting and linking of $a \cup r$. For example, we show for all values of n it is possible for a and r to be unknots. For any even value of n it is possible for $a \cup r$ to be the Hopf link, a trefoil and meridian, or a figure-8 knot and meridian.

1. INTRODUCTION

This paper builds on work done in [13, 8, 16] on Smale flows in which information about a saddle set's topology is used to discern information about the knotting of attracting and repelling closed orbits. These in turn built on work of John Franks on the homology of Smale flows [5, 7]. What is new in the present paper is that we connect information about the dynamics of the saddle set with the link type of $a \cup r$. To do this we limit ourselves to saddle sets that are suspensions of full n-shift spaces. We are able to draw some conclusions about $a \cup r$ for all values of n. It is easy to show that if n is odd $\mathrm{lk}(a, r)$ is even, while if n is even $\mathrm{lk}(a, r)$ is odd (see Corollary 2.4 below). We show for all values of $n \geq 2$ it is possible for a and r to be unknots. For any even value of $n \geq 2$ it is possible for $a \cup r$ to be the Hopf link, a trefoil and meridian, or a figure-8 knot and meridian.

A limitation is that while we show many realizable constructions, we cannot rule out any link type for $a \cup r$ except for the restriction on the linking number's parity. Whether any additional restrictions exist is an interesting open question.

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Section 2 gives background material. The main result is stated and proved in Section 3. A discussion follows in Section 4.

2. Background

Let $\mathcal{M}$ be a compact Riemannian 3-manifold. A flow is a map $\phi : M \times \mathbb{R} \to \mathcal{M}$ such that $\phi(x, s+t) = \phi(\phi(x, s), t)$ and $\phi(x, 0) = x$. We are interested in flows with *hyperbolic chain recurrent sets* and transverse *stable* and *unstable manifolds*; see [4]. Such flow is called a *Smale flow* if the chain recurrent set is at most one dimensional. In fact, we will be concerned with nonsingular Smale flows so the chain recurrent sets will be one dimensional.

By Smale's Spectral Decomposition Theorem [4] a hyperbolic chain recurrent set can be decomposed into a finite collection of disjoint, compact pieces each having a dense orbit. The pieces are called the *basic sets* of the flow. For a nonsingular Smale flow on a 3-manifold each basic set is either an isolated attracting closed orbit, an isolated repelling closed orbit, an isolated closed orbit of saddle type, or a saddle set consisting of infinitely many closed orbits and non-closed orbits some of which are dense. This last type we will refer to as being *nontrivial* or *chaotic*. If there are no nontrivial basic sets the flow is called *Morse-Smale*. On S^3 these were classified by Wada [14] following Morgan [11]. Classifying Smale flows is probably intractable, so instead we study more limited classes.

The nontrivial basic sets are suspensions of *shifts of finite type* (SFT). An SFT is a set of bi-infinite sequences of symbols from a finite alphabet determined by a finite list of *forbidden blocks*, together with a shift map that shifts entries of an sequence once to the left [10, 12]. Two SFT's are equivalent or *topologically conjugate* if there is a homeomorphism that commutes with their shift maps. While an SFT can be finite, those giving nontrivial basic sets are infinite and have a dense orbit. The SFT on n symbols with no forbidden blocks is called the *full n-shift space*.

Any SFT can be determined, non-uniquely, by a square matrix over the non-negative integers. Such matrices are called *incidence matrices* and they can be constructed from a *Markov partition*. The definition is technical (see for example [10, 12, 3]) but a Markov partition of a SFT is a finite collection of subsets, usually called "rectangles", that cover the space, are disjoint and behave well under the shift map. Suppose $(R_1, \ldots, R_n)$ is a Markov partition for (Σ, σ). Then the ij-entry of the $n \times n$ incidence matrix counts how many times $\sigma(R_i)$ "passes through" R_j. For a fine enough Markov partition we can insure the incidence matrix has entries of only 0's and 1's. For the SFT to be infinite, no incidence matrix can be a permutation matrix. For the SFT to have a dense orbit, any incidence matrix must be irreducible.

For the full n-shift the $n \times n$ Matrix of all 1's and the 1×1 matrix $[n]$ are both incidence matrices. In the first case there are n rectangles and the image of each passes once through each of the others. In the second there is one rectangle whose image passes through itself n times.

