A ZETA FUNCTION FOR FLOWS WITH POSITIVE TEMPLATES

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Abstract

A zeta function for a map $f : M \to M$ is a device for counting periodic orbits. For a topological flow however, there is not a clear meaning to the period of a closed orbit. We circumvent this for flows which have *positive templates* by counting the "twists" in the stable manifolds of the periodic orbits.

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1 Introduction

We study smooth flows in S^3 (or R^3) which have one dimensional hyperbolic chain-recurrent sets. See [**BW-II**, section 2] or [**F**] for details. A point $x \in S^3$ is in the chain-recurrent set of a flow ϕ_t if for any $T > 0$ and any $\epsilon > 0$ there exist points in S^3, $x = x_0, ..., x_n = x$, and real numbers $t_0, ..., t_{n-1}$, all greater than T, such that $d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$ for $i = 0, ..., n-1$. Here d is a given metric. S^3 can be replaced by any manifold. The points of any periodic orbit are in the chain-recurrent set. The chain-recurrent set is compact and invariant under the flow.

An invariant set is said to be hyperbolic, or to have a hyperbolic structure, if each orbit has a stable manifold consisting of itself and any other orbits that are attracted to it, and an unstable manifold consisting of itself and any other orbits repelled from it. Thus, in three dimensional flows, each orbit is either an attractor (with a three dimensional stable manifold), a repeller (with a three dimensional unstable manifold) or the stable and unstable manifolds are each two dimensional sheets, as shown in figure 1.1. The notions of chain recurrence and hyperbolicity can also be defined for maps [**F**].

The chain recurrent set can be divided into a finite number of transitive pieces, meaning they contain a dense orbit, called basic sets. A basic set may consist of a single closed orbit or it may contain an infinite mesh of orbits. In the later case, the periodic orbits form a dense subset and all orbits have two dimensional stable and unstable manifolds. The periodic orbits form infinitely many distinct knot types [**FW**].

If such a basic set is one dimensional, as is the case for Smale flows, its periodic orbits can be modeled with a *template* which is a branched 2-manifold with a semi-flow (see definition 3.1). That is the knotting and linking structures in the basic set can be identified with those in the template. Templates have been used to study a variety of flows [**BW-I**, **BW-II**, **HW**, **H1**, **H2**, **H4**, **Su1**, **Su2**]. Here we use templates to model not the knotting of periodic orbits but the twisting of their stable manifolds. See figure 1.2. (This isn't quite true, we will use a nonstandard notion of twist which does incorporate some knot theoretic information.)

The difficulty in applying zeta function theory to flows is that for a topological flow there is not a clear meaning to the period of a closed orbit. We circumvent this for one dimensional hyperbolic chain-recurrent sets by using the twist of a closed orbit as a canonical period. That is we shall define a rational function whose logarithmic series expansion will allow us to count the number of periodic orbits with any given amount of twist, much the way a zeta function for a map counts the periodic orbits. This is done is section 5.

Unfortunately, for many templates the number of periodic orbits with a given amount of twist can be infinite. But we will show in lemma 4.2 that this is not the case for *positive templates*, where only one type of crossing is allowed, and so we will restrict our attention to them.

2 Review of Zeta Functions

For general references to zeta functions see [**F**, chapter 5] or [**Sh**, chapter 10].

Definition 2.1 *The* zeta function *of a map $f : M \longrightarrow M$ is the exponential of a formal power series in t,*

$$\zeta_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m\right),$$

where N_m is the cardinality of the fixed point set of f^m, the m-th iterate of f.

If f has a hyperbolic chain recurrent set then the N_m are all finite and $\zeta_f(t)$ is a rational function. Hence a finite set of numbers determine all the N_m.

If O_l is the number of periodic orbits of length l then

$$N_m = \sum_{l|m} l O_l.$$

We can recover O_l by the Möbius inversion formula [**Sm**, page 765]:

$$O_l = \frac{1}{l} \sum_{m|l} \mu(m) N_{l/m},$$

where μ is the function defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } \exists \text{ a prime } p \ni p^2|m, \\ (-1)^r & \text{if } m = p_1, \ldots, p_r, \text{ for } r \text{ distinct primes.} \end{cases}$$

When a map's chain-recurrent set is hyperbolic and zero dimensional, as is the case for subshifts of finite type, there exists a square matrix A of non-negative integers such that $N_m = \operatorname{tr} A^m$. Then $\zeta_f(t) = 1/\det I - tA$.

3 Knots, Braids and Templates

Standard texts on knots include [**B**], [**BZ**] and [**R**]. The principal references for templates are [**BW-I**] and [**BW-II**].

Definition 3.1 *A* template *is a compact branched 2-manifold with a smooth expansive semi-flow.*

We give an example of a template, known as the Lorenz template, in figure 3.1. The semi-flow goes down from the branch line and then around either of the two branches back to the branch line. This is a semi-flow since backward orbits are not unique and since many orbits exit the template. Figure 3.1 shows a periodic orbit. Its knot type is that of the left-hand trefoil. The Lorenz template has been used to model the Lorenz equations (whose hyperbolicity is still conjectural) [**BW-I**].

