# Quantum Invariants of Templates

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**Abstract**

We define invariants for templates that appear in certain dynamical systems. Invariants are derived from bialgebras and Hopf algebras. Diagrammatic relations between projections of templates and the algebraic structures are used to define invariants. We also construct 3-manifolds via framed links associated to tamplate diagrams, so that any 3-manifold invariant can be used as a template invariant.

## 1 Introduction

For certain flows on 3-manifolds, two dimensional complexes called *templates* are used to model strange attractors and other invariant structures. The most popular example is the Lorenz template [**W**]. The topological types and embeddings of templates are useful in studying such dynamical systems. In particular we can distinguish dynamical systems by proving their templates are different. Furthermore, templates carry all the closed orbits of the system. These aspects have been studied by several authors [**BW-I**, **BW-II**, **FW**, **H1**, **H2**, **HM**, **W**, **Su1**, **Su2**, **Su3**, **Su4**, **G**]. Reference [**H2**] is expository. See also the book [**GHS**].

In this paper we define invariants of templates. A well known such invariant is the Parry-Sullivan invariant [**PS**]. Here we use new relations between algebras and diagrams that have been developed (see for example [**K&P**]) in knot theory after the discovery of new knot invariants by Jones [**Jones**]. Specifically, we use the projections of templates in the plane. Templates can be isotoped to singular surfaces that look like ribbons with branch lines. Three ribbons share a single branch line. Thus a template can be regarded as a thickened trivalent graph. The projections of trivalent graphs in a plane can be interpreted as compositions of homomorphisms between certain algebras as in the case of Jones type invariants. This way we produce invariants of templates that are elements of bialgebras and their generalizations. A remarkable feature of our invariants is a direct relation between certain algebraic structures and local moves that are characteristic for templates. In particular, the compatibility condition between multiplication and comultiplication of a bialgebra corresponds to a *splitting move* of templates along branch lines. Such relations between diagrams and algebras are characteristic in quantum invariants of knots and 3-manifolds.

Further we make a correspondence between templates and 3-manifolds via framed links. We construct a 3-manifold from a given template using framed links, and show that the manifold constructed has unique homeomorphism type associated to the given template. Specifically, we divide template diagrams into basic pieces, and assign a framed tangle to each piece. The tangles are glued together to form a framed link. In this way we obtain a 3-manifold from a given template diagram. We prove that two framed links thus obtained from two template diagrams of the equivalent template are related by Kirby moves, giving the homeomorphic 3-manifold. Thus this procedure assigns a unique 3-manifold to a given template. Hence any 3-manifold invariant can be used as a template invariant.

The paper is organized as follows. In the next section we recall templates and describe local moves for projections of templates. In Section 3 we define invariants of templates derived from bialgebras and give some examples. We study the relation between our invariants and the Parry-Sullivan invariants in Section 4. We discuss using braided Hopf algebras defined by Majid to generalize these ideas to get stronger invariants. In Section 5 we give framed link assignments to template diagrams.

## 2 Templates and Their Moves

In this section we quickly review templates that appear in certain dynamical systems and describe moves of projections of templates. See [**GHS**] for more details about templates.

A *template* $(T, \phi)$ is a smooth branched 2-manifold $T$, constructed from two types of charts, called *joining charts* and *splitting charts*, together with a semi-flow $\phi$. The joining chart (resp. splitting chart) is depicted in Figure 1 left (resp. right). A semi-flow is the same as flow except that one cannot back up uniquely, and in a template one cannot back up uniquely at a branch line. (The middle segment in the joining chart along which three sheets meet is called the *branch line*.) In Figure 1 the semi-flows are indicated by arrows on charts. In the joining chart (resp. splitting chart), there are two segments (resp. a single segment) at the top of the figure along which the semi-flow $\phi$ enters into the chart. These segments are called the *entry* segments (or lines) of the chart. Similarly, there is a segment (resp. three segments) in the joining chart (resp. splitting chart) at the bottom along which $\phi$ goes out of the chart. They are called the *exit segments*. The charts are sown together as follows. The exit segment of each joining chart is attached to an entry segment of a different chart. The left and right exit segments of each splitting chart are attached to entry segments of some other charts. The middle exit segments of the splitting charts are not attaching to anything. The flow exits the template through entry and exit segments. We also regard templates as charts with bands connecting them.

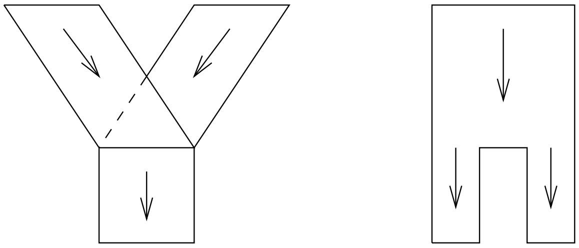

Figure 1: Charts of templates

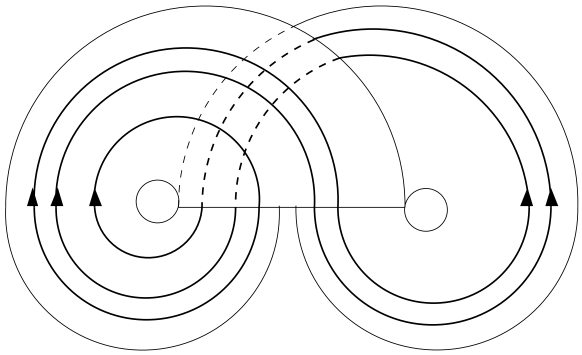

Figure 2: The Lorenz template

In Figure 2 we show the simplest example of a template, the Lorenz Template. It has been used to model the strange attractor believed to be associated with the Lorenz equations [**BW-I**]. The idea is that all the knots and links formed by orbits in a flow, obeying suitable hypotheses, are on a corresponding template. The trefoil shown on the Lorenz template is a periodic orbit of the semi-flow.

Obviously if we isotope a template in a 3-manifold, the knots and links on it are unaltered. Therefore we identify isotopic templates. However, two additional moves are permitted for equivalence among templates. They are depicted in Figures 3 and 4 respectively (called *switch move* and *splitting move* respectively). Notice that the splitting move alters the topological type of the template. This has made finding suitable invariants for templates difficult. One invariant is the Parry-Sullivan invariant [**PS**], but it does not incorporate half twists and it ignores how the template is embedded in 3-space. Another is a zeta function, but its definition is restricted to templates whose orbits have only one type of crossing, i.e. the closed orbits are all positive braids [**Su4**]. See also [**Su5**] for invariants of non-orientable templates. In this paper we define a new class of template invariants.

For the purpose of constructing invariants of templates, we use the idea of defining Jones-type invariants for knots and links diagrammatically. We refer to [**K&P**] for such approaches. We use

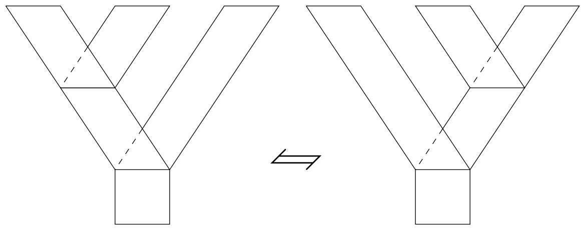

Figure 3: Template move I: switch move

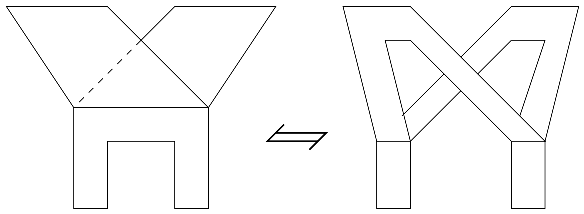

Figure 4: Template move II: splitting move

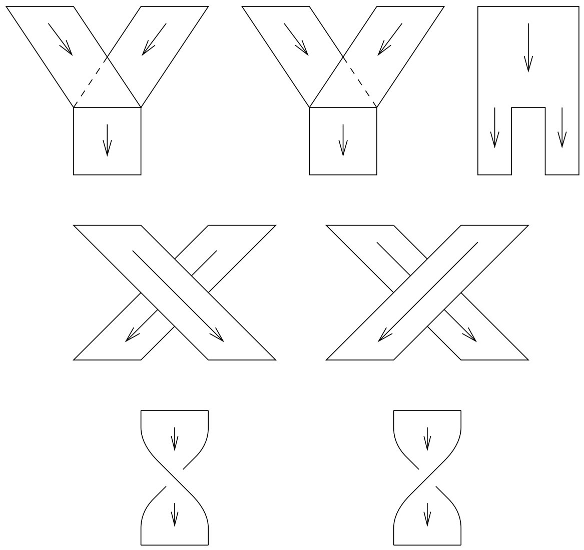

Figure 5: Building blocks of templates

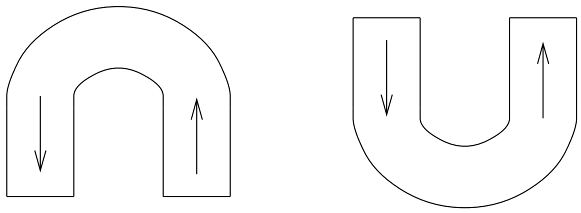

Figure 6: More building blocks of templates with height functions

projections of templates on the plane with crossing information.

