Chapter 1

Knots in Flows

A solution curve to a differential equation may form a loop. Such cyclic or period behavior is of great interest. The classical Poincaré-Bendixson theorem establishes the existence of a periodic solution curve in vector fields in the plane for suitable hypotheses. It is a mainstay of differential equations courses to this day. If the phase space of interest is three dimensional periodic solution curves may form knots and different periodic solution curves may be linked. In 1950 Herbert Seifert proved the existence of closed integral curves for certain flows in S^3 and asked if this was always the case for nonsingular continuous flows on S^3 [40]. This became known as Seifert's conjecture. This question drove a great deal of research. It was answered in the negative by Paul Schweitzer in 1974 [39]. His example was C^1. It is now known that there are C^∞ and analytic examples [33].

In 1963 Edward Lorenz published his work examining a 3×3 ODE with very odd behavior [35]. It had many periodic orbits that are knotted and linked in complex ways. In 1983 Joan Birman and Robert Williams developed a systemic framework to analyze this behavior [9]. This was one of the main motivators for the serious study of knots in flows.

We give some technical definitions. Let M be a compact Riemannian 3-manifold. For our purposes a *flow* on M is a smooth map $f : M \times \mathbb{R} \to M$ such that $f(p, 0) = p$ and $f(p, s+t) = f(f(p, s), t)$ for all $p \in M$ and $s, t \in \mathbb{R}$. Often, M will be S^3. For $p \in M$ the *orbit* of p is $O(p) = \{f(p,t) \,|\, t \in \mathbb{R}\}$. If $O(p) = \{p\}$, then p is a *fixed point* of f. A flow without fixed points is a *nonsingular flow*. If there exists a $T > 0$ such that $f(p, T) = p$, then $O(p)$ is an embedded circle, that is, it is a knot. We say in this case that $O(p)$ is a *periodic orbit* or a *closed orbit*.

As noted above smooth nonsingular flows on S^3 with no closed orbits exist. As we will see there are flows on S^3 in which every knot and link type is realized by closed orbits [16, 19] and such flows can arise as solutions to 3×3 ODEs in $\mathbb{R}^3$ [17].

1.1 Hyperbolic Flows and Basic Sets

The *chain recurrent set* of a flow f is

$$\begin{aligned}\mathcal{R} =&\{p \in M : \forall \epsilon > 0, \exists \{p_0 = p, p_1, p_2, \ldots, p_k\} \subset M, \exists \{t_1, t_2, \ldots, t_k\} \subset \mathbb{R}^+ \\ &\text{such that } d(f(p_i, t_i), p_{i+1}) < \epsilon, i = 1, \ldots, k-1,\ d(f(p_k, t_k), p_0) < \epsilon\}.\end{aligned}$$

The chain recurrent set of a flow includes fixed points and periodic orbits, but also more complicated orbits that come back near themselves over and over.

The chain recurrent set of a flow is said to have a *hyperbolic structure* if the tangent bundle of the manifold can be written as a Whitney sum $T_{\mathcal{R}} = E^u \oplus E^c \oplus E^s$ of sub-bundles invariant under Df where E^c_p is the subspace of TM_p corresponding to the orbit of p and such that there are constants $C > 0$ and $\lambda > 0$ for which $\|Df_t(v)\| \leq Ce^{-\lambda t}\|v\|$ for $v \in E^s$, $t \geq 0$ and $\|Df_t(v)\| \geq 1/Ce^{\lambda t}\|v\|$ for $v \in E^u$, $t \geq 0$.

Steve Smale showed that when $\mathcal{R}$ is hyperbolic it is the closure of the periodic orbits of the flow and that it has a finite decomposition into compact invariant sets each containing a dense orbit; he called these the *basic sets* of the flow [41, 15].