Let A and B be square matrices of non-negative integers. We say an *SSE-move* takes A to B if there are rectangular matrices of non-negative integers R and S such that $A = RS$ and $B = SR$. If there is a finite sequence of SSE-moves taking A to B then A and B are *strong shift equivalent*. It was shown by Williams [15] that two square matrices of non-negative integers determine equivalent (topologically conjugate) SFT's if and only if they are strong shift equivalent. The SSE-move amounts to forming splittings and amalgamations of the rectangles to get a new Markov partition.

An SFT (Σ, σ) determines a *suspension flow* by using the vector field $(0_\Sigma, \partial/\partial t)$ on $\Sigma \times I$ and then identifying $(x, 1)$ with $(\sigma(x), 0)$. This defines a one-dimensional flow. We say two square matrices of non-negative integers are *flow equivalent* if there is a flow preserving topological equivalence between the suspensions of the SFT's they determine. A theorem of Franks [6] determines this for irreducible non-permutation matrices (which are all we need) in terms of easily computed invariants.

Theorem 2.1 (Franks [6])**.** *Let A and B be square matrices of non-negative integers and assume they are irreducible and not permutations. Then A is flow equivalent to B if and only if*

$$\det(I - A) = \det(I - B) \text{ and } \frac{\mathbb{Z}^n}{(I - A)\mathbb{Z}^n} \cong \frac{\mathbb{Z}^m}{(I - B)\mathbb{Z}^m},$$

where n and m are the sizes of A and B, resp.

The first invariant is called the *Parry-Sullivan number* and the second is the *Bowen-Franks group*. A generalization to certain reducible matrices can be found in [9].

Example 1. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is strong shift equivalent to $[2]$, the full 2-shift. $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is not strong shift equivalent to $[3]$ since the trace is known to give the number of fixed points. But, as the reader can check, they are flow equivalent.

In this paper we will restrict ourselves to *simple Smale flows* on S^3 which we define to be nonsingular Smale flows with three basic sets, an attracting closed orbit a, a repelling closed orbit r, and a nontrivial saddle set s. The following proposition follows directly from another result of Franks [7, Theorem 1].

Proposition 2.2. *For any $n \geq 2$ there exists a simple Smale flow on S^3 such the saddle set is the suspension of the full n-shift.*

In this paper we begin to explore the possible link types of the attractor-repeller pair in such flows.

Following Franks [5] we modify the incidence matrix of s for a given Markov partition. Assume we have a Markov partition that is fine enough for the incidence matrix to have only 0's and 1's. We can place an orientation on each rectangle and ask if the first return map is orientation preserving or reserving going from R_i to R_j. If it is orientation reserving we change the 1 in the ij place to -1. This produces the *structure matrix* for a nontrivial basic set. There is a relationship between the structure matrix and the linking number of a and r.

Theorem 2.3 (Franks 1981 [5])**.** *In a simple Smale flow the unsigned linking number of the attractor a and repeller r is the absolute value of the determinant of I minus the structure matrix,*

$$|lk(a, r)| = |det(I - S)|.$$

Henceforth, we use $\mathrm{lk}(a, r)$ to mean the unsigned linking number.

Corollary 2.4. *For a simple Smale flow with saddle set a suspension of the full n-shift we have that $lk(a, r)$ is even if n is odd and is odd if n is even.*

Proof. Assume a Markov partition that is fine enough that the incidence matrix A is a 0-1 matrix and that the structure matrix S is well defined. The incidence matrix is strong shift equivalent to the 1×1 matrix $[n]$. Hence, $\det(I - A) = \det([1 - n]) = 1 - n$. The structure matrix S is mod 2 equivalent to A. The result follows. □

A *template* is a compact branched 2-manifold with boundary and a smooth expansive semi-flow. Templates are formed from *splitting charts* and *joining charts* as shown in Figure 1. Each joining chart contains a *branch line*. The charts are joined together so that flow lines match and orbits only exit templates in the middle portion on the exit set of the splitting charts. The simplest template is the *Lorenz template* also shown in Figure 1.

