SIMPLE SMALE FLOWS WITH A FOUR BAND TEMPLATE

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ABSTRACT. We study simple Smale flows on S^3 and other 3-manifolds modeled by the Lorenz template and another template with four bands but that still has cross section a full 2-shift.

1. Introduction

We study simple Smale flows (defined below) on 3-manifolds. This continues work done in [7], [18] and [1]. We enumerate the 3-manifolds that can support Lorenz-Smale flows extending [1]. We also study embedding types of non-Lorenz-like simple Smale flow in S^3 whose saddle set is modeled by a template with three bands. Most of this work appeared in [12].

2. Background

Background material on Smale flows can be found in [18, 1] and the references therein. We study smooth $\mathbb{R}$ actions on three-dimensional compact manifolds without rest points, *i.e.*, nonsingular flows. A non-singular Smale flow (NSF) is a flow that is structurally stable and whose invariant set is one-dimensional. The invariant set of a Smale flow can be decomposed into a finite number of *basic sets* which are disjoint, compact and transitive (they have a dense orbit). Each basic set is either an attractor, repeller or saddle set. Since the manifold is three-dimensional, each attracting and repelling basic set consists of a single closed orbit. Saddle sets may consist of a single closed orbit or be suspensions of nontrivial irreducible shifts of finite type; such saddle sets are said to be *chaotic* and contain infinitely many closed orbits. (For the definition of *shifts of finite type* see [13] or [5].)

A chaotic saddle S of a Smale flow and its dynamics can be modeled by a *template* [9, 2, 3], which is a branched two-manifold with a semi-flow. A template T for a chaotic saddle set S can be constructed by taking an isolating neighborhood N of S and then identifying points along local stable manifolds. Many orbits of the flow are identified but

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no two periodic orbits are identified. Thus for any finite link of closed orbits in S there is a corresponding link of the same link-type of closed orbits of the induced semi-flow on T.

Templates, originally called *knot holders*, were first used to study the topology and dynamics of the basic sets. However, they can also be used to help understand the global structure of flows. A thickened version of a template model corresponds to an isolating neighborhood of the saddle set. Its boundary is naturally partitioned into an exit set, entrance set and tangent set, where the first two are subsurfaces whose closures intersect to form the one-dimensional tangent set. Thickened templates along with tubular neighborhoods of the attractors and repellers are glued together in a manner compatible with the flow reconstructing the supporting three-manifold. Given a template or templates and some closed orbits marked as attractors, repellers or saddles one can ask what manifolds can support NSFs with corresponding basic sets.

We will restrict our attention to NSFs with three basic sets, an attractor, a repeller and a chaotic saddle set. These are called *simple Smale flows*. If the saddle set can be modeled by a template with just two branches the flow is a *Lorenz-like Smale flow*. Topologically there are three types of Lorenz-like templates denoted $L(0,0)$, $L(1,0)$ and $L(1,1)$ for where both bands are orientable, only one is and neither is respectively; see Figure 1. For an embedded Lorenz-like template we use a pair of integers to denote the number of half twists in each band.

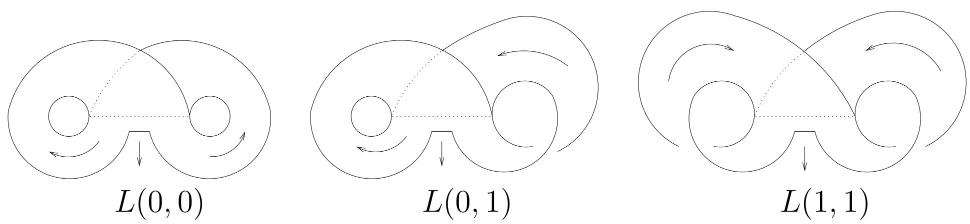

FIGURE 1. Lorenz like templates

We next review some topics from 3-manifold theory. Let $I = [0,1]$. An *i-handle* is a 3-ball represented by I^3 together with a specified subset of its boundary that is called its *attaching set*. For $i = 0$ the attaching set is empty. For $i = 1$ it is $\{0,1\} \times I^2$, that is the top and bottom faces. For $i = 2$ it is $(\partial I^2) \times I$, that is the four side faces. Finally for $i = 3$ the attaching set is the entire boundary 2-sphere of I^3. Given a 3-manifold M one can form a new 3-manifold by attaching an i-handle H to M by gluing the attaching set of H to a portion of the boundary of M for $i > 0$; for $i = 0$ attaching is just a disjoint union and $M = \phi$ is allowed. A *handle body* is a 3-manifold formed by attaching finitely many 1-handles to an initial 0-handle.

If we glue two solid tori together along their boundaries we may form a variety of spaces depending on the isotopy class of the gluing homeomorphism. These include S^3, $S^2 \times S^1$ and the lens spaces. If a meridian of one solid torus in taken to a nontrivial (p, q) curve on the boundary of the other the resulting space is the (p, q) *lens space*, denoted $l(p, q)$.