We define respectively the *stable and unstable manifolds* of an orbit $\mathcal{O}$ in a flow f. See Figure 1.1

$$\begin{aligned}W^s(\mathcal{O}) &= \{y \in M : d(f(y,t), f(x,t)) \to 0 \text{ as } t \to \infty \text{ for some } x \in \mathcal{O}\},\\ W^u(\mathcal{O}) &= \{y \in M : d(f(y,t), f(x,t)) \to 0 \text{ as } t \to -\infty \text{ for some } x \in \mathcal{O}\}.\end{aligned}$$

That these are manifolds is a classical result of Hirsch and Pugh [29] referred to as the Stable Manifold Theorem. A flow is *structurally stable* if it is *topologically equivalent*, i.e., there is a homeomorphism taking orbits to orbits preserving the flow direction, to flows obtained by small enough perturbations.

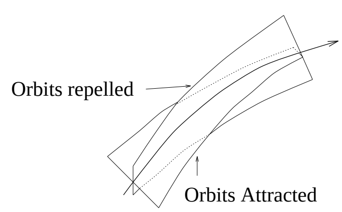

FIGURE 1.1: Stable and Unstable Manifolds

A flow with hyperbolic chain recurrent set $\mathcal{R}$ satisfies the *transversality condition* if the stable and unstable manifolds of $\mathcal{R}$ always meet transversally. A flow that has a hyperbolic chain recurrent set and satisfies the transversality condition is structurally stable; see [15, Theorem 1.10] for references. The converse — known as the C^1 Stability Conjecture — was proposed by Palis and Smale [37] and was proven by Hu [32] for dimension 3 and for arbitrary dimension by Hayashi [28]; see also [48].

1.1.1 Morse-Smale Flows

If the chain recurrent set of a flow is hyperbolic, consists of a finite collection of periodic orbits and fixed points, and satisfies the transversality condition, we have a *Morse-Smale flow*. Daniel Asimov showed that for $n \neq 3$ all n-manifolds (possibly with boundary), subject to certain obvious Euler characteristic criteria, support nonsingular Morse-Smale flows [4]. John Morgan has characterized which 3-manifolds (possibly with boundary) support nonsingular Morse-Smale flows [36] and Masaaki Wada has determined which labeled links can be realized as the invariant set of a nonsingular Morse-Smale flow on S^3. The components are labeled as attractors, repellers and saddles. See [47]; see also [11]. The simplest nonsingular Morse-Smale flow on S^3 is the Hopf flow which has just two closed orbits, one an attractor, the other a repeller, that form a Hopf link. The essence of Wada's result is that given an allowed labeled link one can apply certain allowed moves, involving cableings and connected sums, and the resulting labeled link can be realized as the invariant set of a Morse-Smale flow. Then starting with the labeled link in the Hopf flow, one can generate all other allowed labeled links.

As a possible area for future work, one would like to generalize Wada's theorem to other 3-manifolds, but Wada's proof depends heavily on the triviality of the fundamental group of S^3. Little progress has been made, but see [55].

Wada's links come up in two other types of flows. A certain subset of these links are realized as strands of fixed points for flows on S^3 arising from Bott-integrable Hamiltonian systems and in flows arising from contact structures [18, 45].

1.1.2 Smale Flows

If the chain recurrent set of a flow is hyperbolic, at most one-dimensional and satisfies the transversality condition the flow is known as a *Smale flow*. These were introduced by John Franks who was a student of Smale [15]. Basic sets which are not fixed points or isolated closed orbits are suspensions of nontrivial irreducible *shifts of finite type* (SFTs). (See [34] for definitions of terms for symbolic dynamics, SFTs have infinitely many periodic orbits but rational zeta functions.) These must be saddle sets, referred to as the *chaotic saddle sets*.

These chaotic saddle sets can be modeled by branched 2-manifolds with semi-flows where there is a bijection between any link of closed orbits in the basic set and a link of the same link-type in the semi-flow. These models are referred to as *templates*. Figure 1.2 shows the *Lorenz template*, which is denoted by $L(0,0)$; its boundary is black, the branch line is blue and a the red periodic orbit shown is a trefoil knot. Its periodic orbits were first studied by Joan Birman and Robert (Bob) Williams. [9]. They showed it supports infinitely many distinct knot types as closed orbits, that the set of links that can be realized is a subset of the set of closed positive braids with a full twist. It follows that these knots are fibered and prime. *Lorenz knots*, as they have come to be called, have been studied extensively and we will have more to say about them later.