Let $T$ be a template in 3-space. Choose a subspace $\mathbf{R}^2$ which is disjoint from $T$, and project $T$ into this plane. A template is regarded as a union of “skinny” bands, and each of such bands are locally homeomorphically projected into the plane (except at “twists,” see below). The image of $T$ is not one-to-one at a neighborhood of branch lines, and bands may cross each other on the plane, and there may be twists of bands. Local pictures of such cases are depicted in Figure 5. At branch lines and crossings, we indicate which band lies above the other band by breaking the under path (or by representing them by dotted lines). In Figure 5 top, images of a projection near two types of branch lines and an exit line are depicted. In the middle, two types of crossings of bands are depicted. In the bottom, two types of half twists are depicted. Such projections of templates together with crossing informations specified are called *template diagrams*. Thus a template diagram consists of embedded bands in the plane and building blocks depicted in Figure 5.

We then fix a height function on the plane on which $T$ is projected. With respect to the height function, we have maximal and minimal points of bands. Thus $T$ consists of local diagrams depicted in Figures 5 and 6. We include mirror images with respect to horizontal lines of those depicted in Figure 6 as local diagrams as well. The height function is the vertical direction in these figures. This is essentially the same as diagrammatic approaches of Jones-type invariants (see again [**K&P**]) and the only difference is that we have new charts (or local diagrams, or building blocks): branch lines and exit lines (and half twists of bands).

We arrange the diagrams so that the semi-flows near joining and splitting charts go down with respect to the height function, as depicted in Figure 5. Note also that there are two types of projections near a branch line. However Figure 7 shows that the combination of a crossing and one of two types of projections describes the other type.

**2.1 Remark.** We remark here about the orientations of templates. Each band has local orientations. A template is orientable if there are choices of orientations of bands so that they are consistent under identifications of charts. If a given template is oriented then there is a template diagram without half twist building blocks. This is because full twists of bands can be changed to a small kink of bands as depicted in Figure 8. (This convention is often used for the framed link calculus of 3-manifolds.)