This construction can be viewed as a special case of *Dehn surgery* [15]. Given a 3-manifold M and a knot K one deletes the interior of tubular neighborhood $N(K)$ of K. This *knot complement* space is denoted M_K. Then one glues in a new solid torus V via a homeomorphism $h : \partial N(K)(\subset M_K) \to \partial V$. The gluing homeomorphism is characterized up to isotopy by a (p, q) curve on $N(K)$ that will be identified with a meridian of V. We will call (p, q) the *gluing coordinates*. We note that these are relative to the choice of the longitude for $N(K)$. The standard practice is to chose a *preferred longitude* meaning one that has linking number zero with K. It is easy to show that the order of the first homology group of any (p, q) Dehn surgery on a knot in S^3 is isomorphic to $\mathbb{Z}/q\mathbb{Z}$.

Finally we review *Seifert fibered manifolds* [20, 4, 11]. Let D be the unit disk. Let $C = D \times I$ and construct a solid torus by attaching $D \times 0$ to $D \times 1$ via $2v\pi/u$ rotation for integers v and $u \neq 0$.

This solid torus is naturally fibered by circles which are formed from finitely many copies of I. The center fiber is one copy of I with its ends attached and the others a formed from u copies of I laid end to end. A solid torus together with such a fibration is called a (u, v)-*Seifert fibered solid torus*. If $|u| > 1$ the center fiber said to be *exceptional* of *index* $|u|$.

A *Seifert fibered manifold* is a manifold that can be decomposed into a disjoint union of circles, called the fibers, such that each fiber has a tubular neighborhood that is a Seifert fibered solid torus. If a Seifert fibered manifold is compact the number of exceptional fibers is finite. Compact Seifert fibered manifolds have been classified up to fiber preserving homeomorphism by Seifert [20].

If one quotients out the fibers the result is a surface called the *orbit surface*. For this paper we only need Seifert spaces where the orbit surface is S^2 so we will restrict our discussion to this case; it follows that the 3-manifold must be orientable. The 3-sphere, $S^2 \times S^1$, and the lens spaces are all examples, but there are many others. Each of these can be fibered in infinity many different ways, however for "most"

Seifert fibered manifolds the topological type of the manifold admits only one type of fibration.

The gist of Seifert's classification is that any compact Seifert fibered manifold (with S^2 orbit surface) and n exceptional fibers can be constructed by the following steps. Remove the interiors of $n+1$ disjoint closed disks from S^2; call the result S_{n+1}. Let $M_0 = S_{n+1} \times S^1$ and fiber it using $* \times S^1$ for fibers. Then attach $n+1$ solid tori to n of the tori boundary components with gluing coordinates (β_i, α_i), for $i = 0, \ldots, n$ with $\beta_0 = 1$ and all other $\beta_i > 1$ and all pairs relatively prime. We often express this as a *gluing slope* α_i/β_i.

But gluing coordinates are relative to the coordinates used on the tori. For meridians use a fiber $* \times S^1$. For longitudes we select a cross section of M_0 and take its intersection with the boundary components. Obviously $S_{n+1} \times \theta$ for any $\theta \in S^1$ would do, but there are many other choices of the cross section that give non-equivalent longitudes. One can then fiber the attached solid tori in such a way that the gluing maps are fiber preserving.

We will use the following notation for oriented Seifert fibered spaces over S^2 with n exceptional fibers: $S^2(\alpha_0, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_n}{\beta_n})$. The ambiguity in the choice of the cross section causes the following equivalences: we can add an integer to any of the ratios provided we subtract it from another. This allows us to present a Seifert fibered manifold in a normal form where $0 < \alpha_i < \beta_i$ for $i = 1, \ldots, n$. If $\alpha_0 = 0$ it is often dropped. The sum $\Sigma_{i=0}^{n} \alpha_i/\beta_i$, called the *Euler number* of the manifold, is an invariant under fiber preserving homeomorphisms. If we change the orientation on the manifold we get $S^2(-\alpha_0, -\frac{\alpha_1}{\beta_1}, \ldots, -\frac{\alpha_n}{\beta_n})$. For $n = 3$ order of the first homology group is

$$\pm(\alpha_0\beta_1\beta_2\beta_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3).$$

We can arrange for the sign to be + by our choice of orientation. (Note: we switched the roles of α and β used in [14] to be consistent with [11].)

3. Manifolds realizing Lorenz-like Smale Flows

Theorem 3.1. *[Haynes & Yu]*

a. *The closed manifolds that admit an $L(0,0)$ Lorenz like Smale flow are the following and only the following: S^3, $S^2 \times S^1$, any lens space, the sum $l(3,1)\#\mathbb{R}P^3$, the Seifert fibered manifolds of type $S^2(\frac{1}{2}, \frac{1}{3}, \frac{q_1}{p_1})$ and $S^2(\frac{1}{2}, \frac{1}{3}, \frac{q_1}{p_1}, \frac{q_2}{p_2})$ where $p_i \neq 0$ and $(p_i, q_i) = 1$ for $i = 1, 2$.*

b. *The closed manifolds that admit an $L(1,0)$ Lorenz like Smale flow are just the same as in (a).*

c. *The closed manifolds that admit an $L(1,1)$ Lorenz like Smale flow are the following and only the following: S^3, $l(3,1)$, $l(3,1)\#Y$ where Y is $S^2 \times S^1$ or any lens space, and the Seifert fibered manifolds of type $S^2(\frac{1}{2}, \frac{1}{3}, \frac{q_1}{p_1}, \frac{q_2}{p_2})$ where $p_i \neq 0$ and $(p_i, q_i) = 1$ for $i = 1, 2$.*

Parts (b) and (c) were proven in [1]. We outline the proof given in [12] of (a) for completeness of the theory. It is similar to Bin Yu's proof of (b) which came first.