A knot is a smoothly embedded 1-sphere in a 3-sphere. We can give a knot an orientation, which for us will just be the flow direction. A link is a disjoint union of knots. Any link can be placed (ambiently isotoped) in a braid form, where it wraps around a braid axis (a straight line). This was proven by Alexander, [**B**]. The same is true of templates [**FW**]. That is, any template can be isotoped so that all the closed orbits form braids. Figure 3.2 shows how to braid the Lorenz template.

The crossings in a link, or a template, come in two types, left-handed and right-handed, which we shall call positive and negative, respectively. The crossings of the orbits in the Lorenz template are always positive. Knots and links which can be presented as positive braids are called *positive braids*, and templates which contain only positive braids are called *positive templates.* There are many knots which cannot be presented as positive braids, for example the figure-8 knot, and which therefore do not occur in the Lorenz template.

Of course theorems about positive braids and templates usually carry over to "negative" braids and templates. All our results will do so. One can think of positive braids and templates as including all braids and templates with a presentation that has only one crossing type. Perhaps the name *uniform braids* would be better, but this is not the custom.

Templates are constructed by collapsing out stable manifolds of a flow's chain recurrent set. This process preserves the periodic orbits and how they are linked and knotted. But, it is also the case that the unit normal bundle of a periodic orbit on a template is isotopic to the intersection of a tubular neighborhood of the orbit and its stable manifold in the original flow. The idea is to use the twisting in these *ribbons* as a canonical period for the orbits.

Definition 3.2 *Given a template we construct its* ribbon set *as follows. Remove all orbits which eventually exit the template. Call what is left the* orbit set *of the template. We attach to each of these orbits a ribbon defined by each orbit's unit normal bundle. This whirling collection of ribbons is the* ribbon set.

We shall construct zeta function which is invariant with respect to ambient isotopy of the ribbon set of a template embedded in S^3. This is weaker than just isotopy of the template. The invariance also holds under the two "template moves" in figure 3.3, as they do not disturb the orbit set, compare [**BW-II**, figure 2.4].

4 Positive Ribbons

A closed ribbon, or ribbon for short, is an embedded annulus or Möbius band in S^3. In this section we shall define three notions of twist for ribbons. These are, the *usual twist* τ_u [**K**, section V], the *modified twist* τ_m, and the *computed twist* τ_c.

Like knots and templates, ribbons can be braided. A ribbon which has a braid presentation such that each crossing of one strand over another is positive and each twist in each strand is positive, will be called a *positive ribbon.* The core and boundary of a positive ribbon are positive braids.

We will use the following notation. If R is a ribbon and $b(R)$ is a braid presentation of R, let c be the sum of the crossing numbers of the core of R, using $+1$ for positive crossings and -1 for negative ones. Let t be the sum of the half twists in the strands of $b(R)$ and let n be the number of strands of the core.

Definition 4.1 *Let* $\tau_u = c + t/2$, $\tau_m = n - 1 + t/2$ *and* $\tau_c = 2n + t$.

Lemma 4.1 τ_u *is an isotopy invariant of ribbons over all braid presentations.* τ_m *and* τ_c *are isotopy invariants of positive ribbons over positive braid presentations.*

Proof: The proof uses *linking numbers* and the *genus.* These are standard invariants of knot theory [**BZ**], [**R**]. The linking number of two components of a link is the sum of the crossing types as one of the two knots passes under the other. For an embedded annulus the linking number of the two boundary components is $c + t/2$. The same formula gives one half the linking number of an embedded Möbius band's boundary with its core. In both cases we get that τ_u is an invariant.

The genus of a knot or link is the minimum genus of orientable surfaces whose boundary is the knot or link in question. It was shown in [**BW-I**, Theorem 5.2] that for positive braids the genus is given by,

$$g = \frac{c - n + 1}{2}.$$

The invariance of τ_m for positive ribbons follows from checking that

$$\tau_m = \tau_u - 2g,$$

where g is the genus of the core of R. Lastly we see that $\tau_c = 2(\tau_m + 1)$. ■

For the trefoil orbit in figure 3.1 the reader can check that $g = 1$ and that its unit normal bundle has $\tau_u = 6$, $\tau_m = 4$ and $\tau_c = 10$.

Visually, the conversion of a positive full twist to a loop or writhe decreases t by 2 but creates an extra strand. Since doing this to a negative full twist would increase t by 2 while creating an extra strand, it is easy to show that the invariance of τ_m and τ_c fail for ribbons with mixed crossings. This is a second, and in a sense a more serious, reason for the restriction to positive templates. We also note that $\tau_u = \tau_m$ is equivalent to $g = 0$, which in turn is true if and only if the core of the ribbon is unknotted.