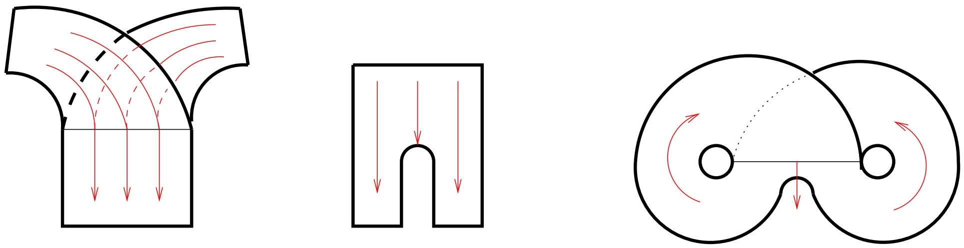

FIGURE 1. A joining chart, a splitting chart and a Lorenz template.

Templates are used to model nontrivial basic sets in the following sense. Suppose $\mathcal{B}$ is a nontrivial basic set of a Smale flow. Then there exists an embedded template T such that there is a one-to-one correspondence between the periodic orbits of $\mathcal{B}$ and those of T that preserves the knot type of each periodic orbit and the link type of any finite link of periodic orbits. This correspondence is constructed by choosing a "nice" neighborhood of foliated by local stable manifolds and then collapsing along the stable direction. The template is not unique as "tighter" neighborhoods yield more complex templates. The basic set can be recovered from T by taking an inverse limit. This is due to Joan Birman and Robert Williams [2] and can also be found in [3].

The neighborhood used to construct the template is called a *thickened template*, but we enlarge it slightly along the edges of the template so that all the periodic orbits are in the interior. It is a handlebody. Its boundary has an *exit set* where the flow exits and an *entrance set* where the flow enters. Usually one works with their closures. Their common boundary is a finite set of circles where the flow is tangent to the boundary of the thickened template. Thus, given a template for the saddle set of a simple Smale flow on S^3, it can be thickened and the exit and entrance sets glued to solid tori neighborhoods of a and r to reconstruct the simple Smale flow from neighborhoods of its basic sets. Figure 2 illustrates the exit and entrance sets for a thickened Lorenz template.

Given a template one can easily choose a collection of cross sectional disks for a Markov partition. Then the incidence matrix can be determined by just following the bands, and the structure matrix is determined by checking the number of half twists in the bands between each pair of disks.

We describe a common operation in "template theory" called *attaching a disk.* Let $\mathbf{D} = D^2\ [-\epsilon, \epsilon]$. Give it the vector field $(0, 0) \times \partial\ \partial t$

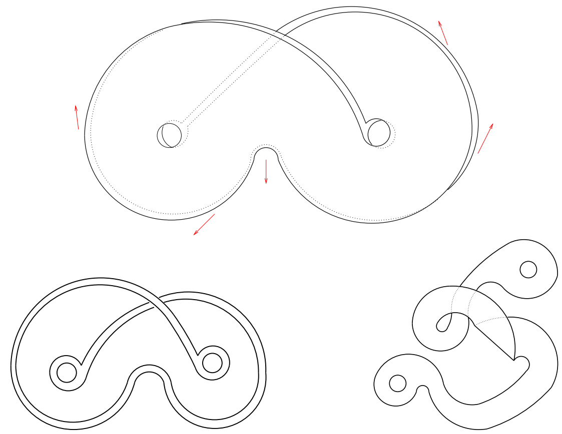

Figure 2. Exit and entrance sets for a thickened Lorenz template

where t is a parameter for the interval $[-\epsilon, \epsilon]$. On a thick template $\mathbf{T}$ select one of the circles C of tangential points in the boundary. Delete a small neighborhood of C in $\mathbf{T}$ as shown in Figure 3. This creates an annulus of points tangent to the flow. Now glue the edge of $\mathbf{D}$, that is $\partial D^2 \times [-\epsilon, \epsilon]$, to this annulus. Do this so as to create a new neighborhood of the same saddle set that has a flow transverse to its boundary. No new invariant orbits or points are created. Figure 3 illustrates this by showing a cross section.

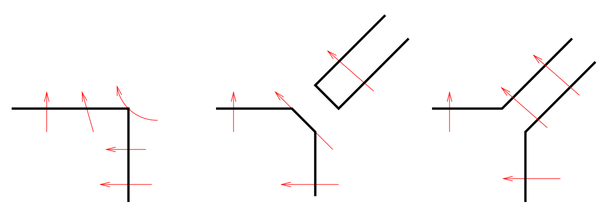

Figure 3. Attaching a disk

3. Results

Theorem 3.1. A. *Let $n \geq 3$ be odd. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, with $a \cup r$ unlinked unknots. This can be done so that the attractor (repeller) links every closed orbit in the saddle set except one and the repeller (attractor) links no other closed orbits, or neither links any other closed orbits.*

B. *Let $n \geq 2$ be even. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $lk(a, r) = 1$*

and the pair $a \cup r$ can be any of, (i) a Hopf link, (ii) a trefoil and meridian, or (iii) a figure-8 knot and meridian.