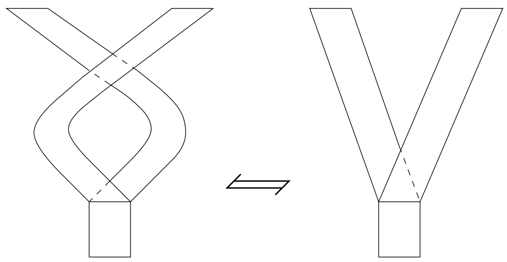

Figure 7: Switching branch lines

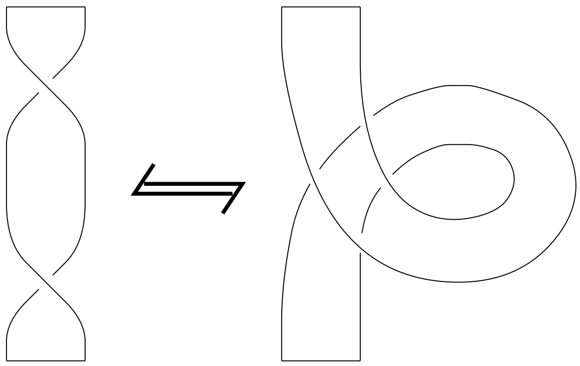

Figure 8: Twists of bands

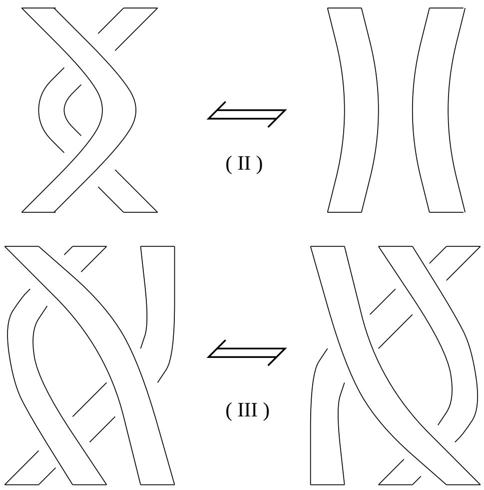


Figure 9: Reidemeister moves

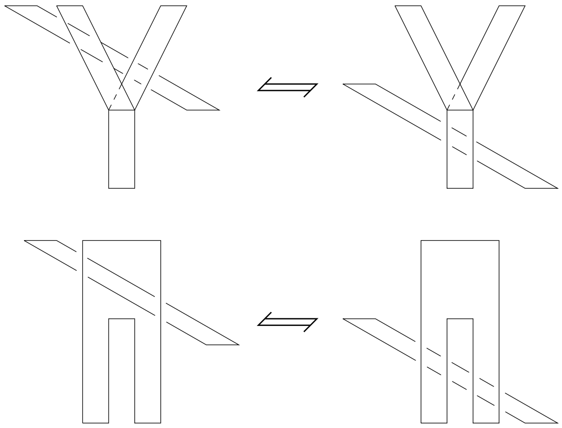

Figure 10: Reidemeister moves involving branch and exit lines

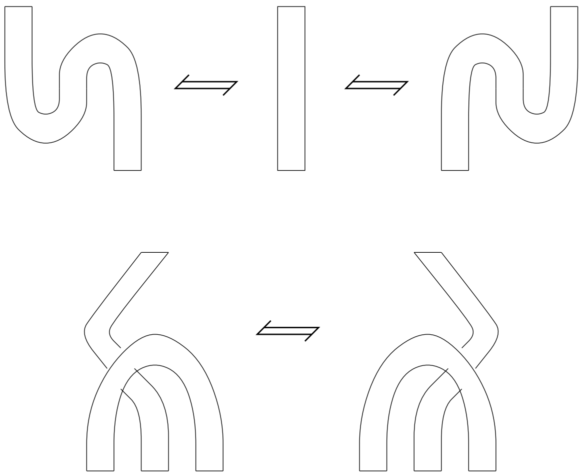

Figure 11: Reidemeister moves in the presence of a height function

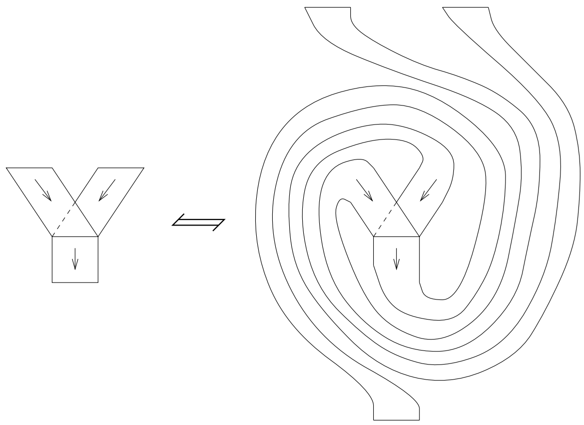

Figure 12: Rotations

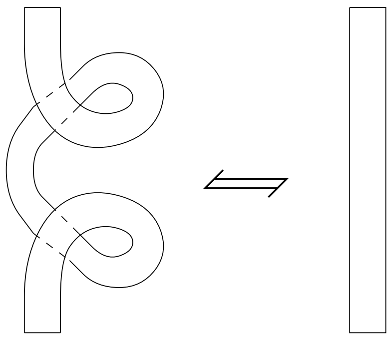

Figure 13: The spherical move of ribbons

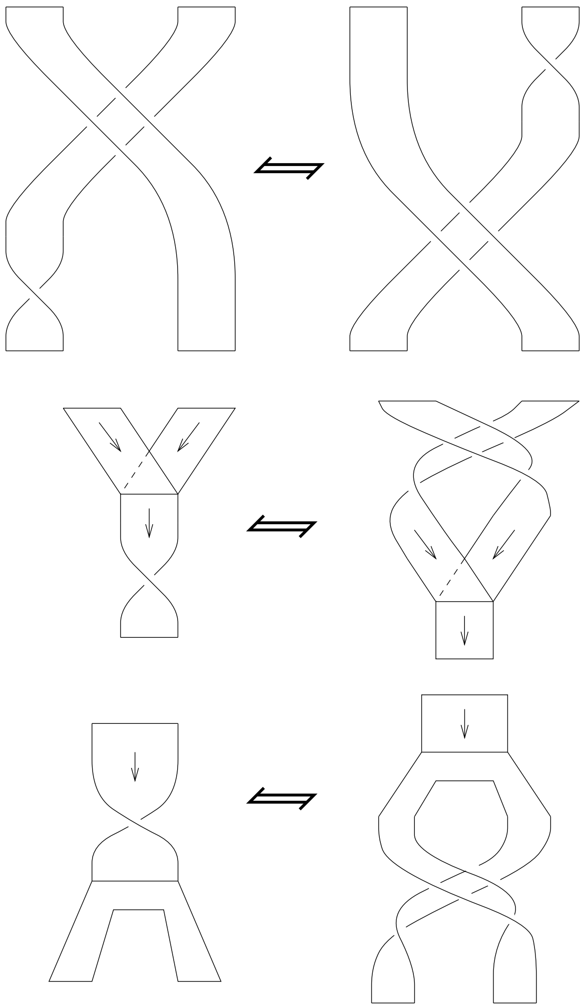

Figure 14: Reidemeister moves for half twists

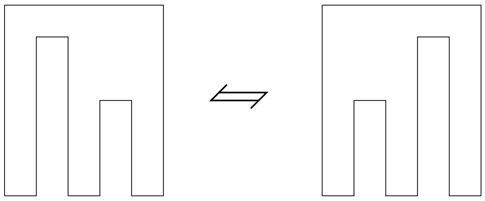

Figure 15: Template move I'

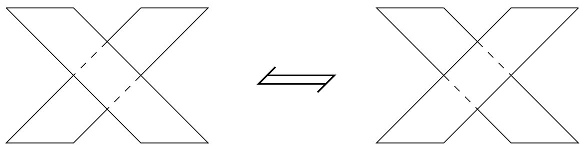

Figure 16: Pass move (crossing change)

Next we recall the Reidemeister moves for knots and links and extend these moves to templates. Figure 9 depicts the Reidemeister moves (type II and III) for bands. Figure 11 and 12 (and the similar rotation moves for the splitting chart) are additional moves for bands when a height function is fixed. Figure 10 depicts additional moves for joining and splitting charts. Another move depicted in Figure 13 is necessary for bands (this move is called a *spherical move*). We also include the mirror images of these moves depicted with respect to the vertical or horizontal lines in our list of moves, whenever the semi-flows of charts match the height function.

There are also moves involving half twists, as depicted in Figures 8 and 14. We call these moves *Reidemeister moves* of templates.

We have other moves depicted in Figures 3 and 4, as well as the move depicted in Figure 15. These are called *template moves.*

If we regard branch and exit lines as trivalent rigid vertices and half twists as 2-valent rigid vertices, we can apply Reidemeister move theory of graphs with rigid vertices [**Ka**, **JM**] to obtain the set of Reidemeister moves of templates (with an additional move in Figure 8). At a branch line, two bands go down into the branch line and one band go out, so that it has the symmetry with respect to one full rotation of the plane and a half twist with respect to the vertical line. The former corresponds to the move depicted in Figure 12 and the latter corresponds to the move in the middle of Figure 14. Thus the combination of Reidemeister moves for graphs with rigid vertices and template moves give

**2.2 Lemma.** Two template diagrams represent the equivalent template if and only if they are related by finite number of Reidemeister moves and template moves and isotopies of the projected templates on the plane.

We also define weaker equivalent relation: two template diagrams are called *pass equivalent* if they are related by a finite sequence of Reidemeister moves of templates, template moves, and *pass moves* depicted in Figure 16.

# 3 Invariants Derived from Bialgebras and Hopf algebras

**3.1 Definitions.** A *bialgebra* over a field $k$ is $(A, m, \eta, \Delta, \epsilon)$ such that

(1) $(A, m, \eta)$ is an algebra where $m : A \otimes A \to A$ is the multiplication and $\eta : k \to A$ is the unit. (i.e., they are $k$-linear maps such that $m(1 \otimes m) = m(m \otimes 1)$, $m(1 \otimes \eta) = 1 = m(\eta \otimes 1)$ where 1 denotes the identity map in this case).

(2) $\Delta : A \to A \otimes A$ is an algebra homomorphism (called the *comultiplication*) satisfying $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$,

(3) $\epsilon : A \to k$ is an algebra homomorphism called the *counit*, satisfying $(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta$.

We refer to [**Abe**] for these definitions.

Here we explain the second condition in the above definition. One of the conditions that the comultiplication is an algebra homomorphism is also called the compatibility condition between the multiplication and the comultiplication and can be written by $\Delta(ab) = \Delta(a)\Delta(b)$ for any $a, b \in A$. (The other conditions are $\Delta \circ \eta = \eta \otimes \eta$, with $k$ identified with $k \otimes k$, $\epsilon \circ \mu = \epsilon \otimes \epsilon$, and $\epsilon \circ \eta = 1_k$.) The LHS of the equality is $\Delta \circ m(a \otimes b)$. For elements $c_1 \otimes c_2, d_1 \otimes d_2 \in A \otimes A$, the multiplication is defined by $(c_1 \otimes c_2) \cdot (d_1 \otimes d_2) = c_1 d_1 \otimes c_2 d_2$. Thus the multiplication on $A \otimes A$ is in fact the map

$$(m \otimes m) \circ P_{23} : (A \otimes A) \otimes (A \otimes A) \to A \otimes A$$

where $P_{23}$ denotes the permutation map $P_{23}(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_1 \otimes x_3 \otimes x_2 \otimes x_4$. Therefore the RHS of the compatibility condition is written as $(m \otimes m) \circ P_{23} \circ (\Delta \otimes \Delta)(a \otimes b)$. In other words the condition in terms of maps is

$$\Delta \circ m = (m \otimes m) \circ P_{23} \circ (\Delta \otimes \Delta).$$

For the image of the comultiplication the notation $\Delta(a) = a_1 \otimes a_2$ is often used. Generally the image under the comutiplication is a sum of such tensors but this notation is commonly used for shorthand. In this notation the LHS and the RHS of the compatibility condition are written as $\Delta(ab) = (ab)_1 \otimes (ab)_2$ and $\Delta(a)\Delta(b) = (a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$ respectively so that the condition becomes $(ab)_1 \otimes (ab)_2 = a_1 b_1 \otimes a_2 b_2$.

**3.2 Example.** Let $G$ be a finite group and $kG$ be the group algebra over a field $k$. As a set $kG$ consists of formal linear sums of elements of $G$ with coefficients in $k$: $kG = \{\sum_{g \in G} \lambda_g g\}$. The addition and the scalar product is defined by

$$\begin{aligned} \sum \lambda_g g + \sum \mu_g g &= \sum (\lambda_g + \mu_g) g \\ c \cdot \sum \lambda_g g &= \sum c \lambda_g g \end{aligned}$$

respectively so that $kG$ is a vector space spanned by the elements of $G$.

The multiplication is defined by $(\sum \lambda_g g) \cdot (\sum \mu_h h) = \sum (\lambda_g \mu_h) gh$. The comultiplication is defined for the basis by $\Delta(g) = g \otimes g$ (and by extending this linearly). One computes $\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h)$ for any elements $g, h \in G$. Thus the compatibility condition is satisfied. The unit is $k \ni a \mapsto a \cdot 1 \in kG$ where $1 \in G$ is the identity element, and the counit is defined by $\epsilon(g) = 1$ for any $g \in G$.