The proof of the next lemma will require the use of Markov partitions. We shall give a brief review, but refer the reader to [**Sh**, Chapter 10] for details. In our context a Markov partition is a finite, disjoint collection of line segments transverse to a template's semi-flow. Each orbit that does not exit the template must pass through some element of the Markov partition in forward time. The flow induces a first return map on the partition elements. This map has the property that each Markov partition element is mapped onto any partition element that its image meets. The thickened line segments in figures 4.1 and 5.1 represent the elements of Markov partitions for the templates shown. Note however, that Markov partition elements do not have to be contained in branch lines.

Lemma 4.2 *For positive templates the number of closed orbits with a given computed twist is finite.*

Proof: Given a positive template we put it into a positive braid form and construct a Markov partition. Let x be the number of partition elements and label the elements $s_1, ..., s_x$. Then each closed orbit is now represented by a cyclic sequence in these symbols, up to cyclic permutations. Because the semi-flow on the template is expansive it is well known that this representation is one-to-one on a subset of bi-infinite symbol sequences [**BW-I**, Section 2.4]. Thus the number closed orbits of a given least cycle length (usually called the word length) is finite.

Given τ_c choose n so that $\tau_c < 2n$. Because the template is braided, a closed orbit that meets any one partition element n times must have wrapped around the braid axis at least n times. Since there are no negative half twists such an orbit's computed twist is bigger than or equal to $2n$. If $w \geq xn$, then any closed orbit with word length w must have traveled around the template's braid axis at least n times. Thus, any closed orbit with computed twist τ_c has word length less than xn. There can only be finitely many such orbits. ∎

The computations in the proof of lemma 4.1 show that lemma 4.2 holds for τ_m and τ_u as well as τ_c. This is clear for τ_m. For τ_u, use the fact $g \geq 0$ implies $\tau_u \geq \tau_m$.

5 Counting Twisted Ribbons

Definition 5.1 *For a given positive template let $T_{q'}$ be the number of closed orbits with computed twist q'. Let $\mathcal{T}_q = \sum_{q'|q} q' T_{q'}$. Then we define the zeta function of the template to be the exponential of a formal power series:*

$$\zeta(t) = \exp\left(\sum_{q=2}^{\infty} \mathcal{T}_q \frac{t^q}{q}\right).$$

Theorem 5.1 *ζ is an invariant of ambient isotopy of the ribbon set for positive templates. It terms of positive templates ζ is invariant under isotopy and the two moves shown in figure 3.3.*

Proof: This is just a corollary of lemma 4.1. ∎

Given a positive template we can put it in positive braid form and consider a Markov partition of the orbit set. We now define a *twist matrix*, $A(t)$, whose entries are nonnegative powers of t and 0's by considering the contribution to

τ_c as an orbit goes from one element of the partition to other. Let $A_{ij} = 0$ if there is no branch going from the i-th to the j-th partition element. $A_{ij} = t^{q_{ij}}$ if the is such a branch, where the q_{ij} is the amount of computed twist orbits pick up as they travel from the i-th to the j-th partition element. It is easy to see that we can, if necessary, isotope the template so that q_{ij} is always integral. This problem could arise if some of the partition elements are outside of the branch lines. (Note: We can always choose the partition so that at most one branch goes from the i-th element to the j-th element for each i and j. However, if one wishes to be more general, we can use polynomials in $A(t)$ instead of just powers of t.)

For example, the template and partition in figure 4.4 give

$$A(t) = \begin{bmatrix} 0 & 0 & 0 & t & t \\ 0 & 0 & 0 & 1 & 1 \\ 0 & t^2 & t^2 & 0 & 0 \\ t^2 & t^2 & t^2 & 0 & 0 \\ t^3 & t^3 & t^3 & 0 & 0 \end{bmatrix}.$$

Theorem 5.2 *For any template and any allowed choice of $A(t)$ we have $\zeta(t) = 1/\det(I - A(t))$. Thus, the zeta function is rational.*

Before proving theorem 5.2 we work through a example. The template in figure 5.1 has been used to model an embedding of the suspension of Smale's horseshoe map and has been extensively studied. See [**BW-II**, Secion 3], [**HW**], [**H1**] [**H2**] and [**H4**].

Using the two element Markov partition indicated by the thick lines in figure 5.1, we have

$$A(t) = \begin{bmatrix} t^2 & t^2 \\ t^3 & t^3 \end{bmatrix},$$

and so,

$$1/\det I - A(t) = 1/(1 - t^2 - t^3).$$

We apply a standard matrix identity (see lemma 5.2 of [**F**] or proposition 10.7 of [**Sh**]) to get

$$\frac{1}{\det I - A(t)} = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{tr}\ A(t)^n}{n}\right). \qquad (*)$$

Let us analyze the first three terms of

$$\sum_{n=1}^{\infty} \frac{\text{tr } A(t)^n}{n} = \frac{t^2+t^3}{1} + \frac{t^4+2t^5+t^6}{2} + \frac{t^6+3t^7+3t^8+t^9}{3} + \cdots$$

There are five closed orbits which pass through the Markov set three or fewer times. All are unknotted, so $\tau_m = \tau_u$. We label each with a word using x and y as the symbols for the left and right partition elements respectively. The five orbits are x, y, xy, xxy, and xyy (up to cyclic permutations).