C. *Let $n \geq 3$ be odd and p be any integer. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, where a (resp. r) has braid word σ^{2p+1} and r (resp. a) is an unknot serving as a braid axis. It follows that $lk(a, r) = 2$.*

D. *Let $n \geq 2$ be even. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $a \cup r$ consists of $(p, 3)$-torus knot and its unknotted core; hence $lk(a, r) = 3$.*

Given a template T we define an operation that we call an *α-move.* One selects a rectangular patch in a band away from any branch lines. The template's flow enters transversely along one edge of the patch, exits transversely along the opposite edge, and the other two edges contain flow lines that are in the boundary of the template. This patch, call it R, is then deleted and replaced with a new structure, R^α, shown in Figure 4. This creates a new template, T^α.

Figure 5 shows the result of applying an α-move to the Lorenz template. Call the result L^α. The figure also shows possible choices of Markov partitions for each template. If we take a first return map using the partition given for the Lorenz template, it is clear that it will be conjugate to the full 2-shift. For the partition of L^α the first return map will be conjugate to a full 4-shift. Clearly, the α-move alters the dynamics, but in a controlled way.

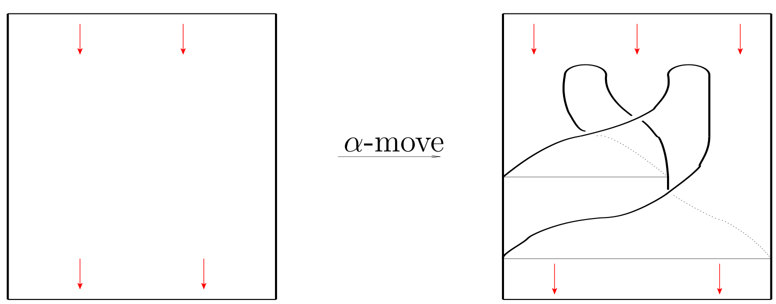

FIGURE 4. Altering a template.

The main tool in the proof is the following lemma.

Lemma 3.2. *Let ϕ_t be a flow on a 3-manifold $\mathcal{M}$ with hyperbolic chain recurrent set $\mathcal{R}$ with basic set decomposition $\bigcup \mathcal{B}_i$. Let T be a template that models a nontrivial 1-d basic set, $\mathcal{B}_j$. Let T^α be a template derived from T by applying an α-move.*

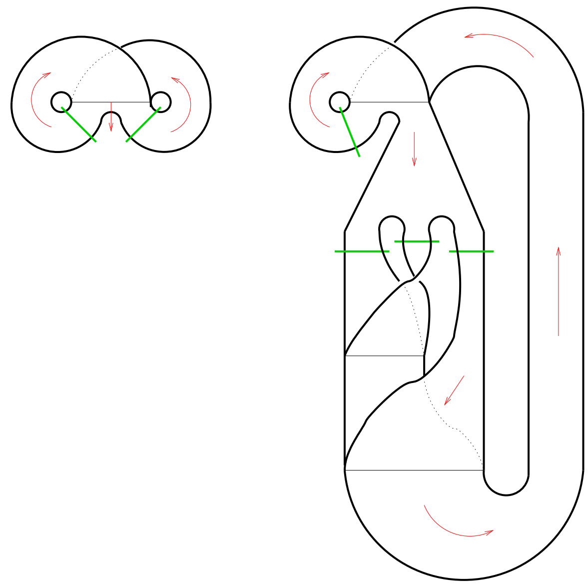

FIGURE 5. α-move applied to the Lorenz template

Then there exists a flow ϕ_t^α on $\mathcal{M}$ with hyperbolic chain recurrent set $\mathcal{R}^\alpha$ that has the same basic sets as $\mathcal{R}$ except that $\mathcal{B}_j$ is replaced by $\mathcal{B}_j^\alpha$ which is modeled by T^α. Further, the global topology of $\mathcal{R}^\alpha - \mathcal{B}_j^\alpha$ is the same as $\mathcal{R}$'s; in particular, the knotting, linking and stability types of these closed orbits are unchanged and no singularities are created.