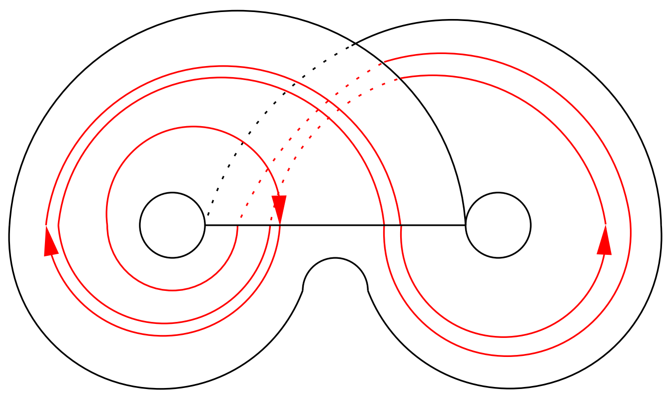

FIGURE 1.2: Lorenz Template

Figure 1.3 depicts a Smale flow on S^3 with three basic sets, an attracting closed orbit that is a trefoil, a repelling closed orbit that is a meridian of the attractor, and a chaotic saddle set modeled by a Lorenz template. In [44] all possible ways that the Lorenz template can be realized in a nonsingular Smale flow on S^3 with three basic sets are given.

There are many other templates one can construct. In "template theory" one generally asks how a given template can be realized in Smale flows, including the knotting and linking of the isolated closed orbits, and what types of knots and links are realized as closed orbits in an embedded template. A Lorenz-like template $L(m,n)$ is like the Lorenz template but there are m and n half twists in the respective bands. The template $L(0,1)$ has been used to model a suspension of Smale's horseshoe map [10, 31, 30].

Up to homeomorphism there are three types of Lorenz-like templates: $L(0,0), L(0,1)$ and $L(1,1)$. Bin Yu studied how the latter two can be realized as models for Smale flows on S^3 and how each can be realized in Smale flows on certain other 3-manifolds [53]. He later showed that every orientable 3-manifold without boundary supports a nonsingular Smale flow with three basic sets, an attracting closed orbit, a repelling closed orbit and a chaotic saddle set [54]. Elizabeth Haynes and Kamal Adhikari each studied the realization in flows of more complex templates [27, 1].

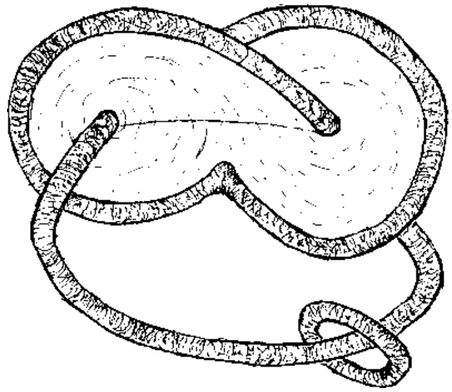

FIGURE 1.3: Smale flow with three basic sets

The templates $L(0, n)$ for $n \geq 0$ (n positive means the band has the same crossing type as the crossing above the branch line) were shown by Robert F. Williams to contain only prime knots [51]. He and Joan Birman in the aforementioned foundational paper [9] conjectured that there would be a bound on the number of prime factors on any template. This was shown to be false in [42]. Robert Ghrist went further and showed that many templates, including $L(0, -1)$, contain all knot and link types [16, 52]. Templates with this property are called *universal templates*.

Ghrist and Todd Young constructed a family of flows that have a bifurcation from being Morse-Smale flows to Smale flows containing all links [20].

For flows where the hyperbolic chain recurrent set is of dimension two or three, one can "split" along one or two, respectively, periodic orbits and create a one dimensional basic set whose periodic orbits are the same as in the original flow, with one or two exceptions [10, 19]. This provides a connection between Smale flows and Anosov flows.