The t^2 and the t^3 of the first term of the sum correspond to the orbits x and y respectively. In the second term, x and y are counted again, by t^4 and t^6 respectively, but they have been traversed twice. The $2t^5$ corresponds to xy, where the 2 is the product of number of orbits that pass through the Markov set twice (just 1 in this case) and 2, the number of passes.

The reader should check that $3t^7$ corresponds to xxy and $3t^8$ to xyy. The t^6 and the t^9 again count x and y respectively, this time making three trips on each. In general the exponents give the τ_c times the "trip number." It is worth noting that tr $(A(1))^n$ is the number of intersection points of the Markov set with the link of closed orbits which meet the Markov set n' times, where n' divides n.

Proof of theorem 5.2: Since equation $(*)$ applies to any twist matrix all that is left to do is rearrange the terms of the infinite sum by collecting the powers of t.

Let $T_{q'n'}$ be the number of closed orbits with $\tau_c = q'$ that pass through the Markov set n' times, i.e those orbits whose symbol sequence has least period n'. We shall set $T_{q'n'} = 0$ if either index is nonintegral.

If

$$\text{tr } (A(t))^n = \sum_{q=m_n}^{M_n} p_{qn} t^q,$$

for some positive integers M_n and m_n, then

$$p_{qn} = \sum_{n'|n} n' T_{q'n'},$$

where $q' = \frac{n'}{n} q$. Note that $q/q' = n/n'$ is the "trip number", when q' is an integer.

That is, p_{qn} is the number of points in the intersection of the Markov set with those closed orbits that meet the Markov set n' times for some $n'|n$ and whose τ_c divides q. This can be proved by induction on n as in lemma 2 of [**BL**].

We can rewrite this as

$$p_{qn} = \sum_{q'|q} n' T_{q'n'},$$

where now $n' = \frac{q'}{q} n$.

Now we show that for a fixed q only finitely many of the p_{qn} are nonzero. Let x be the number of elements in the Markov partition. Let $N > xn$ for some $n \geq q/2$. Allowed symbol sequences of (not necessarily least) period N represent orbits that have traveled around the braid axis at least n times, taking the number of "trips" into account. Thus the smallest exponent of $\mathrm{tr}\ (A(t))^N$ is bounded below by $2n$. (This can be thought of as an application of lemma 4.2.)

If we let

$$\mathcal{T}_q^* = q \sum_{n=1}^{\infty} \frac{p_{qn}}{n},$$

then

$$\sum_{n=1}^{\infty} \frac{\mathrm{tr}\ (A(t))^n}{n} = \sum_{q=2}^{\infty} \frac{\mathcal{T}_q^*}{q} t^q.$$

It is only left to show that $\mathcal{T}_q^* = \mathcal{T}_q$.

$$\sum_n \frac{p_{qn}}{n} = \sum_n \sum_{q'|q} \frac{n'}{n} T_{q'n'} = \sum_n \sum_{q'|q} \frac{q'}{q} T_{q'n'} = \frac{1}{q} \sum_{q'|q} q' \sum_n T_{q'n'}.$$

But, $n' = \frac{n}{q/q'}$. As n ranges over all positive integers so does $\frac{n}{q/q'}$, since q/q' is an integer. This allows us to write

$$\sum_n T_{q'n'} = \sum_n T_{q'n} = T_{q'}.$$

Thus, $\mathcal{T}_q^* = \mathcal{T}_q$ and the proof is complete. ■

6 Applications

We regard two templates as equivalent if one can be taken to the other by a combination of isotopy through 3-space and a finite number of the two template moves depicted in figure 3.3. This implies their respective ribbon sets are isotopic and we conjecture that the converse is true as well.

Distinguishing between different templates or telling if one template is a subtemplate of another can be a highly nontrivial affair [**Su2**]. However, for positive templates at least, the twist approach makes the task much easier.

Let $L(m,n)$ denote a *Lorenz-like* template, which is similar to the Lorenz template, except that the left branch has m half twists and the right branch has n half twists. The corresponding zeta functions, for $m, n \geq 0$ are

$$\zeta_{m,n} = 1/(1 - t^{m+2} - t^{n+2}).$$

This distinguishes between these templates except that $\zeta_{m,n} = \zeta_{n,m}$. The reader can check that $L(m,n)$ is equivalent to $L(n,m)$ via a 180^o rotation.

Template A in figure 6.1 contains the Lorenz template as a *subtemplate*; just consider those orbits which do not make use of the lower loop on the left. Is it possible that A is also a subtemplate of $L(0,0)$? The answer is no, since A has three loops with $\tau_m = 0$ while the Lorenz template has only two. It was the generalization of this observation that led to the discovery of our zeta function for positive templates. In fact there are infinitely many orbits of A which are not to be found on $L(0,0)$. Table 1 lists some of them.