Proof. Let $I = [0, 1]$ and $I_\epsilon = [-\epsilon, \epsilon]$. Let $\mathbf{T}$ and $\mathbf{T}^\alpha$ denote the thickened versions of T and T^α, resp. Let $R \cong I \times I$ be the rectangular patch in T and R^α be its replacement in T^α as in Figure 4. Let $\mathbf{R} \cong R \times I_\epsilon$ be the subset of $\mathbf{T}$ corresponding to R. We assume all orbit segments of the saddle set s are within $R \times [-\epsilon/10, \epsilon/10]$ and that no other orbits of the chain recurrent set enter $\mathbf{R}$. Let $\mathbf{R}^\alpha$ be the portion of $\mathbf{T}^\alpha$ corresponding to R^α. We can make $\mathbf{T}^\alpha$ thin enough that it is within $\mathbf{T}$ and with the two vector fields identical outside of $\mathbf{R}$.

For $\mathbf{R} \cong I \times I \times I_\epsilon$ suppose that the first factor is going horizontally left to right, the second as vertically going from top to bottom and the last factor is the thickness. Write $\partial\mathbf{R} = F_1 \cup \cdots \cup F_6$ where,

$$\begin{aligned} F_1 &= \{(0,t,s) \,|\, t \in I,\ s \in I_\epsilon\}, \\ F_2 &= \{(1,t,s) \,|\, t \in I,\ s \in I_\epsilon\}, \\ F_3 &= \{(t,0,s) \,|\, t \in I,\ s \in I_\epsilon\}, \\ F_4 &= \{(t,1,s) \,|\, t \in I,\ s \in I_\epsilon\}, \\ F_5 &= \{(t,s,\epsilon) \,|\, t \in I,\ s \in I\}, \\ F_6 &= \{(t,s,-\epsilon) \,|\, t \in I,\ s \in I\}. \end{aligned}$$

Now, F_1 and F_2 are subsets of the exit set of $\mathbf{T}$, while F_5 and F_6 are subsets of the entrance set of $\mathbf{T}$. The flow within $\mathbf{T}$ enters $\mathbf{R}$ through F_3 and exits through F_4. Let F_i^α, $i = 1, \ldots, 6$ be the corresponding parts of $\partial\mathbf{R}^\alpha$. See Figure 6.

Now $\mathbf{T}$, $\mathbf{T}^\alpha$ and $\mathbf{R}^\alpha$ are handle bodies. The genus of $\mathbf{R}^\alpha$ is two, while the genus of $\mathbf{T}^\alpha$ is two plus the genus of $\mathbf{T}$. We will attach two thickened disks $\mathbf{D}_1$ and $\mathbf{D}_2$ to $\mathbf{R}^\alpha$ along annuli in F_1^α and F_2^α, respectively, with vector fields as described in Section 2. The cores of these annuli are closed curves in the boundary of the exit set and are shown darkened in Figure 6. Now, $\mathbf{R}^\alpha \cup \mathbf{D}_1 \cup \mathbf{D}_2$ is a topological 3-ball, and $\mathbf{T}^\alpha \cup \mathbf{D}_1 \cup \mathbf{D}_2$ is a neighborhood of the saddle set s^α whose exit set can be attached to ∂A exactly as the exit set of $\mathbf{T}$ was. All this can be done inside the region $\mathbf{R}$ without affecting the rest of the flow. □

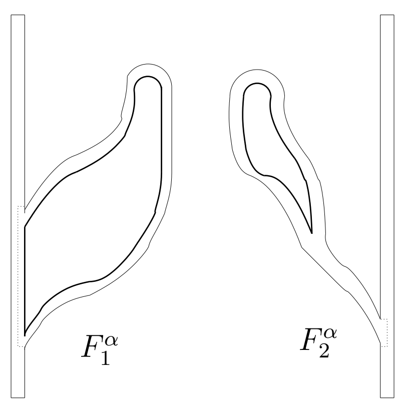

FIGURE 6. Portion of the exit set of $\mathbf{T}^\alpha$ that is in $\mathbf{R}^\alpha$

Proof Theorem 3.1. Case A. Figure 7(a) shows a template T with a diagram that depicts the attractor in blue (or dark), a solid torus neighborhood in green (or gray) and the repeller in red (or dark). A three member Markov partition (shown as black bars) with a disk cutting across each of the bands coming down from the upper branch line (dotted) has first return map conjugate to the full 3-shift. Taking the thickened template and attaching the green solid torus along its exit set as suggested by the figure results is a new solid torus that is unknotted in S^3. The attractor and repeller are unknotted and unlinked. Apply the α-move to the right most band coming from the branch line. The right most Markov disk can be divided into three disks and one sees that the first return map is conjugate to the full 5-shift. Repeat. The proof follows by induction.

