Symbolic Dynamics and its Applications: AMS Short Course, January 4-5, 2002, San Diego California, in the series Proceedings of Symposia in Applied Mathematics, Vol. 60. Edited by Susan G. Williams. American Mathematical Society, 168 pp., hardcover. $39.00.
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This volume consists of a preface and six chapters based on lectures given in an AMS Short Course on symbolic dynamics organized by Susan Williams in San Diego 2002. Each article was refereed. Symbolic dynamics is a field of growing importance. The range of its applications and the depth of its theoretical base are expanding rapidly as this volume attests.

We briefly introduce the reader to the field. Let $\mathcal{A}$ be a finite set. Let $\mathcal{F}$, the set of forbidden words, be a list of finite words made from elements (symbols) of $\mathcal{A}$. Let X be the subset set of $\mathbb{Z}^{\mathcal{A}}$ (bi-infinite sequences of symbols from $\mathcal{A}$) such that no member of $\mathcal{F}$ appears as a subword of a sequence. We give X the natural topology as a subspace of a product space. The *shift map*, $\sigma : X \to X$ increments the indices by one: $(\sigma(\bar{x}))_i = x_{i+1}$. The pair (X, σ) is a dynamical system where one studies the behavior of iterates of points under σ. We call X a *shift space*. If $\mathcal{F}$ is a finite list, we call X a *shift of finite type*. Shifts of finite type can be defined via a directed graph or an incidence matrix and are probably the most studied symbolic dynamical systems. Variations on this theme allow for $\mathcal{A}$ to be countably infinite, producing *countable shifts*, and for sequences indexed by the nonnegative integers, producing *one-sided shifts*.

There are two primary textbooks, one by Doug Lind and Brian Marcus [10], who each have contributions to this collection, and another by Bruce Kitchens [9], whose works are referenced by most of the papers in this collection. Two other important collections of introductory articles on symbolic dynamics are [1] and [8].

Susan Williams provides an eleven page introduction based on her lecture that surveys the early history of the field, starting with Hadamard's paper in 1898, and goes on to introduce many of the topics to be covered in the mini-course. She provides basic definitions, key concepts and problems, and two examples to get the reader started.

Ever wonder what is going on when your modem is screeching? The chapter by Marcus introduces us to the world of data transmission and storage. We learn how various physical restraints determine lists of forbidden words. The numerous examples go well beyond the material in Section 2.5 of the textbook [10]. This chapter could be a useful resource for students using [10] who wish to pursue this area further, but it can certainly be read by itself.

Paul Blanchard, Bob Devaney and Linda Keen, have contributed a chapter on the interplay between complex dynamics and symbolic dynamics. The corresponding mini-course was presented by Devaney. This chapter studies the iteration of polynomials on the complex plane. Since the mappings are not one-to-one, the corresponding symbolic dynamical systems are one-sided shift spaces. An *automorphism* of a shift space is a homeomorphism that commutes with the shift map. The collection of automorphisms of a shift space is a group under composition. It is shown that the study of the asymptotic behavior of complex polynomials yields a complete set of generators for certain automorphism groups (Theorem 2). The chapter concludes by asking about the automorphism group of the two-sided full 2-shift (all possible bi-infinite sequences of two symbols), and conjectures that the answer is closely tied to the dynamics of the *Henón map*.

Devaney *et al* are well known for their colorful computer graphics: [6], [5], [7]; so it was a bit disappointing to find only black and white figures. For more on the automorphism groups of shift spaces see [9]. For a more elementary treatment of complex dynamics don't forget to read Chapter 3 of [6].

If we replace $\mathbb{Z}$ by $\mathbb{Z}^d$ we advance to the domain of *Multi-Dimensional Symbolic Dynamics*, the title of Lind's chapter. The infinite sequences are replaced by infinite latices of symbols. Instead of allowed words and index shifting, *allowed patterns* and lattice translations are used to define the higher dimensional analog of a shift of finite type. Save for a class called *textile systems*, matrix methods ubiquitous in the study of one-dimensional shifts of finite type have met with only limited success in the study of higher-dimensional shifts.

The study of higher dimensional shift spaces is a daunting task. Quite understandably the examples given are for $d = 2$ and symbols set $\mathcal{A} = \{0, 1\}$. Yet even here we quickly wind up in what Lind dubs the *Swamp of Undecidability*: determining when a d-dimensional shift of finite type is nonempty is often undecidable. But our intrepid guide takes us down one path around the Swamp. One can define many interesting d-dimensional shifts with an additional group structure imposed on them. These *algebraic* $\mathbb{Z}^d$*-actions* are often amenable to analysis using powerful tools from algebra and algebraic geometry. This part of the chapter gets quite technical, but references are supplied to those wishing to explore this terrain further. Connections to ergodic theory, cellular automata and tiling theory are also discussed. Lind's chapter dovetails nicely with the next chapter on tiling theory by Arthur Robinson.

Robinson's chapter, at 39 pages, is the longest in the collection. It is an excellent introduction to tiling theory. Charles Radin's *Miles of Tiles* [12] is good for undergraduates and Martines Queffélec's book [11], while essential for the experts, is somewhat technical for those just testing the waters. The chapter has many examples, proofs of shorter results and exercises. One could build a very nice graduate seminar around it.