In [**Su1**] and [**Su2**] it was shown that certain templates with mixed crossing types contained themselves as proper subtemplates. This represents a self-similarity structure in these systems. For positive templates such a situation is impossible. There would be infinitely many isotopic copies of every orbit in the template. But, the number of orbits of a given modified twist number is finite, by lemma 4.2.

Alas, the next example shows our zeta function is not a complete invariant. Template B shown in figure 6.2 has the same zeta function as A. However, they are distinguishable in at least two ways. While they both have three closed orbits of zero twist, in A these are all in the boundary, but one of B's is in its interior. In [**Su2**] it was shown that all the knots in A are prime while B contains composite knots.

Robert Ghrist has pointed out to us that positive templates arise in the setting of forced oscillators. Let $\ddot{x} = g(x, \dot{x}, t)$, where g is smooth and periodic

τ_m	$L(0,0)$	$L(0,1)$	A
0	2	1	3
$\frac{1}{2}$	0	1	0
1	1	0	2
$\frac{3}{2}$	0	1	0
2	2	0	5
$\frac{5}{2}$	0	1	0
3	3	1	10
$\frac{7}{2}$	0	1	0
4	6	1	24
$\frac{9}{2}$	0	1	0
5	9	1	50
$\frac{11}{2}$	0	2	0

Table 1: Number of orbits listed by τ_m for different templates.

in t. This can be converted to a first order autonomous system:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= g(x, \dot{x}, t)\\ \dot{t} &= 1,\end{aligned}$$

with phase space the real plane cross a circle. The chain-recurrent set, however, will live in a solid torus. The condition $\dot{x} = y$ causes the twisting in the flow to be monotonic. Thus if the flow can be modeled by a template, the template will be positive. This is the case for the Duffing equation, where $g = \delta\dot{x} + x - x^3 + \gamma\cos\omega t$, and is very likely the case for the Josephson equation, where $g = \delta\dot{x} - \sin x + \nu - \beta\cos\omega t$, for certain parameter values [**H3**].

7 Future Work

In many applications one is not concerned with how the template is embedded in 3-space. For example, the mapping torus of a suspension flow can be embedded with any number of full twists. As long as the template exists and is positive we can define ζ for any particular embedding. However, it seems reasonable to define $A(t)$ only up to multiplies of even powers of t. Then $\{\det I - t^{2n}A(t) | n \geq 0\}$ is an invariant that doesn't change with (positive) embedding.

More generally, different embeddings of a template can only change the twisting in the branches by full twists. Thus we redefine $A(t)$ taking the powers of t to be 0 or 1 as the number of half twists is even or odd. The reciprocal of $\det I - A$ is no longer a zeta function, but it is an invariant for positive templates and we strongly suspect that it is a general template invariant, though perhaps it is necessary to compute the determinant $\bmod t^2$.

In [**PS**] B. Parry and D. Sullivan show that $\det I - A(1)$ is a flow invariant. The key idea in their paper is the invariance of $\det I - A(1)$ when one adds a new element to a Markov partition that is parallel, in the sense of the flow, to an already existing element. This would change the usual zeta function $1/\det I - tA(1)$ since the number of passes orbits make through the Markov partition has changed.

Intuitively, their invariant works because while it keeps track of whether an orbit is periodic, it does not compute the period. That is it "knows" if you can get from one element of the Markov partition to another, but it does not "remember" how many partition elements a closed orbit goes through, since $1 \times 1 = 1$. Invariance arises by suppressing information. In our case, invariance arises by adding information about the behavior of the orbits between elements of the partition.

Let $a + bt = \det I - A(t)$ computed $\bmod t^2$. How should a and b be interpreted? They seem to be telling us something about the orientation preserving and reversing "parts" of the flow, respectively. Can this rather vague intuition be made more precise?

Applying this function to the Lorenz-like templates we see that they fall into just three classes, up to embedding. These can be represented by $L(0,0)$, $L(0,1)$ and $L(1,1)$, whose invariants are, respectively, -1, $-t$ and $1-2t$. Or as ordered pairs in a and b, $(-1,0)$, $(0,-1)$ and $(1,-2)$, respectively. Notice that they all have the same Parry-Sullivan invariant, -1.

In another vein, we notice the apparent contrast in the growth rates in table 1. This warrants further study. In particular we hope to define a consistent notion of entropy for flows with positive templates.

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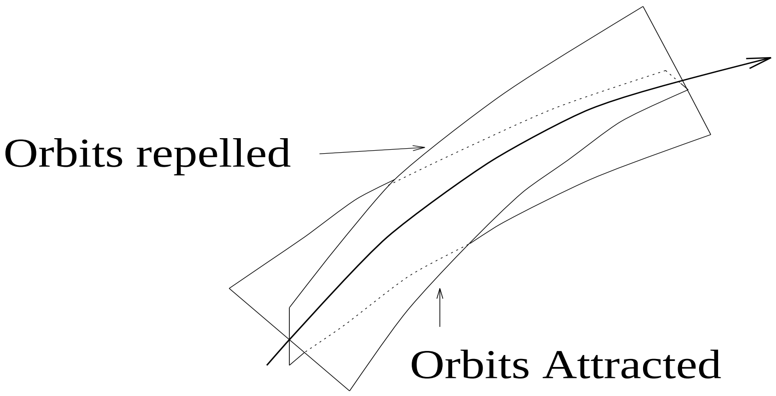

Figure 1.1: Stable and Unstable manifolds for an orbit.