Proof of a. Let T be a tubular neighborhood of $L(0,0)$, that is T is a thickened up version of $L(0,0)$. It is to have a smooth flow which is the inverse limit of the semi-flow on $L(0,0)$. Then ∂T has an exit set and an entrance set. Denote their closures by Ex and En respectively. Then $Ex \cap En$ is a set of circles where the flow on the completed manifold will be tangent to ∂T.

Let A be a tubular neighborhood of the attracting orbit a and let R be a tubular neighborhood of the repeller r, each with the obvious vector field. We assume A, T and R are disjoint and then glue them together with diffeomorphisms that match the vector fields to create M. Thus we can write

$$M \cong (A \cup_\phi T) \cup_\psi R,$$

where ϕ and ψ are attaching maps; $\phi : Ex \to \phi(Ex) \subset \partial A$ and $\psi : \partial R \to \partial(A \cup T)$.

Partition Ex into three pieces, two annuli, C_1 and C_2, and a rectangle L. Let c_1, c_2 and l be the cores of these respectively as shown in Figures 2 and 3.

Let $AT = A \cup_\phi T$. Up to diffeomorphisms AT is determined by the images of c_1 and c_2. Let $c_i' = \phi(c_i)$ for $i = 1$, 2. We divide the proof into three cases. In each the roles of a and r can be reversed.

Case 1. Suppose c_1' and c_2' are inessential in ∂A. We claim AT is a solid torus. Either c_1' and c_2' are concentric on ∂A or not. First suppose they are not. Let D_i be the disk in ∂A with boundary c_i', $i = 1, 2$. Let $B_i = D_i \times [0, \epsilon]$ be disjoints balls in A such that $A' = \text{cl }(A - B_1 - B_2)$ is still a solid torus. We can regard B_i as a 2-handle and attach it to C_i, $i = 1, 2$. Then $T' = T \cup B_1 \cup B_2$ is a 3-ball. Now $D_1 \times \{\epsilon\}$, $D_2 \times \{\epsilon\}$, and L form a disk in $\partial T'$. Then $AT = A' \cup T'$ where the gluing is along this disk. Hence AT is a solid torus as claimed.

Next suppose c_1' and c_2' are concentric and that $c_1' \subset D_2$. We attach B_1 to T as before. Clearly $B_1 \cup T$ is a solid torus and c_2 is a longitude. Then let $B_2' = D_2 \times [0, 2\epsilon] - \text{int }(B_1)$; it is a ball. We treat it as a

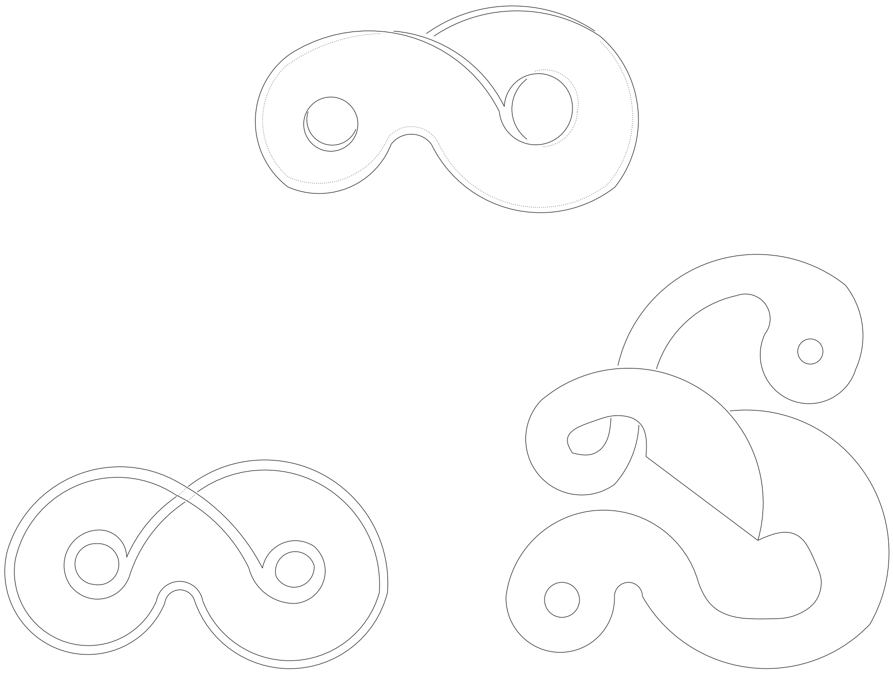

FIGURE 2. The entrance and exit sets of a thick Lorenz template

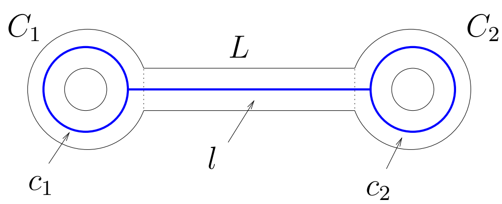

FIGURE 3. Core of the exit set

2-handle and attach it to $B_1 \cup T$ along C_2 and see that $T \cup B_1 \cup B_2'$ is a ball. As before see have that AT is a solid torus.