1.2 Lorenz Knots

The subject of Lorenz knots and links has sparked quite a bit of interest [7, 22]. Birman and Ilya Kofman have characterized Lorenz links, actually a sight generalization, as T-links. A T-link is the closure of a braid of the form

$$(\sigma_1\sigma_2 \cdots \sigma_{r_1})^{s_1}(\sigma_1\sigma_2 \cdots \sigma_{r_2})^{s_2} \cdots (\sigma_1\sigma_2 \cdots \sigma_{r_k})^{s_k}$$

where $1 \leq r_1 \leq r_2 \leq \cdots \leq r_k$, $1 \leq s_i$ for $i = 1, \ldots, k$ and σ_j denotes a positive crossing of the j and $j+1$ strands of the braid. They make two interesting observations. Following Étienne Ghys they note that of the 1,701,936 prime knots with sixteen or fewer crossings only twenty are Lorenz knots. Yet, among the 201 simplest hyperbolic knots, at least 107 are Lorenz knots. They then pose the question, "Why are so many geometrically simple knots Lorenz knots?" [8]

Polynomial invariants are very important in knot theory and there are interesting things to be said about polynomial invariants and Lorenz links. In there work [8], Birman and Kofman noted: "The Jones polynomials of Lorenz links are very atypical, sparse with small nonzero coefficients, compared with other links of an equal crossing number." Pierre Dehornoy has showed that the zeroes of the Alexander polynomial of a Lorenz knot all lie in an annulus whose width depends on the genus and the braid index of the knot [12]. This is a deep area of on going work.

In one of the most surprising developments Ghys has discovered that Lorenz knots arise in a seemingly unrelated problem. He studied the geodesic flow on the unit tangent bundle of the quotient of the Poincare disk by $PSL(2,\mathbb{Z})$. The periodic orbits are exactly the Lorenz knots excluding the two boundary unknots [21]. See [23] for a visually stunning exposition.

1.3 Partially Hyperbolic Flows and Strange Attractors

Historically, the study of Lorenz knots, and the reason they are called Lorenz knots, started with attempts to understand the "strange attractor" arising in the flow for a 3×3 nonlinear ODE introduced by Edward Lorenz [35].

$$\begin{aligned}
\dot{x}(t) &= -10x + 10y \\
\dot{y}(t) &= 28x - y - xz \\
\dot{z}(t) &= -8z/3 + xy
\end{aligned}$$

Figure 1.4 shows three orbits with different initial conditions converging toward the attractor.

It was in this context that Williams introduced a *geometric Lorenz attractor* in the 1970's to study the periodic solution curves for the Lorenz equations [50]. See also [49, 26]; similar ideas were developed independently in the Soviet Union [2]. The model differs from $L(0,0)$ in two important ways. First, it has a saddle fixed point that cannot be isolated from all the periodic orbits which implies that it is not hyperbolic. Thus, it was dubbed a *strange attractor*. Second, the two unknotted periodic orbits in the boundary of $L(0,0)$ are not realized. See Figure 1.5.

Proving that the Lorenz equations actually have a Lorenz attractor is very

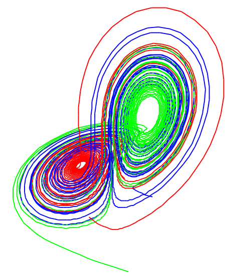

FIGURE 1.4: Lorenz Attractor

hard and took many years before this was established by Warwick Tucker in 1999 [46].

Notice the flow line coming from the left end point of the branch line (blue) next meets the branch line at α a little to the right of the left end point. Likewise the flow line coming from the right end point of the branch line next meets the branch line at β a little to the left of the right end point. It has been shown the small changes in the parameters of the Lorenz equations shift the location of α and β. Thus, many periodic orbits are created or destroyed in such a perturbation. Hence, the geometric Lorenz attractor is not structurally stable. Yet, the basic form of the attractor remains under small enough perturbations. This phenomena is called *robustness*. The details of this are beyond the scope of this article, but we refer the reader to the books [5] and [56].