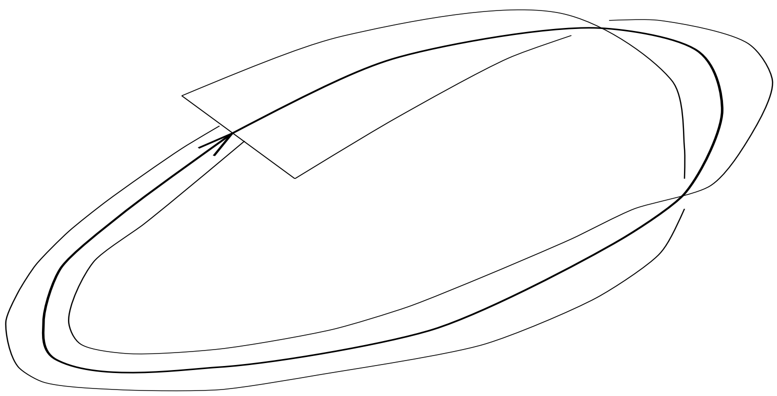

Figure 1.2: Stable manifold for a closed orbit with a full twist.

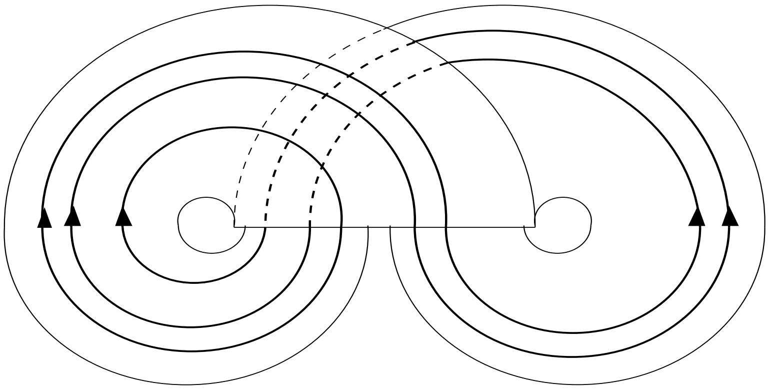

Figure 3.1: Lorenz template with a trefoil orbit shown.

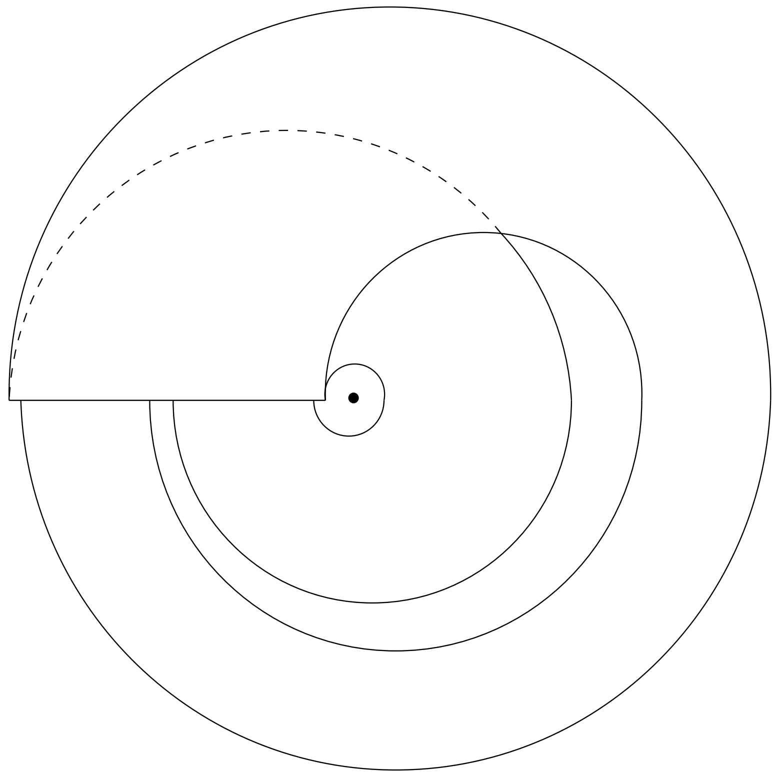

Figure 3.2: Lorenz template in braid form.