In the construction above neither the attractor nor the repeller links other closed orbits. Figure 7(b) gives an example where the repeller links every closed orbit in the saddle set except one and the attractor still links no other closed orbits. In Figure 7(c) both the attractor and repeller link closed saddle orbits. We can apply α-moves to each of these as before.

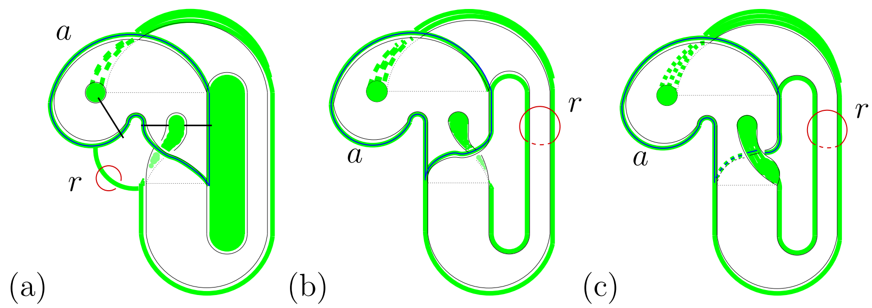

FIGURE 7. Templates for Case A.

Case B(i). This is known for $n = 2$ [13]. See Figure 8(a). The two element Markov partition gives a two-by-two incidence matrix of all ones. So, this is a suspension of a full 2-shift. The entire exit set of the thickened template is glued to a topological disk in the boundary of a tubular neighborhood of the attractor a. This neighborhood is shown in green (or gray), the attractor is blue (or dark), the repeller r is in red (dark). This tubular neighborhood of r is not shown, but is the closure if the complement of the union of the thickened template and the tubular neighborhood of a. Apply the α-move to either band of the template and use induction as in Case A. It is clear that each application of the α-move increases n by two.

Case B(ii). This is known for $n = 2$ [13]. See Figure 8(b). Now the exit set of the thickened template is attached to the boundary of a tubular neighborhood of a so that the two annuli are glued along meridians. Apply the α-move and use induction.

Case B(iii). This is known for $n = 2$ [8]. See Figure 9. We apply the α-move to the right band below the lower branch line. Subsequent α-moves are applied to the right most band below the lower branch line. However, it is not obvious that we still have a full shift. Figure 10 shows the result of applying the α-move once with our choice for the Markov partition. To see that we do still have a full shift first return map notice that the incidence matrix after applying k α-moves is $2k+2 \times 2k+2$ of the form

$$Q = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $Q = RS$ where

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \end{bmatrix}.$$

But then $SR = \begin{bmatrix} 1 & 1 \\ 2k-1 & 1 \end{bmatrix}$. The Parry-Sullivan number and Bowen-Franks group of $I - SR$ are $1 - 2k$ and $\frac{\mathbb{Z}}{(2k-1)\mathbb{Z}}$, respectively. These are identical to the invariants for the full $2k$-shift. Hence the flow on the modified template is flow equivalent to the suspension of the full $2k$-shift as claimed.

Note: If we change the lower "Lorenz ear" to be in front of, instead of behind, the band from the upper branch line, we could give a different proof of Case B(ii) [8]. In fact the template of Figure 9 can be realized by a simple Smale flow with $a \cup r$ a Hopf link [8], but we thought the Lorenz template was a cleaner approach to B(i) and B(ii).

Case C. Study Figure 11. Clearly $n = 3$. We can apply the α-move repeatedly to the right most band below the branch line. It is clear the value of n increases by two each time. Any integer value for p, the number of full twists in the middle band, is allowed. This causes the attractor (or repeller if we reverse time) to be σ^{2p-1}.

