Appeared in the March 1996 issue of College Mathematics Journal.

Knots about Stokes' Theorem

Michael C. Sullivan, Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA, msulliva@math.siu.edu

Stokes' Theorem is a standard topic in vector calculus. Usually it is the final topic in a difficult course. It states that given a differentiable vector field $\mathbf{F}$ and a smooth oriented surface S with single boundary component C, all in R^3, we then have

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} dS.$$

Here $\mathbf{N}$ denotes a continuous unit vector field normal to S, and $\mathbf{T}$ is a unit vector field tangent to the boundary curve C related to $\mathbf{N}$ by the *right hand rule* (RHR). That is when you point with the index finger of your right hand along C in the direction of $\mathbf{T}$, with your middle finger tangent to the surface and pointing inward from the boundary, as in Figure 1, your thumb will point in the direction of $\mathbf{N}$. The choice of a normal vector field for S together with the right hand rule define an *orientation* for S. The tangent field $\mathbf{T}$ of C gives C an orientation as well.

I wanted to raise the problem of non-orientable surfaces with my class in a way that would be fun. For motivation, assume $\mathbf{F}$ is an electric field that is irrotational. That is $\nabla \times \mathbf{F} = \mathbf{0}$. For any closed loop of wire does if follow that

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = 0,$$

implying there is no net current around the wire? Now $\nabla \times \mathbf{F} = \mathbf{0}$ implies the vector field is the gradient of a scalar function and hence line integrals are path independent. But I wanted to do it using Stokes' Theorem. Thus, our problem is, given a curve C can we find an orientable surface S with boundary C?

We started with the curve in Figure 2a, oriented counter-clockwise. The most natural surface to try is, of course, a Mobius band. See Figures 2b & c. I had my students bring scissors and tape to class, and I showed them how to make a Mobius band. They put an X on one spot and placed a normal vector on it (a short pencil or pen cap will do). Sliding their normal vector once around the strip, they discovered that upon returning to X the vector had reversed its direction, as in Figure 2c. Thus the Mobius band is not orientable – the surface is *one-sided.* One can also see this using the RHR, as indicated in Figure 2b. Applying the RHR to the outer loop the orientation $\mathbf{T}$ produces a normal vector $\mathbf{N}$ pointing out of the paper toward the viewer, while doing likewise on the inner loop of C makes $\mathbf{N}$ point the other way. So, this Mobius band just will not do. Is there a surface that does work?

The answer is yes. Cut out a small disk and a big disk, and place the small one above the big one as on the left in Figure 3a. Now connect the two disks with a rectangular strip with a half twist. This surface S clearly has the curve C as its boundary. And since we can unfold S and lay it in the plane, as shown in Figure 3b, this surface is orientable. Thus, we can now apply Stokes' Theorem.

Next we did the trefoil, shown in Figure 4a. It is the simplest example of a knot. One might be tempted to try a band with three half twists. But, the same reasoning we used with the Mobius band shows this surface is non-orientable. To build an orientable surface for the trefoil cut out two disks, one smaller than the other and place the smaller disk above the larger disk just as before. Now, connect them with three half-twisted strips as in Figure 4b. One of my students got this on his own. The boundary is our trefoil. After some experimenting the class could see that this surface is orientable. One can visualize a continuous unit normal vector field $\mathbf{N}$ and see that it produces a consistent orientation for the boundary. See Figure 4c.

It is possible to construct an orientable surface for any knot, no matter how complex. This fact is know as Seifert's Theorem [3] and the surfaces obtained are called Seifert surfaces. They play an important role in knot theory.

Using Seifert's Theorem, which I stated but did not prove, and Stokes' Theorem, we now know that

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = 0,$$

for any smooth simple closed curve C.

Incidently, George Stokes got the idea for Stokes' Theorem from an 1850 letter from Lord Kelvin [2, page 948], the founder of knot theory [1, section 1.3].

References

[1] Michael Atiyah, *The Geometry and Physics of Knots*, Cambridge University Press, 1990.

[2] Edwards & Penny, *Calculus with analytic Geometry*, Fourth Edition, Prentice Hall, 1994.

[3] Dale Rolfsen, *Knots and Links*, Publish or Perish, Inc., 1976.