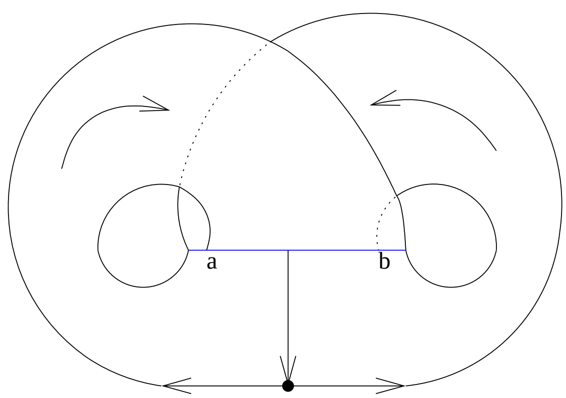

FIGURE 1.5: Geometric Lorenz Attractor

Clark Robinson constructed differential equations with an attractor modeled by Lorenz-like templates contained in $L(-1,-1)$ [38]. Here the knots are positive but are not necessarily positive braids. They need not be prime but it is conjectured that there is a bound of two prime factors [43]. Although positive knots need not be fibered, as is the case with positive braids, Ghazwan Alhashimi has shown that the knots in $L(-1,-1)$ are fibered [3].

Physicist Robert Gilmore has pioneered the empirical study of strange attractors. He has studied how the existence of certain knotted periodic orbits in flows force the existence of many others. Templates play a key role. With Marc Lefranc he has written a book on this work, *Topology in Chaos* [24]. See also the collection [25] in celebration of his 70$^{\text{th}}$ birthday.

1.4 Euler Flows

John Etnyre and Ghrist wrote a series of papers on flows arising in theoretical fluid dynamics. They prove that any steady solution to the C^ω Euler equations on a Riemannian S^3 must possess an unknotted periodic orbit. They employed the relationship between contact structures to show the existence of solutions whose flow lines trace out closed curves of all possible knot and link types by way of a universal template. [13, 14]

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Bibliography

[1] Kamal M. Adhikari and Michael C. Sullivan. Further Study of Simple Smale Flows Using Four Band Templates, *Top. Proc.* 50:21–37, 2017.

[2] V. S. Afraimovich, V. V. Bykov and L. P. Shilnikov. On the origin and structure of the Lorenz attractor, *Akademiia Nauk SSSR Doklady*, vol. 234, 336–339, May 11, 1977 and *Soviet Physics Doklady,* vol. 22, 253–255, May 1977.

[3] Ghazwan Alhashimi and Michael C. Sullivan. Knots in Robinson's Attractor (tentative title), in preparation.

[4] Daniel Asimov. Round handles and non-singular Morse-Smale flows, *Ann. of Math. (2)* 102(1):41–54, 1975.

[5] Vtor Araújo and Maria José Pacifico. *Three-Dimensional Flows*, A Series of Modern Surveys in Mathematics, Vol. 53, Springer, 2010.

[6] Ivar Bendixson. Sur les courbes définies par des équations différentielles, *Acta Mathematica*, 24(1):1–88, 1901.

[7] Joan S. Birman. The mathematics of Lorenz knots. *Topology and dynamics of chaos: In celebration of Robert Gilmore's 70th birthday* , 127–148, World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises, 84, World Sci. Publ., Hackensack, NJ, 2013.

[8] Joan S. Birman and Ilyn Kofman. A new twist on Lorenz links. *Journal of Topology*, 2(2):227–248, 2009.

[9] Joan S. Birman and Robert F. Williams. Knotted Periodic Orbits in Dynamical Systems I: Lorenz Knots. *Topology* **22** (1983), 47–82.

[10] Joan S. Birman and Robert F. Williams. Knotted periodic orbits in dynamical system. II. Knot holders for fibered knots. *Low-dimensional topology*, 1–60, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.