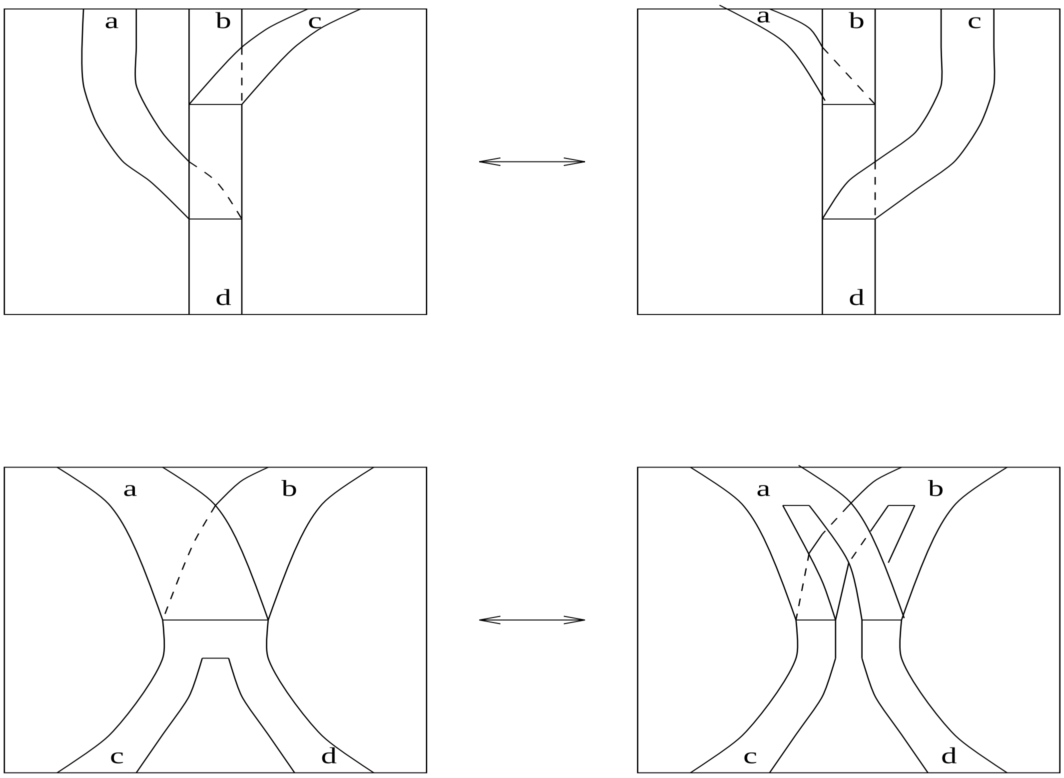

Figure 3.3: Template moves; portions of template outside boxes unchanged.

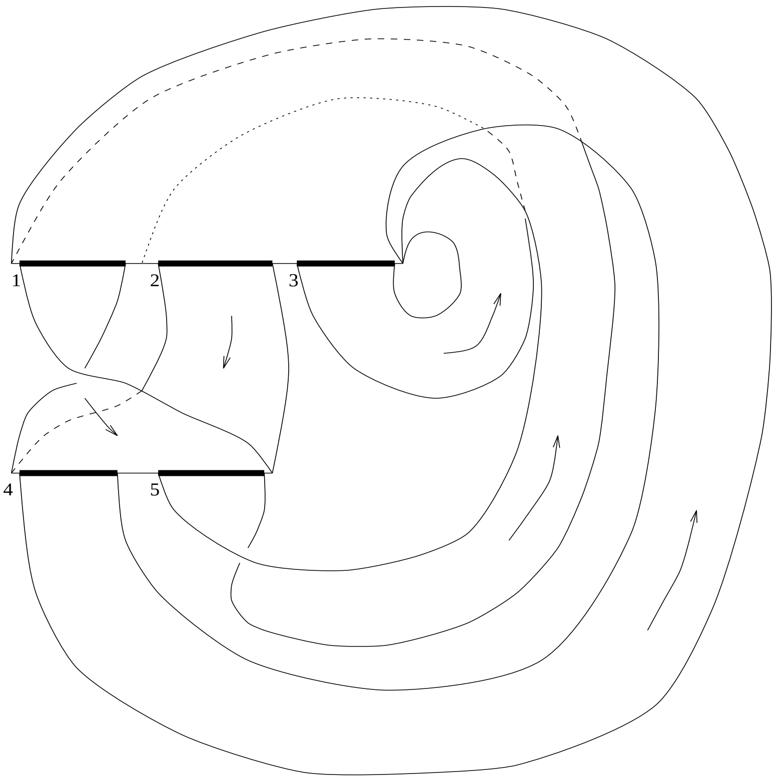

Figure 4.1: A template with a Markov partition indicated by thick lines.