**3.3 Example.** In fact this example is something between bialgebras and braided bialgebras that are defined by Majid [**Maj1**], since it is a "$(\mathbf{Z}/2)$-graded" bialgebra. (The map $P$ which appear in the conpatibility condition is replaced by the map $R$ in this example as described below, which is not the permutation map, but still satisfies $R^2 = 1$. See Remark 4.4 for discussions on braided bialgebras and for more details on this point.)

Let $V$ be a vector space over a field $k$ of a finite dimension $d$. Let $A = \wedge V$ be the exterior algebra of $V$. Thus $\wedge V$ is isomorphic to $\oplus_{p=0}^{d} \wedge^p V$ where $\wedge^p V$ consists of elements $\{v_1 \wedge \cdots \wedge v_p | v_i \in V\}$ ($p$ is called the degree) where $v \wedge v = 0$ for any $v \in V$. Any linear map $f : V \to W$ between vector spaces is extended to $\wedge f : \wedge V \to \wedge W$ by defining the map on the basis elements by $\wedge f(v_1 \wedge \cdots \wedge v_p) = f(v_1) \wedge \cdots \wedge f(v_p)$.

The multiplication is defined by $\wedge V \otimes \wedge V \ni a \otimes b \mapsto a \wedge b \in \wedge V$.

The comultiplication is defined by extending the map $V \ni a \mapsto (a \otimes 1 + 1 \otimes a)$ to the exterior product.

The unit is the inclusion $\eta : k \to k = \wedge^0 V \subset \wedge V$.

The counit is the map induced from $V \ni a \mapsto 0 \in k$.

The map $R : \wedge V \otimes \wedge V \to \wedge V \otimes \wedge V$ which serves the role of the permutation is defined by

$$R(a \otimes b) = (-1)^{|a||b|} b \otimes a$$

where $|x|$ denotes the degree of $x \in \wedge V$. This extends to tensor products and for the dual spaces. Note that $R = \wedge P$ where $P(x, y) = (y, x)$ is the permutation map $P : V \oplus V \to V \oplus V$ and $\wedge P$ is the induced map on the exterior product. This $R$ satisfies the compatibility condition

$$\Delta \circ m = (m \otimes m) \circ R_{23} \circ (\Delta \otimes \Delta).$$

We now define invariants of templates using bialgebras as follows. Let $A$ be a bialgebra over a field $k$. We assign $A$ (resp. $A^*$, the dual of $A$, i.e., $A^* = \mathrm{H}om_k(A, k)$) to a band with the downward (resp. upward) orientation.

To each building block we assign a map. Figure 17 illustrates the correspondence. In the diagrams, the maps go from bottom to top of the sheet as depicted. We choose specific maps as follows. Let $A$ be a bialgebra. At a branch line assign the comultiplication of $A$, $\Delta : A \to A \otimes A$. At an exit line assign the multiplication $m : A \otimes A \to A$. At a local minimum/maximum assign the evaluation/co-evaluation maps respectively as follows. The evaluation map $\psi : A \otimes A^* \to k$ is defined by $\psi(a, b^*) = b^*(a) \in k$, the value of $a$ evaluated by a dual element $b^* \in A^*$. The co-evaluation map is defined by $\psi(1) = \sum_{i=1}^{n} e_i \otimes e_i^*$ where $\{e_i\}$ is a basis of $A$. At a crossing point, choose the permutation map $P(a, b) = (b, a)$. Note that $a$ and $b$ can be elements in $A$ or $A^*$ and the type of crossing (positive or negative) does not matter when we use the permutation map. At a half twist, assign a map $S : A \to A$, the condition for which is stated in the following theorem.

Thus a projection $T$ of a template represents as a composition of these maps a linear homomorphism $T_* : k \to k$ which is a multiplication by a scalar $B(T) \in k$, $T_*(x) = B(T) \cdot x$. This element $B(T)$ is our invariant.

**3.4 Theorem.** The element $B(T) \in k$ gives an invariant of templates if the map $S$ assigned to half twists satisfies the following conditions: $S^2 = 1$, $S(ab) = S(b)S(a)$ for any $a$, $b \in H$, and

$$\Delta \circ S = P \circ (S \otimes S) \circ \Delta,$$

where $P$ is the permutation.

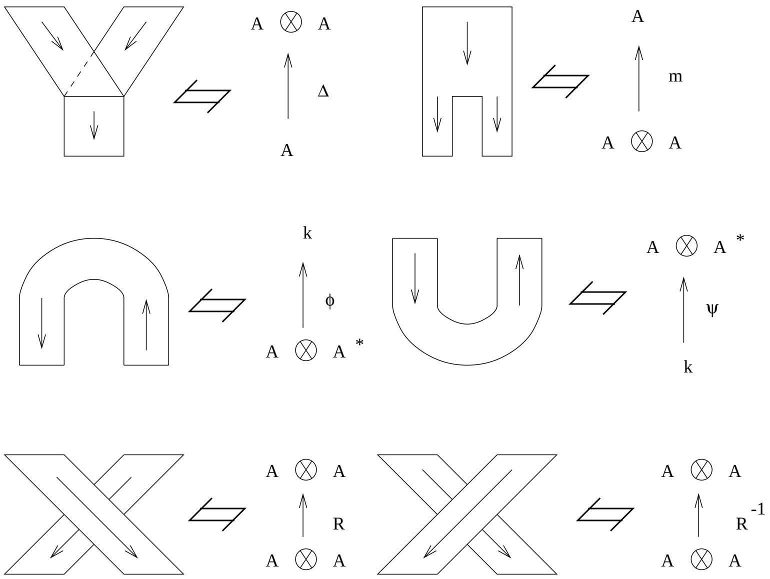


Figure 17: Maps corresponding to Building blocks

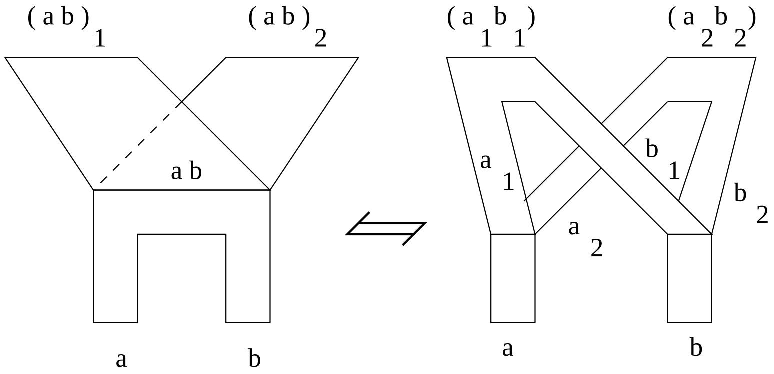


Figure 18: The splitting move and the compatibility condition

*Proof.* We prove that the above defined $B(T)$ is invariant under local moves discussed in the previous section.

Since we have chosen (co-)evaluation maps and permutation maps for local max/min and crossing points respectively, it is routine to check the invariance under the Reidemeister moves. Thus it remains to prove the invariance under the template moves.

The invariance under the switch move follows from the coassociativity of the comultiplication: $\Delta(\Delta \otimes 1) = \Delta(1 \otimes \Delta)$.

The change of level of exit lines follows similarly from the associativity of the comultiplication.

The invariance under the splitting move follows from the compatibility between multiplication and comultiplication: $\Delta(ab) = \Delta(a)\Delta(b)$. This can be seen on the diagrams as follows. After the splitting move, the segment labeled $a$ (resp. $b$) is divided into two segments labeled $a_1$ and $a_2$ (resp. $b_1$ and $b_2$). Regard $a$ and $b$ as elements in $A$, so that $\Delta(a) = a_1 \otimes a_2$ (resp. $\Delta(b) = b_1 \otimes b_2$) under the correspondence between separation lines and comultiplications. Then on the one hand $\Delta(a)\Delta(b) = a_1b_1 \otimes a_2b_2$ by definition (in the bialgebra $A$), in terms of diagrams two branch lines join segments labeled $a_1$ and $b_1$ (resp. $a_2$ and $b_2$). See Figure 18 for this correspondence.