Thus, $M = AT \cup R$ can be S^3, $S^2 \times S^1$ or any lens space.

Case 2. Suppose c_1' is inessential and c_2' is essential in ∂A. Then we can define B_1 as before, $A - \text{int}(B_1)$ and $T \cup B_1$ are solid tori with c_2 a longitude on the latter. It is not hard to show that gluing two solid tori together along longitudinal annuli in their boundaries yields another solid torus. Thus for $AT \cup R$ we get the same list as in Case 1.

Case 3. Suppose c_1' and c_2' are essential in ∂A. In S^3 one can use the Seifert–van Kampen theorem and some basic 3-manifold theory to show that $T \cup R$ must be a trefoil complement with a a trefoil knot, [18, 1]. Thus Dehn surgery on r can be preformed to yield $S^2 \times S^1$ or any lens space. But we can also perform Dehn surgery on a. The resulting spaces have been classified by a theorem of Louis Moser's [14, 19]. Let $\sigma = 6p + q$. There are three cases to consider.

If $|\sigma| > 1$ then we get $S^2(\alpha_0, \frac{1}{2}, \frac{\alpha_2}{3}, \frac{\alpha_3}{\sigma})$. The values of the α_i are determined the equation derived by the two computations of the order of the first homology group:

$$6\sigma\alpha_0 + 3\sigma + 2\sigma\alpha_2 + 6\alpha_3 = |q|.$$

Given any choice for the α_i we can solve for p and q. It is also possible to rewrite the manifold in the form $S^2(\frac{1}{2}, \frac{1}{3}, \frac{\alpha}{\beta})$.

Example 1. For $(p, q) = (7, -3)$ we have $\sigma = 39$. We then find the manifold is $S^2(-1, \frac{1}{2}, \frac{1}{3}, \frac{7}{39})$. For $(p, q) = (5, 2)$ we get $S^2(-2, \frac{1}{2}, \frac{2}{3}, \frac{9}{32})$.

For $|\sigma| = 1$ the result is a the lens space $l(|q|, 9p)$. For the last case $\sigma = 0$ the result is $l(3, 2)\#l(2, 3)$, which is known to be homeomorphic to $l(3, 1)\#\mathbb{R}P^3$.

We can also do Dehn surgery on both a and r. If $|\sigma| > 1$ then r being a meridian causes it to be a regular fiber after doing the surgery on a. So when we doing the surgery on a tubular neighborhood of r we get $S^2(\alpha_0, \frac{1}{2}, \frac{\alpha_2}{3}, \frac{\alpha_3}{\beta_3}, \frac{\alpha_4}{\beta_4})$.

If $|\sigma| = 1$ then r is a regular (non-exceptional) fiber in the fibration of $l(|q|, 9p)$ and the result is just another lens space (see [16, Section 1.6]). If $\sigma = 0$ then a and r are parallel after the surgery on a. Thus doing the r surgery just sends us to a space of the form $S^2(\alpha_0, \frac{1}{2}, \frac{\alpha_2}{3}, \frac{\alpha_3}{\beta_3})$. □

4. Realizing U in S^3

Let U be the template shown in Figure 4 on the left. Using this figure as a specific embedding of U into S^3 it was been shown by Rob Ghrist that U contains every finite link as set of periodic orbits [8, 9]. Thus, the constructions in Theorem 4.1 give examples of nonsingular structurally stable flows in which every finite link is realized as a set of periodic orbits. Let U^+ be the same as U but with the crossing on the top and bottom branch lines having the same sign as shown on the right side of Figure 4.1. It is known that all knots on U^+ are prime [17]. In both cases there is a Markov partition whose first return map is topologically conjugate to the full 2-shift.

Theorem 4.1. *For a simple Smale flow on S^3 with saddle set modeled by U the link $a \cup r$ is either a Hopf link or a figure-8 knot and meridian. In the latter case the bands are untwisted, unknotted, and unlinked. In*