Case D. The case for $n = 2$ is known [16]. Figure 12 shows a realization for $p = 1$. Apply the α-move. Use induction. $\square$

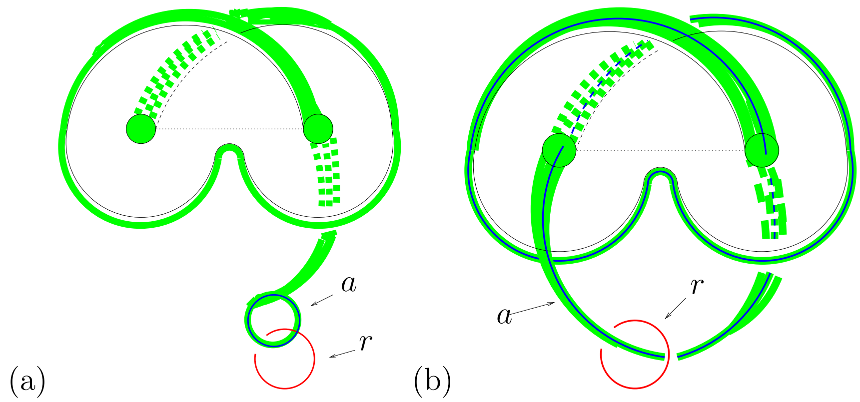

FIGURE 8. (a) Full 2-shift, with Hopf link. (b) Full 2-shift, linking number 1 with trefoil.

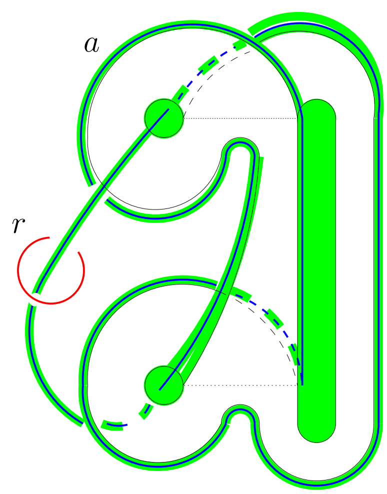

FIGURE 9. Full 2-shift, linking number 1 with figure-8 knot.

4. DISCUSSION

Figure 13 gives a table summarizing Theorem 3.1. The rows correspond to linking numbers l and the columns to number of symbols n for the full shift space. The shaded cells are those for which no examples can be constructed by Corollary 2.4. The link diagrams depict the

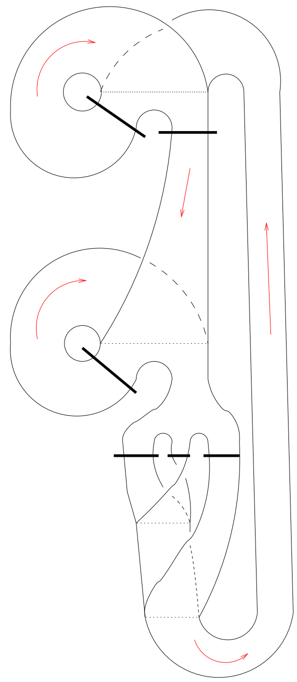

FIGURE 10. Markov partition after one α-move on template in Figure 9.

constructions in Theorem 3.1. We observe that in each case it is possible for both the attractor and repeller to be unknots. Is this true for $l \geq 4$? We also observe that in every construction either the attractor or repeller is an unknot. Is this necessary? Is this true for $l \geq 4$? In the $l = 0$ row is it possible to get Whitehead links?

Up until recently it was unknown if there were any restrictions on the set of two components links that could be realized as an attractor-repeller pair for a nonsingular Smale flow on S^3. However, a paper in preparation by Francois Beguin, Christian Bonatti and Bin Yu [1] shows that it is not possible to have two unlinked nontrivial knots as an attractor-repeller pair. This is the only restriction we know of for the $l = 0$ row of the table in Figure 11.

We chose to work in S^3, but one can do Dehn surgery on the attractor and the repeller to get simple Smale flows on other manifolds. The figure-8 knot in the $l = 1$ row means such surgeries would generate many hyperbolic manifolds. What about the other rows? Anosov flows can generate simple Smale flows by doing two orbit surgeries [3] but the manifold cannot be S^3 and will often be hyperbolic.