Finally we check the local moves involving half twists. The top picture in Figure 14 is easily checked. The equality corresponding to the middle picture is $\Delta \circ S = P \circ (S \otimes S) \circ \Delta$ where $P$ is the permutation, which is satisfied by the assumption. The bottom picture gives the equality $S(ab) = S(b)S(a)$ for any $a$, $b \in H$, which is also satisfied by the assumption. Two half twists constitutes a full twist, which corresponds to the identity map by the assumption $S^2 = 1$. This gives the invariance under the move depicted in Figure 8. □

We remark here that the relation between templates and bialgebra structures has also been discovered by Hillman [**Hill**] independently.

**3.5 Definition.** A bialgebra $A$ is *commutative* if $m(a \otimes b) = m(b \otimes a)$ for any $a, b \in A$, *cocommutative* if $\Delta = P \circ \Delta$ where $P$ is the permutation.

**3.6 Corollary.** If $A$ is commutative and cocommutative, by assigning $S = 1$ (the identity map) to half twists, $B(T)$ gives an invariant of templates up to pass equivalence.

**3.7 Definition.** A *Hopf algebra* is a *bialgebra* over a field $k$ $(A, m, \eta, \Delta, \epsilon)$ with a mapping $S: A \to A$ called the *antipode* satisfying

$$m \circ (S \otimes 1) \circ \Delta = \eta \circ \epsilon = m \circ (1 \otimes S) \circ \Delta.$$

**3.8 Corollary.** For a Hopf algebra with $S^2 = 1$ for the antipode $S$, by assigning $S$ to half twists, the element $B(T) \in k$ defines an invariant of templates up to the pass equivalence.

*Proof.* We check the local moves involving half twists. The equality $\Delta \circ S = P \circ (S \otimes S) \circ \Delta$, where $P$ is the permutation, is satisfied for an antipode ([**Abe**], p63). The equality $S(ab) = S(b)S(a)$ for any $a$, $b \in H$, is also satisfied ([**Abe**], p62). We note that the condition $S^2 = 1$ is satisfied if the Hopf algebra is commutative or cocommutative ([**Abe**] p63). □

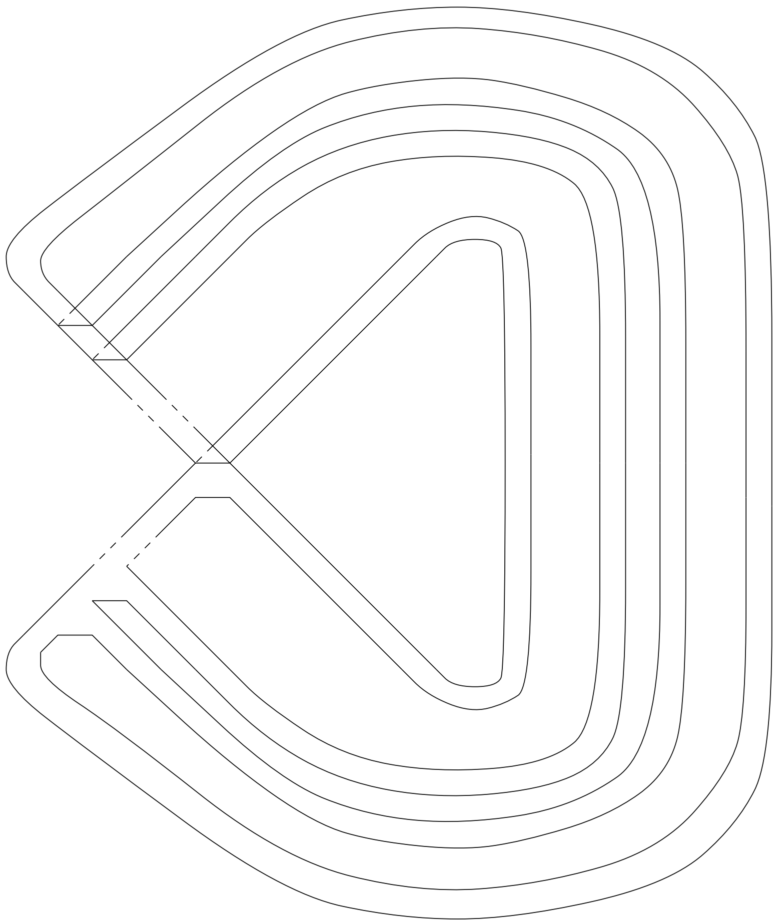

Figure 19: Some examples of templates

The use of Hopf algebras here is related to 3-manifold invariants defined in [**ChFS**, **Kup**]. In particular, the cone move defined in [**ChFS**] for triangulations of 3-manifolds is related to the splitting move of templates.

**3.9 Example.** Let $C_n$ be the cyclic group of order $n$ for a positive integer $n$ and $A = kC_n$ be the bialgebra constructed in Example 3.2 for the group algebra over a field $k$.

Let $T_s$ be the template depicted in Figure 19 where there are $s$ branch lines for a positive integer $s$.

We compute the invariant $B_n(T_s)$ for this particular bialgebra. In this case the bialgebra is commutative and cocommutative, so that we use the Cor. 3.6 by setting $S = 1$.

Let $g \in C_n$ be a generator so that $A = kC_n$ has a basis $\{e = g^0, g, g^2, \cdots, g^{n-1}\}$.

For an exit line and a branch line we use the multiplication and comultiplication defined in general for group algebras in Example 3.2.

The Figure 19 is of the form of the closure of a diagram which joins $(s+1)$ bands by $s$ branch lines into one band and separates it again by $s$ exit lines into $(s+1)$ bands. Reading it from bottom to top, this diagram defines a map $f : A^{\otimes(s+1)} \to A^{\otimes(s+1)}$ which is in fact equal to $f = \Delta^s \circ m^s$. Here $\Delta^s = (\Delta \otimes 1^s)(\Delta \otimes 1^{s-1}) \cdots \Delta$ where $1^s$ denotes the identity map on $A^{\otimes s}$, and similar for $m^s$.

We choose the basis $g^{u_1} \otimes g^{u_2} \otimes \cdots \otimes g^{u_{s+1}}$ of $A^{\otimes(s+1)}$ where $0 \leq u_j < n$, $j = 1, \cdots, s+1$. Let $u = \sum_{j=1}^{s+1} u_j \pmod{n}$. (Thus $u$ is an integer $0 \leq u < n$.) Then the image under $f$ of the basis are $f(g^{u_1} \otimes g^{u_2} \otimes \cdots \otimes g^{u_{s+1}}) = \Delta^s(g^u) = g^u \otimes \cdots \otimes g^u$.

By the definition of the pairing and the copairing taking the closure of this diagram correspond to taking the trace of $f$ (see [**K&P**]). Thus $B_n(T) = \#\{u \in \mathbf{Z}/n : u(s+1) = u\} = \#\{u : us = 0\}$.

Thus we can conclude that $T_s = T_{s'}$ iff $s = s'$ using these invariants.

# 4 A relation to Parry-Sullivan invariants

An alternate definition of the Jones polynomial and generalizations is given by taking certain traces of linear representations of braid groups. In this section we give such an interpretation for the Parry-Sullivan invariant [**PS**]. The linear maps whose trace we compute come from bialgebras we discussed in the preceding section. First we review the Parry-Sullivan invariant.

Two flows are said to be *topologically equivalent* if there is a homeomorphism between them that takes orbits to orbits, preserving orientation. The Parry-Sullivan invariant is an invariant of flows under topological equivalence. Although templates are semi-flows, there is no problem in applying the Parry-Sullivan invariant to them.

Before defining the Parry-Sullivan invariant, we shall give a brief review of Markov partitions, but refer the reader to [**Shub**, Chapter 10] for details. In our context a Markov partition is a finite, disjoint collection of line segments transverse to a template's semi-flow. Each orbit that does not exit the template must pass through some element of the Markov partition in forward time. The flow induces a first return map on the partition elements. This map has the property that each Markov partition element is mapped onto any partition element that its image meets. The thickened line segments in Figure 20 represent the elements of a Markov partition for the template shown.