In tiling theory one works with tiles instead of symbols and patches instead of words. The tiles are to fit together and fill $\mathbb{R}^d$. There are two types maps to consider. One has translations of $\mathbb{R}^d$ and *substitutions*, also called *inflations*. In the former case we get *tiling dynamical systems*, which include $\mathbb{Z}^d$ shifts. In the latter case we get *substitution tiling dynamical systems*. Here each tile is partitioned into smaller tiles, and the *substitution map* is a rescaling that enlarges the smaller tiles. Applications to topological dynamics and ergodic theory are discussed in some depth, and the theory of quasicrystallography is touched on.

The last chapter is *Strong Shift Equivalence Theory*, by Jack Wagoner, one of the pioneers of a new approach to symbolic dynamics. He along with Fred Roush, Ki Hang Kim, Mike Boyle, Lind and others have been applying K-theory to symbolic dynamics. Let A and B be square matrices over the nonnegative integers, $\mathbb{Z}_+$. If there are matrices R and S over $\mathbb{Z}_+$ such that $A = RS$ and $B = SR$ we say there is an *SSE-move* from A to B. If there is a sequence of SSE-moves from A leading to B, we say A and B are *strong shift equivalent* (SSE). In 1974 Bob Williams showed that the classification of shifts of finite type up to topological conjugacy is equivalent to classifying their incidence matrices up to strong shift equivalence. Determining SSE is a difficult problem and may be undecidable. Williams had conjectured that SSE was equivalent to a more tractable notion call *shift equivalence*. Kim and Roush have shown that shift equivalence is decidable but that Williams' conjecture is false. This chapter gives the story of these developments along with many ramifications.

In the new K-theoretic approach the matrices are over polynomials and the SSE-moves are replaced by row and column operations. The new methods can be used without having to know a great deal about K-theory. Readers needing a little help getting through this chapter may wish to consult the survey articles [2] and [4], referenced by Wagoner.

Although Wagoner's chapter is the most theoretical in this collection it turned out to be the most applicable for this reviewer. In the early 90s I developed the concept of *twistwise flow equivalence*, a generalization of *flow equivalence* of shifts of finite type. The incidence matrices now contain information about orientation of a first return map and are taken over $\{a + bt \,|\, a\&b \in \mathbb{Z}_+\} \mod t^2 = 1$. By 1997 several invariants were known but then progress stalled. At the symposium I attended Wagoner's talks and meet Boyle (a former student of Lind's). Using the new framework we reduced classification up to twistwise flow equivalence to a purely algebraic problem and generalized it to other applications [3]. The K-theory framework for doing symbolic dynamics will likely open many doors.

Each Chapter in this collection is by a leading expert or experts in the field. All are well written and contain extensive references. There is a common index. This collection will be of great service to the dynamics community and those seeking entrance.

References

[1] F. Blanchard, A. Maass, and A. Nogueira, editors. *Topics in symbolic dynamics and applications*, volume 279 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000. Papers from the CIMPA-UNESCO Summer School held at Universidad de la Frontera, Temuco, Chile, January 6–24, 1997.

[2] M. Boyle. Positive K-theory and symbolic dynamics. In *Dynamics and randomness (Santiago, 2000)*, volume 7 of *Nonlinear Phenom. Complex Systems*, pages 31–52. Kluwer Acad. Publ., Dordrecht, 2002. (`http://www.math.umd.edu/~mmb/papers/index.html`)

[3] M. Boyle & M. Sullivan. Equivariant flow equivalence for shifts of finite type. Preprint (2004): `http://www.math.siu.edu/sullivan`

[4] M. Boyle & J. Wagoner Positive Algebraic K-Theory and Shifts of Finite Type. (Katok Festschrift, to appear.) Preprint (2003): `http://www.math.umd.edu/~mmb/papers/index.html`

[5] R. Devaney. *Chaos, fractals and dynamics*. Science Television, New York, 1989. Computer experiments in mathematics.

[6] R. Devaney. *An introduction to chaotic dynamical systems*. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989.

[7] R. Devaney and L. Keen, editors. *Chaos and fractals*, volume 39 of *Proceedings of Symposia in Applied Mathematics*. American Mathematical Society, Providence, RI, 1989. The mathematics behind the computer graphics, Lecture notes prepared for the American Mathematical Society Short Course held in Providence, Rhode Island, August 6–7, 1988, AMS Short Course Lecture Notes.

[8] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[9] B. Kitchens. *Symbolic dynamics*. Universitext. Springer-Verlag, Berlin, 1998. One-sided, two-sided and countable state Markov shifts.

[10] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[11] M. Queffélec. *Substitution dynamical systems—spectral analysis*, volume 1294 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.
[12] C. Radin. *Miles of tiles*, volume 1 of *Student Mathematics Library*. American Mathematical Society, Providence, RI, 1999.
[13] J. Wagoner. Strong shift equivalence theory and the shift equivalence problem. *Bull. Amer. Math. Soc. (N.S.)*, 36(3):271–296, 1999.

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