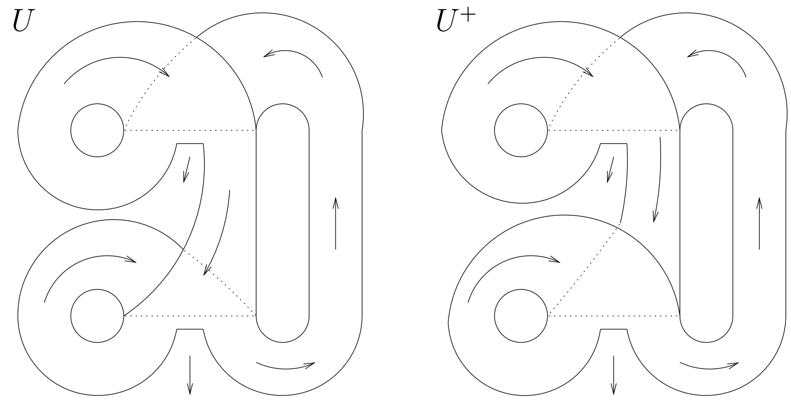

Figure 4. Templates U and U^+

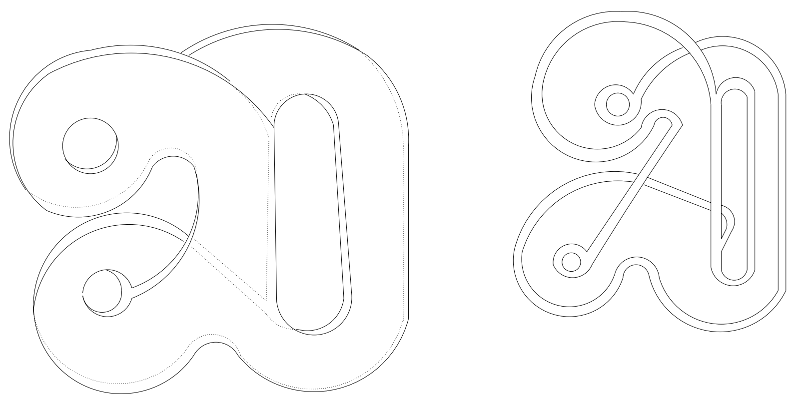

Figure 5. Thickened template U and its exit set

the Hopf link case one or two bands may form (p, q) torus knots about a or r; however the two looped bands on the left of Figure 4 cannot both be knotted, twisted, linked.

Proof. A thickened version of U is a genus 3 handlebody which we will still call U. The exit Ex can be partitioned into three annuli and two rectangular strips: C_1, C_2, C_3, L_1 and L_2 respectively. We partition the core of the exit into three loops and two line segments, c_1, c_2, c_3, l_1 and l_2. Let $h : Ex \to \partial A$ be the attaching map. The topology of $A \cup R$ is determined by the image of the core. Let $c_i' = h(c_i)$ and $l_j' = h(l_j)$ for $i = 1, 2, 3$ and $j = 1, 2$. See Figures 6.

We divide the problem into cases based on how many of the c_i''s are essential in ∂A. Note that in the subcases we can always switch c_1' and c_3' since U has the following symmetry: rotation about the axis marked by a dot in Figure 5 so as to switch the upper and lower "ears" then

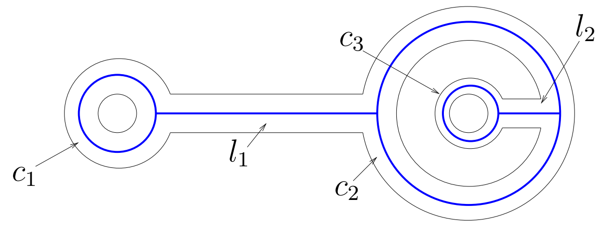

Figure 6. The core of U's exit set.

take a mirror image. Also notice that in any allowed configuration l_1' and l_2' must attach to opposite sides of c_2'; for example "O-O-O" is not allowed. We shall assume if c_2' bounds a disk that c_3' is inside that disk and c_1' is not.

Case O. All three are inessential. We construct $A \cup U$ in stages. There two subcases as shown in Figure 7: either c_2' is inside the disk bounded by c_1' or it is not. Either way c_3' bounds a disk in ∂A. Thicken it slightly to create a thin ball B_3 in A. Let $A' = \text{cl }(A - B_3)$; it is a solid torus. Let $U' = U \cup B_3$ where the attaching is as a 2-handle with attaching set the inner half of C_3'. Now U'' is a genus two handlebody. Next, notice that c_2' bounds a disk in $\partial A'$ (l_2' is in this disk). We repeat the procedure: $A'' = \text{cl }(A' - B_2)$ is a solid torus and $U'' = U' \cup B_2$ is a solid torus too. Note that the gluing is still as a 2-handle with attaching set now formed by the inner half of C_2', L_2', and $B_2 \cup B_3$.

Subcase a. If the disk in $\partial A''$ bounded by c_1' is disjoint from B_2 then we can create a thin 3-ball B_3 by pushing into A a little. Then define $A''' = \text{cl }(A'' - B_3)$, which is a solid torus, and $U''' = U'' \cup B_3$, which is a 3-ball. We can now form $A \cup U$ by gluing A''' to U''' by identifying a disk in $\partial U'''$ consisting of $B_1 \cap \partial U'''$, L_1, and $B_2 \cap \partial U'''$ with the corresponding disk in $\partial A'''$. Thus we see that $U \cup A$ is a solid torus and a can be taken as its core. Therefore, if this subcase can be realized it must be that $a \cup r$ is a Hopf link. We will see later that this configuration and the others below can be realized.

Subcase b. If the disk in ∂A bounded by c_1' contains c_2' then we can still carve out a 3-ball B_1 from A'' by digging in a little deeper. In this case L_1 is in the boundary of B_1. We let $A''' = \text{cl }(A'' - B_1)$ and $U''' = U'' \cup B_1$ where B_1 is a 2-handle. Again A''' is a solid torus and U''' is a 3-ball. We form $U \cup A$ by gluing A''' to U''' by identifying the disk in $\partial U'''$ consisting of $\partial B_1 \cap \partial U'''$ with the corresponding disk in $\partial A'''$. Thus we see that $U \cup A$ is a solid torus and that a can be taken as its core. Therefore, if this subcase can be realized it must be that $a \cup r$ is a Hopf link.