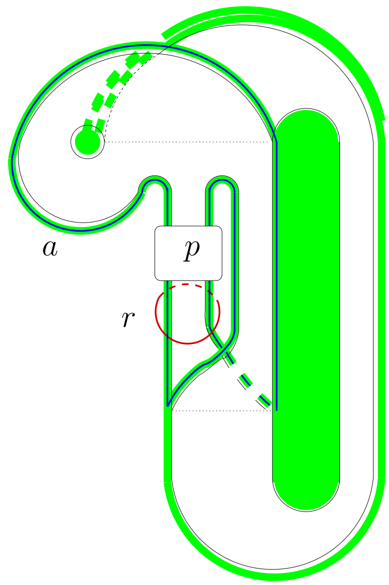

FIGURE 11. Full 3-shift with linking number 2. There are p full twists in the middle band.

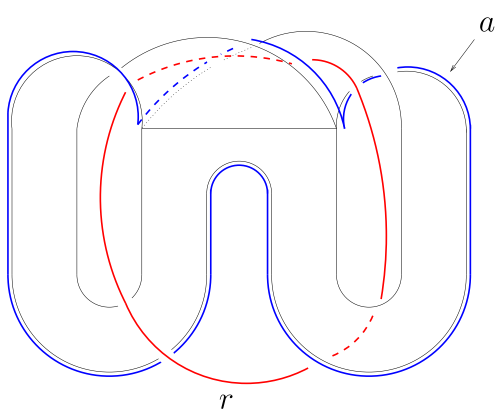

FIGURE 12. Full 2-shift with linking number 3.

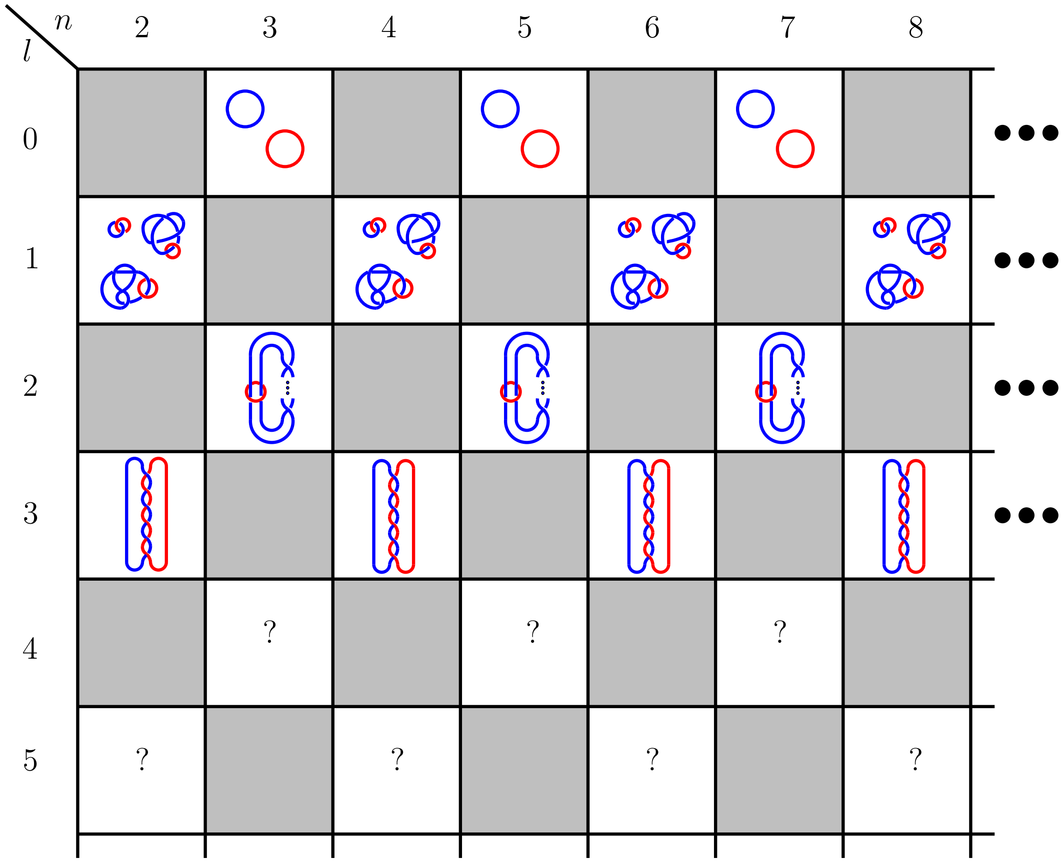

FIGURE 13. Summary of Theorem 3.1.

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