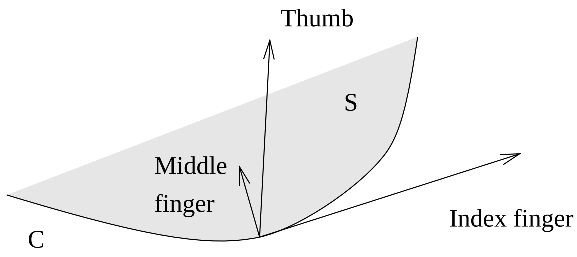


Figure 1: The right hand rule (RHR)

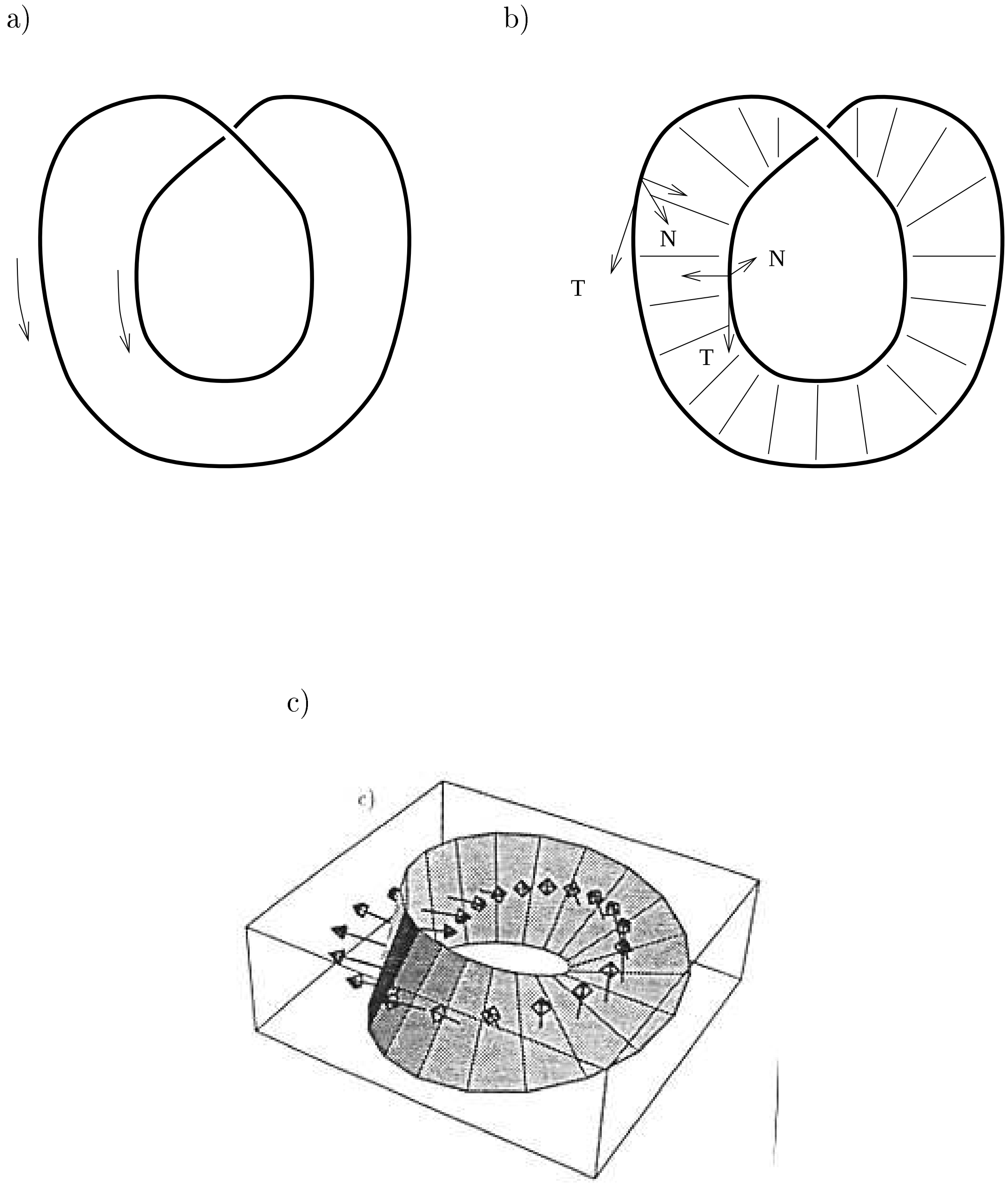
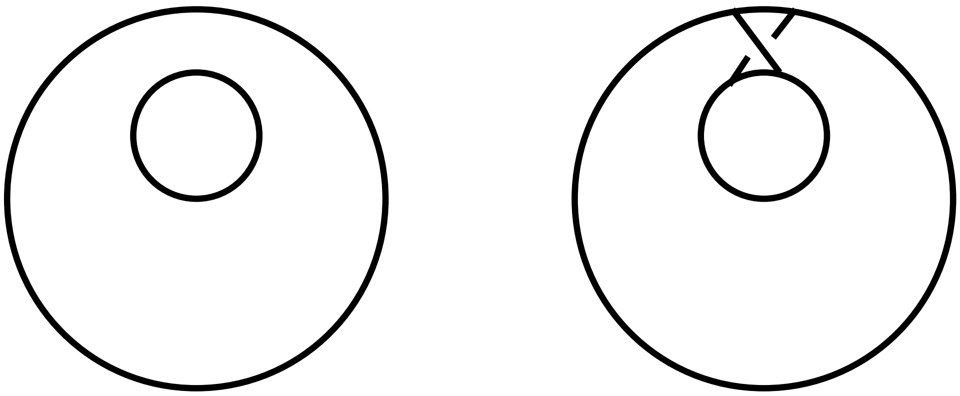