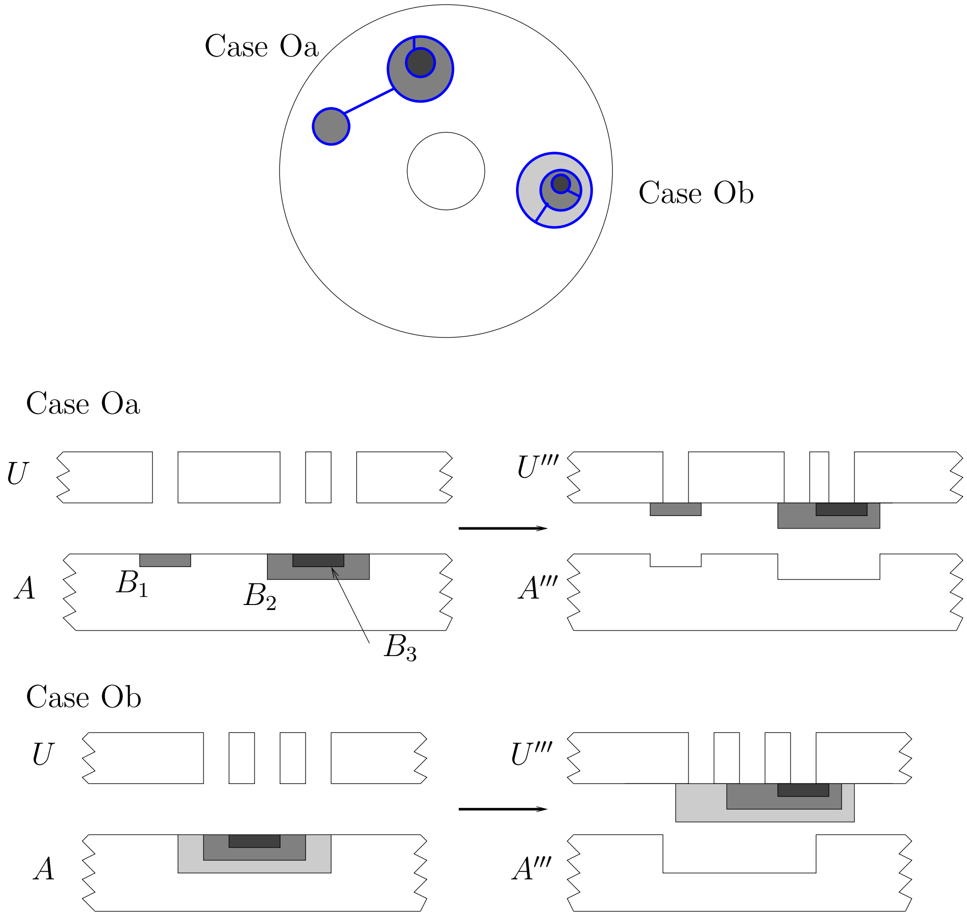

FIGURE 7. Case O

Case I. Now we suppose only one of the c_i''s is essential. It could be any (p, q) curve on ∂A. There are two subcases, both illustrated in Figure 8.

Subcase a. First suppose c_1' is essential with the others being inessential. We can define thin 3-balls B_3 and B_2 as before. Let $A'' = \text{cl}\,(A - B_1 - B_2)$ and $U'' = U \cup B_1 \cup B_2$. Both A'' and U'' are solid tori. Now we form $U \cup A$ by gluing U'' and A'' by identifying the annulus in $\partial U''$ formed C_1, L_1 and $B_2 \cap \partial U''$ with the corresponding annulus in A''. While the latter can be any (p, q) curve the former is a longitudinal annulus. Thus the union is a new solid torus and that a can be taken as its core. This forces $a \cup r$ to be a Hopf link.

Subcase b. Now suppose c_2' is essential and the others are not. Now c_1' and c_3' bound disk is ∂A. Thicken these to get thin 3-balls B_1 and

B_3 in A. Let $A'' = \text{cl } A - B_1 - B_3$ and $U'' = U \cup B_1 \cup B_3$. Again both are solid tori. Now c_2 is a longitude on U'' so the gluing of A'' and U'' is a solid torus equal to $U \cup A$ and that a can be taken as its core. Again we conclude that $a \cup r$ is a Hopf link.

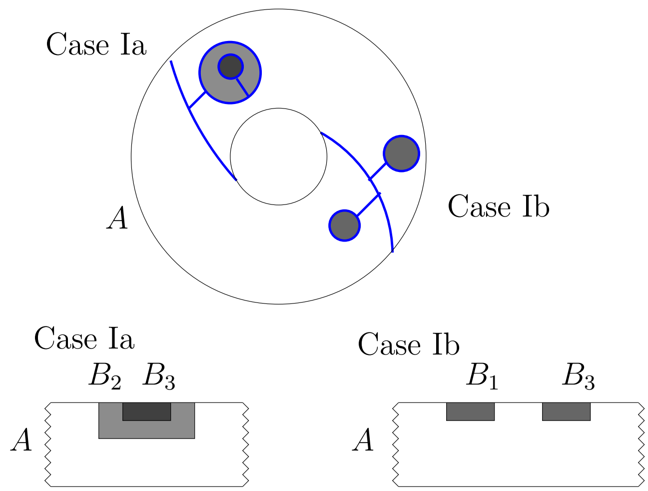

FIGURE 8. Case I

Case II. Now we suppose only two of the c_i''s are essential. If c_1' and c_3' were essential then l_1 and l_2 would attach to the same side of c_2' as it bounds a disk on the other side. Thus we can exclude this subcase. We assume that c_2' and only one of c_1', c_3', are essential. Without lose of generality assume it is c_1' that is essential. The two essential curves must be parallel (p, q) curves on ∂A. Since l_2' must attach to the opposite side of c_2' from l_1' there is only one configuration to consider; it is shown in Figure 9.