[11] J. Casasayas, J. Martinez Alfaro, and A. Nunes. Knots and links in integrable Hamiltonian systems. *Journal of Knot Theory and Its Ramifications*, 7(2):123–153, 1998.

[12] Pierre Dehornoy. On the zeroes of the Alexander polynomial of a Lorenz knot. *Ann. Inst. Fourier, Grenoble*, 65(2):509–548, 2015.

[13] John Etnyre and Robert W. Ghrist. Contact topology and hydrodynamics I: Beltrami fields and the Seifert Conjecture, *Nonlinearity*, 13(2):441–458, 2000.

[14] John Etnyre and Robert W. Ghrist. Contact topology and hydrodynamics III: knotted orbits, *Trans. Amer. Math. Soc.*, 352(12):5781–5794, 2000.

[15] John Franks. *Homology and dynamical systems.* CBMS Regional Conference Series in Mathematics, 49. Published for the Conference Board of the Mathematical Sciences, Washington, D.C.; by the American Mathematical Society, Providence, R.I., 1982. viii+120 pp. Reprinted, with corrections, 1989.

[16] Robert W. Ghrist. Branched two-manifolds supporting all links. *Topology*, 36(2):423–448, 1997.

[17] Robert W. Ghrist, and Philip Holmes. An ODE whose solutions contain all knots, *Intl. J. Bifurcation and Chaos*, 6(5):779–800, 1999.

[18] Robert W. Ghrist and R. Komendarczyk. Topological features of inviscid flows, in *Introduction to the Geometry and Topology of Fluid Flows*, NATO-ASI Series II, vol. 47, Kluwer Press, 183–202, 2002.

[19] Robert W, Ghrist, Philip Holmes and Micheal C. Sullivan. *Knots and links in Three-Dimensional Flows*, Lecture Notes in Mathematics, Vol. 1654, Springer-Verlag, Berlin, 1997.

[20] Robert W. Ghrist and Todd Young. From Morse-Smale to all links, *Nonlinearity*, 11(4):1111–1125, 1998.

[21] Étienne Ghys. Knots and dynamics. *International Congress of Mathematicians*, vol. I, 247–277, Eur. Math. Soc., Zrich, 2007.

[22] Étienne Ghys. The Lorenz attractor, a paradigm for chaos. *Chaos, Poincaré Seminar 2010*, 1–54, Part of the Progress in Mathematical Physics book series, volume 66, Birkhuser/Springer, Basel, 2013.

[23] Étienne Ghys and J. Leys. Lorenz and modular flows: a visual introduction, *Amer. Math. Soc. Feature Column,* November 2006. Available at http://www.ams.org/featurecolumn/archive/lorenz.html.

[24] Robert Gilmore and Marc Lefranc. *The Topology of Chaos: Alice in Stretch and Squeezeland*, 2nd Ed., Wiley, 2012.

[25] *Topology and dynamics of chaos: In Celebration Of Robert Gilmore's 70th Birthday* , 127–148, World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises, 84, World Sci. Publ., Hackensack, NJ, 2013.

[26] John Guckenheimer and Robert F. Williams. Structural stability of Lorenz attractors. Inst. Hautes Études Sci. Publ. Math. No. 50, 59–72, 1979.

[27] Elizabeth L. Haynes and Michael C. Sullivan. Simple Smale Flows with a Four Band Template, *Topology and It's Applications*, 177(1):23–33, 2014.

[28] Shuhei Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. of Math. (2)* 145(1):81–137, 1997. Errata: *Ann. of Math. (2)* 150(1):353–356, 1999.

[29] Morris W. Hirsch and Charles C. Pugh. Stable manifolds and hyperbolic sets. *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, 133–163, Amer. Math. Soc., Providence, R.I. 1970.

[30] Philip Holmes. Knotted periodic orbits in suspensions of Smale's horseshoe: Period multiplying and cabled knots, *Physica D: Nonlinear Phenomena*, 21(1):7–41, August 1986.