Figure 2: a) The curve C. b) The curve C bounds a Mobius band. c) Mobius band is not orientible.

a)



b)

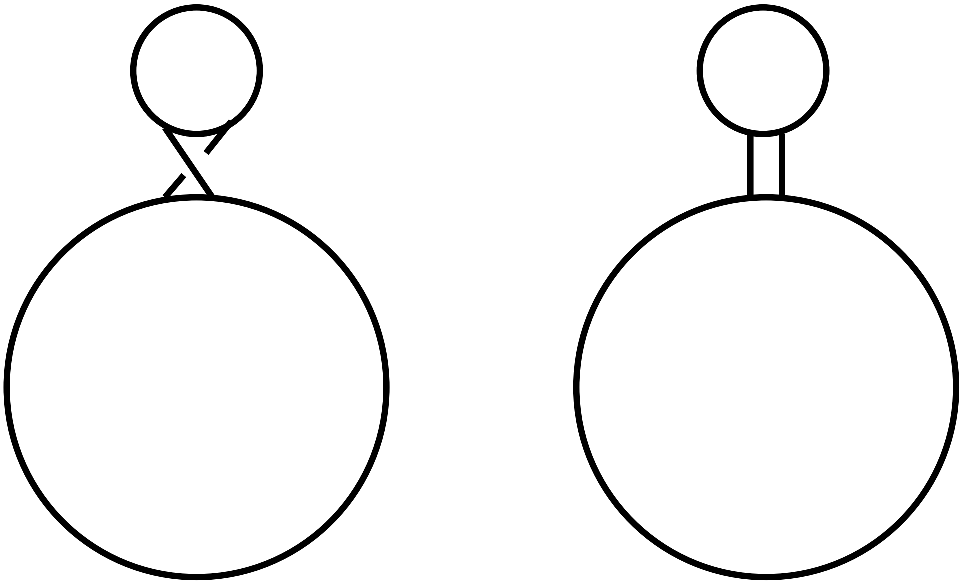
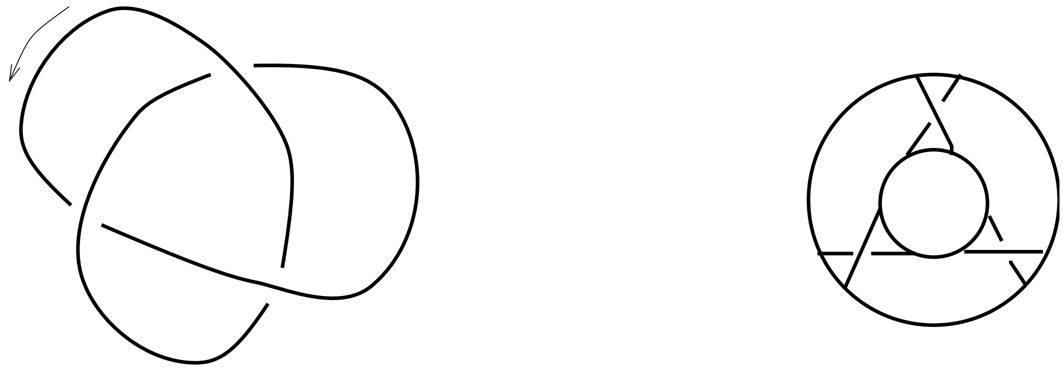


Figure 3: a) An orientable surface S for the curve C. b) The surface S can be placed in the plane.

a)



b)

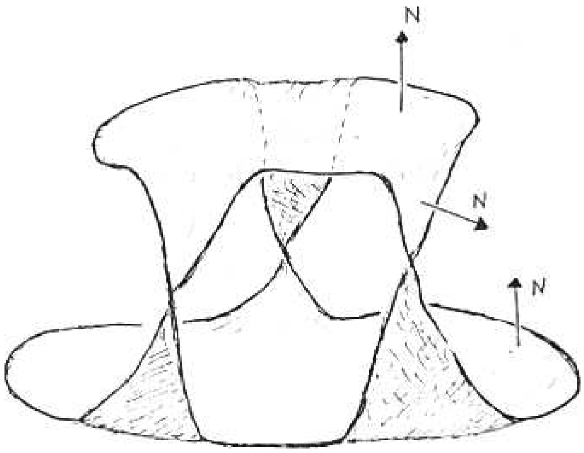


Figure 4: a) An orientable surface S for the trefoil. b) The surface S viewed from the side.