As before c_3' is the boundary of a disk in ∂A which we can push in a little to create thin 3-ball in A such that $A' = \text{cl } (A - B_3)$ is still a solid torus. Then we let $U' = U \cup B_3$ attached along the original disk. Then U' is a genus two handle body.

Next we consider the open disk in $\partial A'$ bounded by c_1', c_2' and on two sides by l_1'. We can shrink the disk a little bit away from l_1' so its closure is still a disk. From the closure of this disk push into A' a little to create a thin 3-ball B_x in A' such that $A'' = \text{cl } (A' - B_x)$ is still a solid torus. Now attach B_x to U' as a 2-handle to get U'' which is a solid torus. Then $U'' \cup A''$ is formed by gluing along a (p, q) annulus in $\partial A''$ that consists of C_1', C_2', L_1', $\partial A'' \cap B_x$, L_2' and $\partial A' \cap B_3$ to a corresponding annulus in $\partial U''$. This annulus in $\partial A''$ is homotopic to a c_1'. Therefore the corresponding annulus in U'' is homotopic to c_1, a longitude. Hence the union $U'' \cup A'' = U \cup A$ is a solid torus and that a can be taken as its core. We conclude that again $a \cup r$ is a Hopf link.

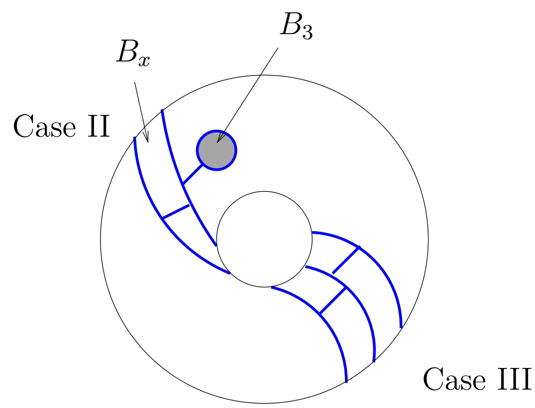

FIGURE 9. Cases II & III

Case III. Suppose all three are essential. They must be parallel (p, q) curves on ∂A. We will use the Seifert van Kampen theorem to find $\pi_1(U \cup A)$. Using the generators shown in Figure 10 we get

$$\left\langle a, x, y, z \,|\, a^p = xz^{-1}xzx^{-1}, a^p = xyx^{-1}, a^p = z \right\rangle \cong \left\langle a, x \,|\, a^{-p}xa^{-p}xa^px^{-1} \right\rangle.$$

Using Fox's free differential calculus [6] we find that the Alexander polynomial is $2t^{-1} - t^{p-1}$. But this cannot be the Alexander polynomial of a knot unless $p = 0$.

For $p = 0$ this group is infinite cyclic. Thus $U \cup A$ is a solid torus. Thus r is unknotted.

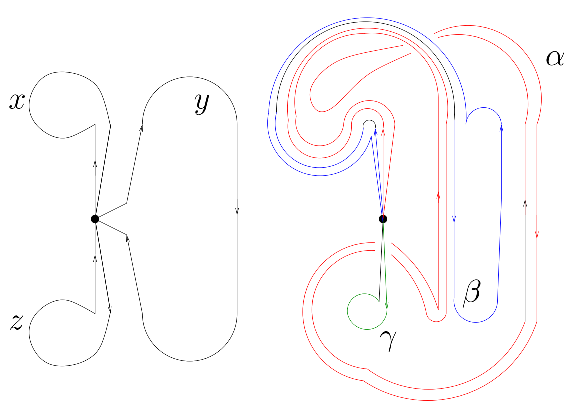

FIGURE 10. Generators for U and Ex

Realizations. All cases above can be realized. To show this we attach a big ball, B, to U along the exit set as shown in Figure 11. In Figure 12 we show just B (in green) with the exit set of ∂U in thick red. The complement of the exit set in ∂B can be divided into four disjoint open disks which we denote α, β, γ and δ. We also show in Figure 12 several one-handles (blue) and how they could be attached to $\partial B - Ex$ to form a solid torus which will serve as the tubular neighborhood of the attractor. We use the following notation for the result of attaching a one-handle. If both feet land in α we call this $\alpha\alpha$. If one foot lands in α and the other in β we call this $\alpha\beta$, and so on. Of course $\alpha\beta = \beta\alpha$ and all the symbols commute. Also by symmetry we can see that $\alpha\beta = \delta\gamma$, *etc.* Table 1 lists all the non-redundant pairs and which of the Cases each gives us.