[31] Philip Holmes and Robert F. Williams. Knotted periodic orbits in suspensions of Smale's horseshoe: torus knots and bifurcation sequences. *Arch. Rational Mech. Anal.* 90(2): 115–194, 1985.

[32] Sen Hun. A proof of C^1 stability conjecture for three-dimensional flows. *Trans. Amer. Math. Soc.* 342(2):753–772, 1994.

[33] G. Kuperberg and K. Kuperberg. Generalized counterexamples to the Seifert conjecture, *Ann. of Math. (2)* 143(3):547–576, 1996.

[34] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge 1995.

[35] Edward Lorenz. Deterministic non periodic flow, *J. Atmosph. Sci.* 20, 130–141, 1963.

[36] John W. Morgan. Nonsingular Morse-Smale flows on 3-dimensional manifolds. *Topology* 18(1):41–53, 1979.

[37] J. Palis and S. Smale. Structural stability theorems, *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, Amer. Math. Soc., Providence, R.I., 223–231, 1970.

[38] Clark Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. *Nonlinearity*, 2(4):495–518, 1989.

[39] Paul A. Schweitzer. Counterexamples to the Seifert conjecture and opening closed leaves of foliations, *Ann. of Math. (2)*, 100(2):386–400, 1974.

[40] Herbert Seifert. Closed integral curves in 3-space and isotopic two-dimensional deformations, *Proc. Amer. Math. Soc.*, 1(3):287–302, 1950.

[41] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73(6):747–817, 1967.

[42] Michael C. Sullivan. Composite knots in the Figure-8 knot complement can have any number of prime factors, *Topology and Its Applications*, 55(3):261–272., 1994.

[43] Michael C. Sullivan. Positive Knots and Robison's attractor, *Journal of Knot Theory and its Ramifications*, 7(1):115–121, 1998.

[44] Michael C. Sullivan. Visually building Smale flows in S^3. *Topology and Its Applications*, 106(1):1–19, 2000.

[45] Michael C. Sullivan. Transverse Foliations to nonsingularMorse-Smale flows on the 3-sphere and Bott-integrable Hamiltonian systems, *Qual. Th. Dyn. Syst.* 7(2):411–415, January 2009.

[46] Warwick Tucker. The Lorenz attractor exists, *C. R. Acad. Sci. Paris Sér. I Math.*, 328(12):1197–1202, 1999.

[47] Masaaki Wada. Closed orbits of nonsingular Morse-Smale flows on S^3. *J. Math. Soc. Japan*, 41(3):405–413, 1989.

[48] Lan Wen. On the C^1 stability conjecture for flows, *J. Differential Equations*, 129(2):334–357, 1996.

[49] Robert F. Williams. The structure of Lorenz attractors, Inst. Hautes Études Sci. Publ. Math. No. 50, 73–99, 1979.

[50] Robert F. Williams. The structure of Lorenz attractors, *Turbulence Seminar* (Univ. Calif., Berkeley, Calif., 1976/1977), pp. 94–112. Lecture Notes in Math., Vol. 615, Springer, Berlin, 1977.

[51] Robert F. Williams. Lorenz knots are prime, *Ergodic Theory and Dynamical Systems*, 4(1):147-163, March 1984.

[52] Robert F. Williams. The universal templates of Ghrist, *Bulletin of the American Mathematical Society*, 35(2):145–156, April 1998.

[53] Bin Yu. Lorenz like Smale flows on three-manifolds, *Topology and Its Applications*, 156(15):2462–2469, 2009.

[54] Bin Yu. Every 3-manifold admits a structurally stable nonsingular flow with three basic sets. *Proc. Amer. Math. Soc.* 144(11):4949–4957, 2016.

[55] Bin Yu. Behavior 0 nonsingular Morse Smale flows on S^3. *Discrete Contin. Dyn. Syst.* 36(1):509–540, 2016.

[56] Elhadj Zeraoulia and Sprott Julien Clinton. *Robust Chaos and Its Applications*, World Scientific, 2012.