Given a Markov partition for a template we construct an incidence matrix, $A = [a_{ij}]$, as follows. Number the Markov partition elements, 1 through $n$. Let $a_{ij}$ be the number of bands going from the $i$-th element to the $j$-th element. Thus, the incidence matrix for partition in Figure 20 is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The Parry-Sullivan invariant is given by $\det(I - A)$. It is independent of the choice of the Markov partition.

We give an alternate description of the Parry-Sullivan invariant using the braided form of templates as follows.

**4.1 Definition.** A template $T$ is *braided* if there exists an unknotted circle $X$ (called the *braid axis*) in $S^3$ such that (1) $X$ is disjoint from $T$, (2) fix an orientation of the circle $S^1$ in $S^3 \backslash X \cong S^1 \times D$

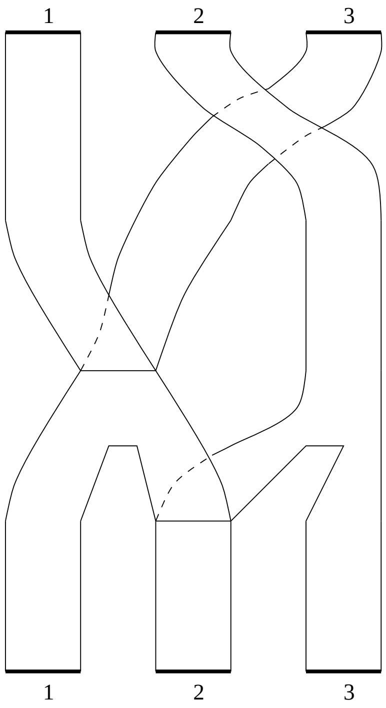


Figure 20: A markov partition of a template

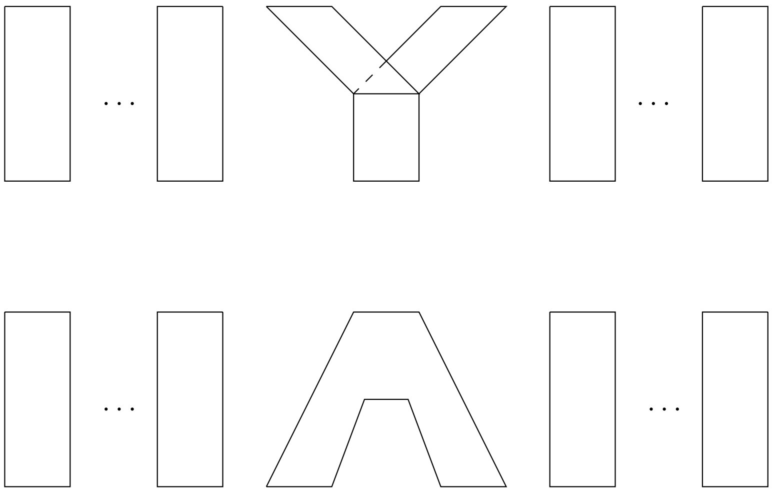

Figure 21: Generators of braided templates

where $D$ is an open disk, then the semi-flow of $T$ matches the orientation of $S^1$. Specifically, consider the projection $p : S^3 \setminus X \cong S^1 \times D \to S^1$. Then the second condition above means that the direction of the semi-flow of $T$ always matches the orientation of $S^1$ via the projection $p$.

**4.2 Lemma [FW].** Any template is isotopic to a braided one.

In terms of template diagrams, this implies that any template can be isotoped to the following form (see Figure 20 for an example).

- All the exit lines, branch lines, and crossings lie in the interior of the rectangular region $[0,1] \times [0,1]$.
- Semi-flows restricted to $T \cap [0,1] \times [0,1]$ are always downward.
- The portion of $T$ outside of this rectangular region is nested trivial bands.

We call the portion $T \cap [0,1] \times [0,1]$ the *braided template tangle* and attaching the trivial bands to recover $T$ is called taking the *closure*. This is similar to knots and links (see for example [**B**]).

By slicing a braided template tangle $T$ we can express $T$ as the juxtaposition of $T_1, \cdots, T_n$ where each of $T_i, i = 1, \cdots, n$ consists of parallel bands and one of the template charts (see Figure 21), or ordinary braid generators for bands (*i.e.,* parallel bands and a crossing between two bands).

To express the Parry-Sullivan invariant as an invariant we discussed in the preceding section using trace, recall the Example 3.3. Let $V$ be a one-dimensional vector space and assign to the above generators the linear maps constructed by exterior algebra defined in Example 3.3. More specifically, to the generator in Figure 21 top, assign the map $I \otimes \cdots \otimes I \otimes \Delta \otimes I \otimes \cdots \otimes I$, and to the bottom figure assign $I \otimes \cdots \otimes I \otimes m \otimes I \otimes \cdots \otimes I$, where $m$ (resp. $\Delta$) denotes the multiplication (resp. comultiplication) and the identity $I$ corresponds to trivial bands to the left and right of the building blocks. Let $T_\otimes$ denote the composition of such linear maps assigned to a given braided template tangle $T$.

**4.3 Theorem.** The Parry-Sullivan invariant $PS(\hat{T})$ of the closure of a braided template tangle $T$ is equal to the trace $Tr(\mathcal{S}T_\otimes)$ where $\mathcal{S}$ is defined by $\mathcal{S}|_{\wedge^k W}(x) = (-1)^k x$ for any template $T$.

*Proof* . Choose a Markov partition of a braided template so that the segments are the top and bottom segments of each generator of the braided template.

Then the matrix for each generator is as follows.

- $I \oplus \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus I$ for a branch line,
- $I \oplus [1,1] \oplus I$ for an exit line, and
- $I \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I$ for a crossing, where $I$ denotes the identity matrices of some size.

Each $T_i$ is one of these generators, and we assign the above matrices to $T_i$ s. Let $M_i$ be the assigned matrices. Assign the product $M = M_1 \cdots M_n$ to the template tangle $T$. Then the Parry-Sullivan invariant is computed by $\det(I - M)$.

Let $V$ be a vector space over $\mathbf{C}$ of dimension one generated by $\xi$. Then the above matrices are regarded as the following linear maps.

- A branch line : $\beta : V \ni \xi \mapsto (\xi, \xi) \in V \oplus V$.
- An exit line : $\eta : V \oplus V \to V$ where $\eta(\xi, 0) = \xi$, $\eta(0, \xi) = \xi$.
- A crossing : $\gamma : V \oplus V \to V \oplus V$ where $\gamma(\xi, 0) = (0, \xi)$, $\gamma(0, \xi) = (\xi, 0)$.

These maps induce maps on the exterior algebra $\wedge V$ as follows, respectively. (Note that $1 \in \wedge^0 V \subset \wedge V$ is always sent to $1 \otimes \cdots \otimes 1 \in (\wedge V)^{\otimes n}$.)

- A branch line : $\wedge\beta : \wedge V \to \wedge V \otimes \wedge V$ where $\wedge\beta(\xi) = (\xi \otimes 1 + 1 \otimes \xi)$.
- An exit line : $\wedge\eta : \wedge V \otimes \wedge V \to \wedge V$ where $\wedge\eta(a \otimes b) = a \wedge b$
- A crossing : $\wedge P : \wedge V \otimes \wedge V \to \wedge V \otimes \wedge V$ where $\wedge P(x \otimes y) = (-1)^{|x||y|} y \otimes x$.

Let us prove this correspondence. Let $\xi_1 = (\xi, 0)$, $\xi_2 = (0, \xi) \in V \oplus V$. The space $\wedge(V \oplus V)$ is generated by $1 \in \wedge^0 V$, $\xi_1, \xi_2 \in \wedge^1 V$, and $\xi_1 \wedge \xi_2 \in \wedge^2 V$. The isomorphism $\Phi : \wedge(V \oplus V) \to \wedge V \otimes \wedge V$ is given by $\Phi(1) = 1 \otimes 1$, $\Phi(\xi_1) = \xi \otimes 1$, $\Phi(\xi_2) = 1 \otimes \xi$, and $\Phi(\xi_1 \wedge \xi_2) = \xi \otimes \xi$. By this isomorphism $\Phi$ the above correspondence follows.