Now Figure 12 is abstracted from S^3. Going back to Figure 11 we see that, except in Case III, adding the one-handle, H, can be done so as to make $B \cup U \cup H$ an unknotted tube. Thus we can glue in R to make S^3. For Case III $A \cup U$ is a solid torus with a inside as figure-8 knot as shown in Figure 13. Gluing in R we form S^3. Then $U \cup R$ is the complement of a figure-8 knot; since the complement determines the knot [10] we see that a can only be the figure-8 knot.

$\alpha\alpha$	$\rightarrow$	Case Ob
$\alpha\beta$	$\rightarrow$	Case Ia
$\alpha\gamma$	$\rightarrow$	Case II
$\alpha\delta$	$\rightarrow$	Case III
$\beta\beta$	$\rightarrow$	Case Oa
$\beta\gamma$	$\rightarrow$	Case Ib

Table 1

□

Remark 1. Clearly the template U can be realized on any 3-manifold produced by surgery on r or a. This would include all lens spaces but there is as yet no succinct way to list all the 3-manifolds produced by surgery on the figure-8 knot.

Theorem 4.2. *For a simple Smale flow on S^3 with saddle set modeled by U^+. Then the link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the bands are untwisted, unknotted, and unlinked. In the Hopf link case one or two bands may form (p, q) torus knots about a or r; however the two looped bands on the left of Figure 4 cannot both be knotted, twisted, linked.*

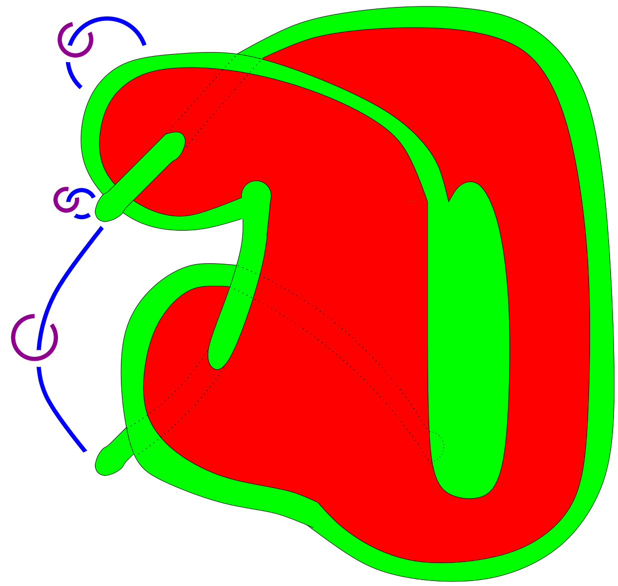

FIGURE 11. $B \cup U$

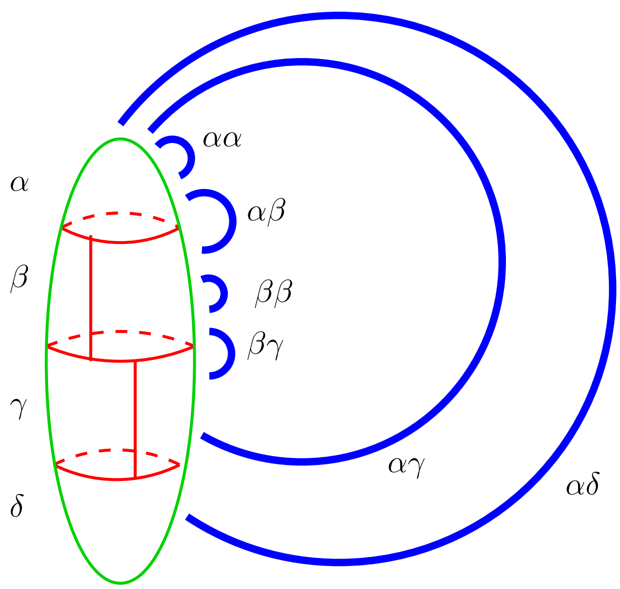

FIGURE 12. Ball, exit set and various 1-handles

Sketch of the proof. The proof breaks down into the same cases as the proof of Theorem 4.1. The arguments are the same in each case except

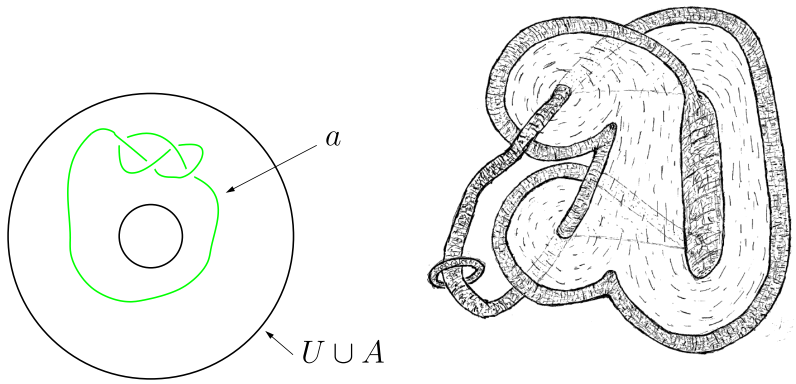

FIGURE 13. Left: $U \cup A$ with a inside. Right: $U \cup A \cup R$

Case III where the differences are minor. We leave the details to the reader. □

Corollary 4.3. *The set of manifolds which can support a simple Smale flow with saddle set modeled by* U^+ *is the same as in Theorem 3.1.a.*

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