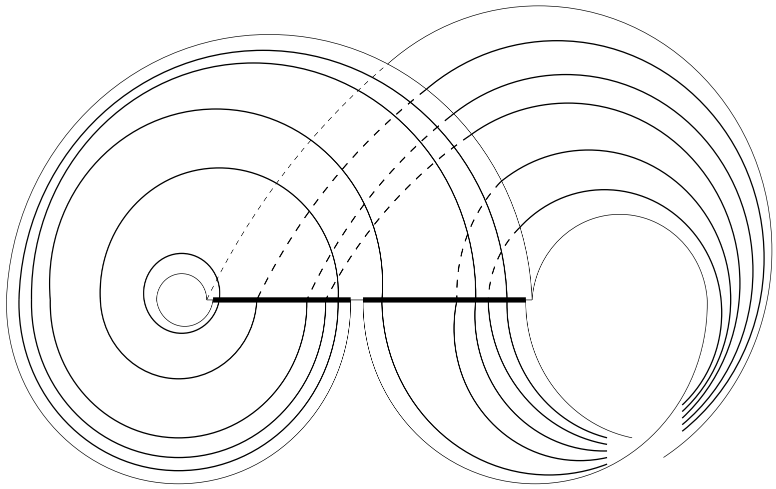

Figure 5.1: The Horseshoe template with a Markov partition (thick lines). The periodic orbits x, y, xy, xxy, and xyy are shown.

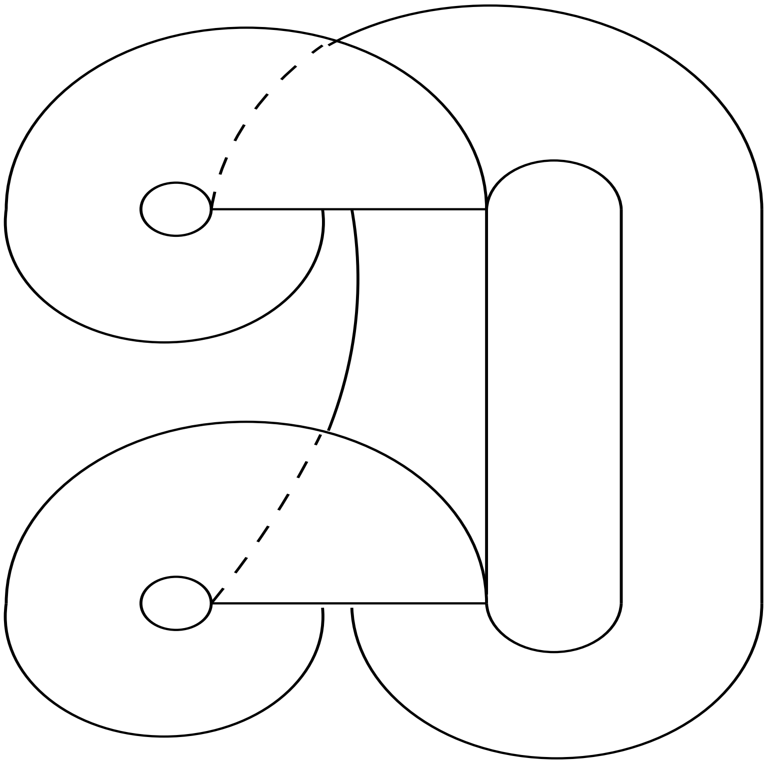

Figure 6.1: Template A.

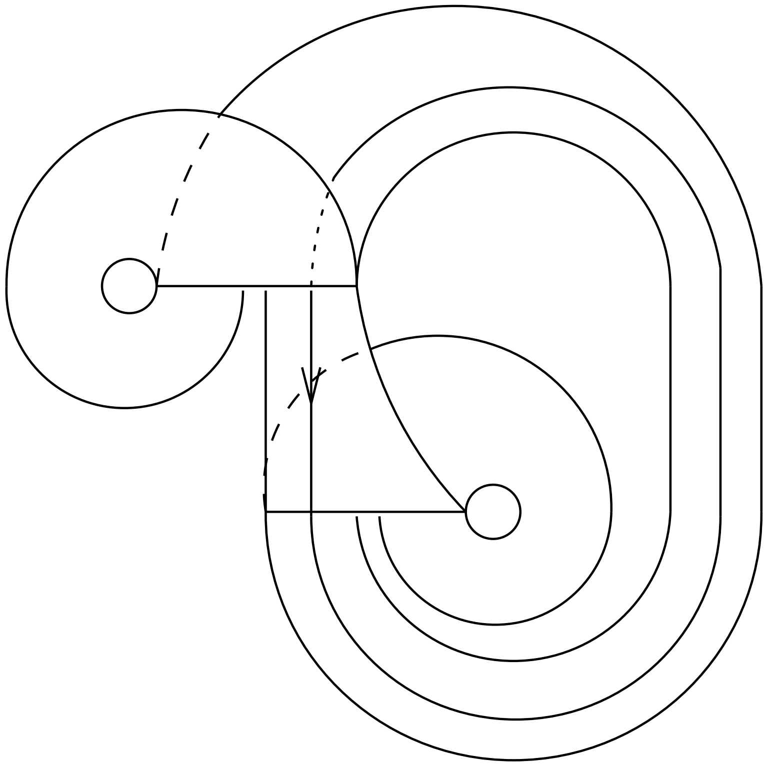

Figure 6.2: Template B.