Let $T_0$ be the template tangle with $n$ bands on the top and bottom whose closure is a given braided tangle diagram $T$. Then $T_0$ represents a map $T_\oplus : \oplus_n V \to \oplus_n V$ as a composition of maps corresponding to building blocks via correspondence given above. It induces an endomorphism $T_\otimes$ on $\wedge(\oplus_n V) \cong (\wedge V)^{\otimes n}$ via the correspondence listed above.

Thus the Theorem follows from the fact $\det(T_\oplus - I) = Tr(\mathcal{S}T_\otimes)$ where $\mathcal{S}$ is defined by $\mathcal{S}|_{\wedge^k W}(x) = (-1)^k x$ [**K&P**, **KS**]. $\square$

**4.4 Remark.** We constructed template invariants using bialgebras. For bialgebras, we do not expect to derive as strong invariants as Jones type invariants for knots and links since the pass equivalence does not detect the crossing information. However we have to require invariance under the pass move since in the compatibility condition we only have the permutaion map:

$$\Delta \circ m(a \otimes b) = (m \otimes m) \circ P_{23} \circ (\Delta \otimes \Delta)(a \otimes b)$$

where $P_{23}(w \otimes x \otimes y \otimes z) = w \otimes y \otimes x \otimes z$, and the permutation map does not distinguish positive and negative crossings.

Thus to get stronger invariants we need new algebraic structures with a generalized compatibility condition

$$\Delta \circ m(a \otimes b) = (m \otimes m) \circ R_{23} \circ (\Delta \otimes \Delta)(a \otimes b)$$

where $R$ denotes a *braiding* homomorphism on $A \otimes A$ without the condition $R^2 = 1$ but satisfying the Reidemeister moves. Such an algebraic structure called a *braided Hopf algebra* or *braided groups* has been defined and studied by S. Majid [**Maj1**, **Maj2**, **Maj3**]. Such algebraic objects would define stronger invariants for templates, as they capture the braiding and is not invariant under pass moves.

Majid [**Maj1**] constructed braided Hopf algebras from quantum groups as follows. We refer the reader to [**Maj1**] for more details. Let $(H, \Delta, \epsilon, S, R)$ be a quantum group (a quasitriangular Hopf algebra). (Here $\Delta$ is a comultiplication, $\epsilon$ is a counit, $S$ is an antipode, and $R$ is the universal $R$-matrix.) Let $\mathcal{C} = Rep(H)$ be the category of representations of the quantum group. Then $H$ gives rise to a braided Hopf algebra. As an algebra set $A = H$. If $\Delta(x) = x_1 \otimes x_2$ in the original comultiplication of $H$, then the new comultiplication for $A$ is defined by $\Delta_A(x) = x_1 R_2 S_A(R'_2) \otimes R'_1 x_2 R_1$ where $S_A x = R_2 R'_2 S(R_1 x S R'_1)$ is a new antipode ($S$ is the antipode of $H$) and $R' = R'_1 \otimes R'_2$ is another copy of the universal $R$-matrix.

It is expected that the template invariants be computed explicitly for such braided Hopf algebras defined from quantum groups at roots of unity. Further studies of invariants constructed in this paper and generalizations using Majid's braided groups are expected.

## 5 Framed links associated to template diagrams

In this section we associate a framed link to a given template diagram and prove that two diagrams representing the equivalent template give framed links representing the homeomorphic 3-manifold. We take the common convention of using the projection direction as the framing of framed links (this is the same as taking the parallel strings on the plane for framing). We also use the diagrammatic convention of using dots to represent small kinks as depicted in Figure 22 following [**MP**].

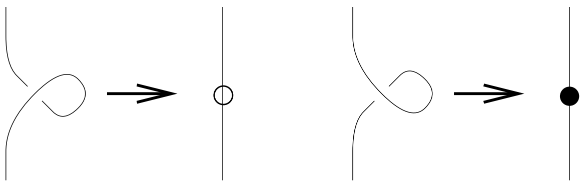

Figure 22: Dots convention for small kinks by Matveev-Polyak

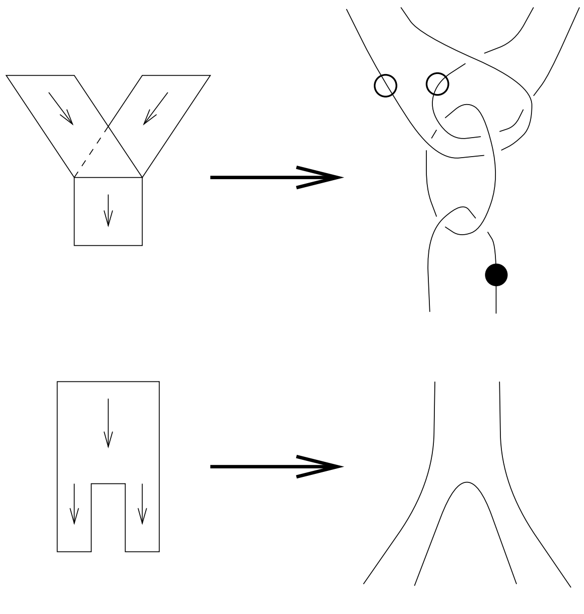

Figure 23: Assigning framed links to templates

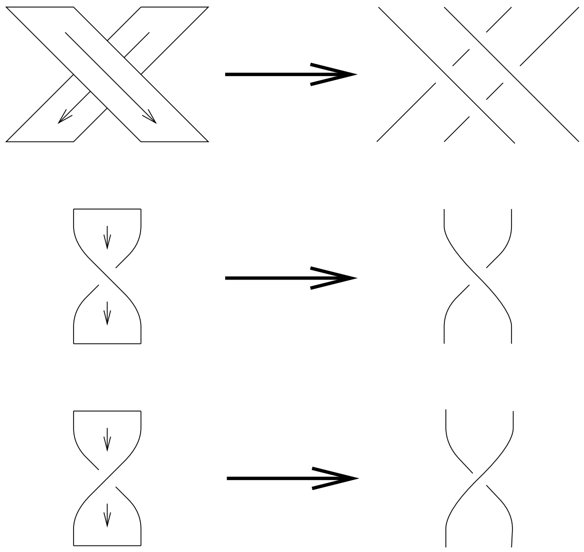

Figure 24: Assigning framed links to templates

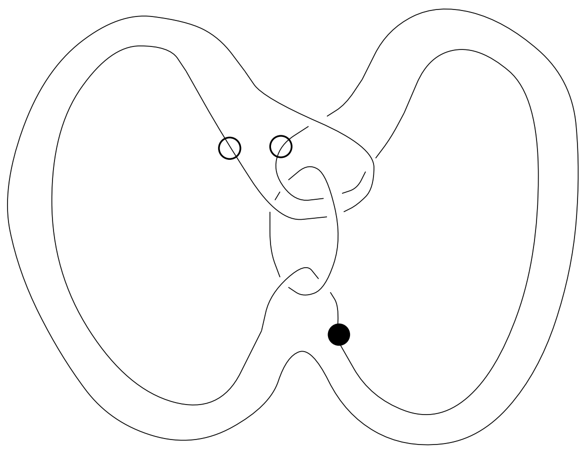

Figure 25: The framed link assigned to the Lorenz template

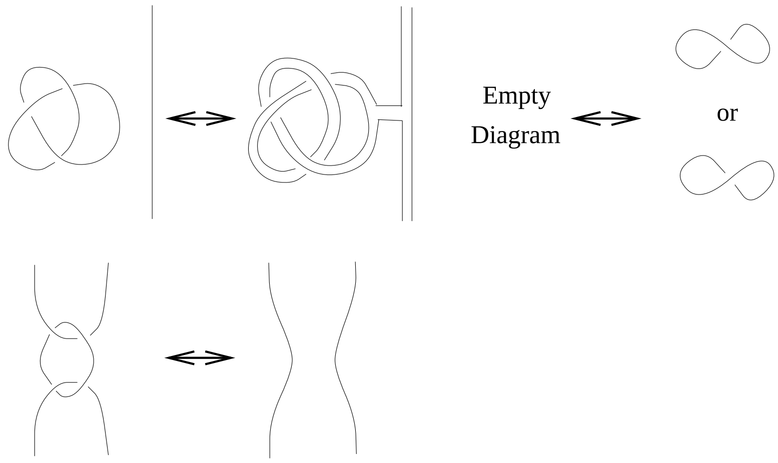


Figure 26: Kirby moves

Let a template diagram $T$ be given. As in the preceding section, fix a height function on the plane and cut $T$ into building blocks consisting of copies of maxima/minima/crossings, splitting/joining charts, and half twists. To a band, assign a parallel arcs. To each building block, assign framed tangles as depicted in Figure 23 and 24. In Figure 24 top, only one of two crossing types is depicted, but the other type is similar, except the crossing information of parallel strings are reversed. In other words, the boundary of bands of the template diagram together with the framed tangles assigned to splitting/joining charts form a framed link which is assigned to the given template diagram. (Hence we do not have to fix a height function on the plane in this section.) Figure 25 depicts the framed link assigned to the Lorenz template.

We now prove

**5.1 Theorem.** If two template diagrams represent equivalent templates, then the framed links assigned to the diagrams represent homeomorphic 3-manifolds.

*Proof.* It is known that two framed links represent homeomorphic 3-manifold if and only if they are related by Kirby moves [**Ki**] depicted in Figure 26 top two figures. The bottom figure is a consequence of the Kirby moves that we use often in the proof (the top and bottom strings in this figure represent distinct component of framed links).

Thus we show that framed tangles assigned to before/after each template/Reidemeister move are related by Kirby moves. Then, since equivalent template diagrams are related by these moves, and corresponding moves on associated framed links are related by Kirby moves, the Theorem follows. The proofs of such Kirby moves are depicted in Figures 27 and 28. Other moves directly follow from Reidemeister moves of framed links. □

**5.2 Examples.** It is easy to see that the framed link corresponding to the Lorenz template, as depicted in Figure 25, represents the 3-sphere $S^3$.

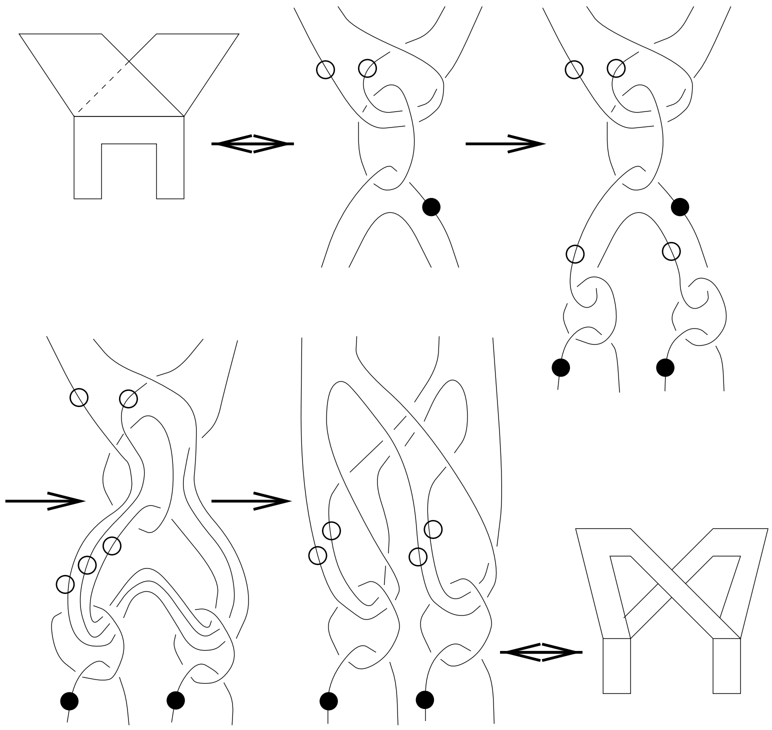

Figure 27: Splitting move and Kirby moves

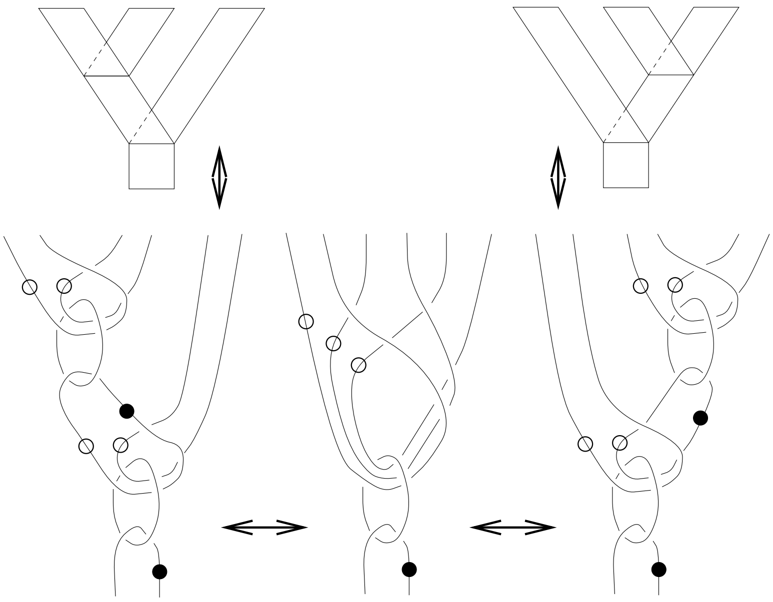

Figure 28: Another template move and Kirby moves

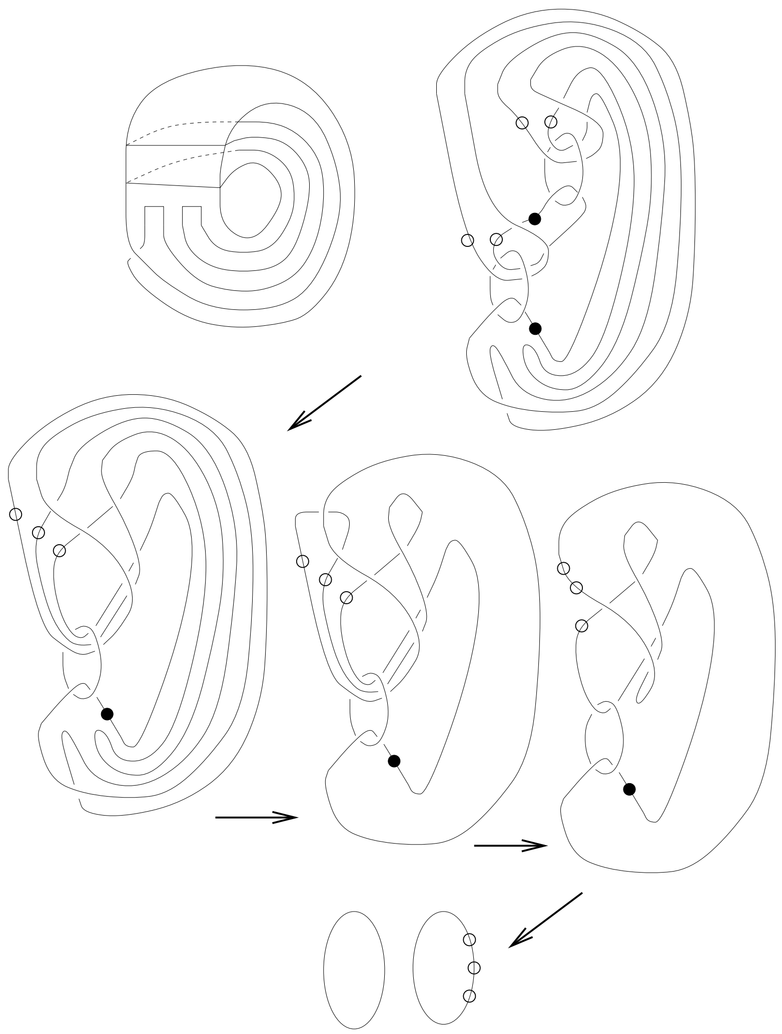

Figure 29: An example

Figure 29 top left shows an example of a template, and the top right shows the corresponding framed link. The bottom sequence of Kirby moves from left to right simplifies to a 3-manifold not homeomorphic to $S^3$.

**5.3 Remark.** In [**CY**], the braided Hopf algebra structure was constructed from Topological Quantum Field Theories (TQFTs) in dimension 3. Thus the correspondence between template moves and braided Hopf algebras gives rise to the above correspondence between templates and 3-manifolds via framed links.

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