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Qualitative Theory of Dynamical Systems

Transverse Foliations to nonsingular Morse-Smale flows on the 3-sphere an d Bott-integrable Hamiltonian systems

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Abstract. In this note we apply results of Goodman, Yano and Wada to determine which nonsingular Morse-Smale flows on S^3 have transverse foliations. We then observe that there is a connection to flows arising from certain Hamiltonian systems and from certain contact structures.

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1. Introduction

Masaaki Wada [6] determined the set of indexed links that can form the set of periodic orbits in a nonsingular Morse-Smale flow on the 3-sphere; the indexing is 0 for an attractor, 1 for a saddle, and 2 for repeller. Sue Goodman [5] and Koichi Yano [7] independently derived a simple criteria for when a nonsingular Morse-Smale flow on a closed orientable 3-manifold has a transverse foliation; we will use Goodman's terminology. We shall determine the set of indexed links that can be realized as the set of periodic orbits of a nonsingular Morse-Smale flow on S^3 that have transverse foliations. This turns out to be the same set of indexed linked realized as the indexed link of fixed points of flows on S^3 induced by Bott-integrable Hamiltonian systems and by certain flows given by connect structures on S^3.

2. Results

By a flow on a manifold M we mean a smooth function $\phi : M \times \mathbb{R} \to M$ such that $\phi(\phi(x,s),t) = \phi(x, s+t)$ and $\phi(x,0) = x$. Definitions and properties of *chain recurrent sets*, *hyperbolic structures*, *basic sets*, and *stable and unstable manifolds* can be found in [3], or most any graduate level textbook in dynamical systems. For basic facts about *transverse foliations* see [5].

Definition 2.1. A flow ϕ on a manifold M is a *Morse-Smale flow* if the following hold.

- The chain recurrent set is hyperbolic.
- The stable and unstable manifolds of basic sets meet transversely.
- Each basic set consists of a single closed orbit or fixed point.

For M a compact manifold, it follows that Morse-Smale flows have a finite number of periodic orbits and fixed points. A *nonsingular flow* is a flow without fixed points.

Definition 2.2. A flow on a 3-manifold satisfies the *Linking Property* if every periodic orbit that bounds a disk, the disk has nonzero algebraic intersection number with another periodic orbit. On the 3-sphere this is equivalent to requiring every periodic orbit that bounds a disk to have nonzero linking number with another periodic orbit.

Theorem 2.3 (Goodman – Yano). *A nonsingular Morse-Smale flow on a closed orientable 3-manifold has a transverse foliation if and only if it satisfies the Linking Property* [5, 7].

Notation. If L_1 and L_2 are links in S^3 (each in its own copy of S^3), then $L_1 \circ L_2$ is their *split sum*. It is constructed by forming the connected sum of the two copies of S^3, using 3-balls that miss the links. If L is a link with at least two components and K is a component of L, then $L - K$ denotes the link formed from L by removing K. Finally let L_1 and L_2 be links with at least two components in distinct 3-spheres S_1 and S_2 respectively. Let K_1 and K_2 be components of L_1 and L_2 respectively. For $i = 1, 2$ choose a ball neighborhood B_i of a point of K_i, that misses all other components of L_i and so that $(B_i, K_i \cap B_i)$ is diffeomorphic to the standard ball pair (B^3, B^1). Take a diffeomorphism $\zeta : (\partial B_1, K_1 \cap \partial B_1) \to (\partial B_2, K_2 \cap \partial B_2)$. Let $K = K_1 \# K_2$. Let $S = (S_1 - \mathrm{int}(B_1)) \cup_\zeta (S_2 - \mathrm{int}(B_2))$. We define $(L_1, K_1)\#(L_2, K_2)$ to be the link $(L_1 - K_1) \cup (L_2 - K_2) \cup K$ is S. Notice that no component of $L_1 - K_1$ links any component of $L_2 - K_2$.

Definition 2.4. Let $\mathcal{W}$ be the collection of indexed links determined by the following axioms:

W0: The Hopf link indexed by 0 and 2 is in $\mathcal{W}$.

W1: If $L_1, L_2 \in \mathcal{W}$ then $L_1 \circ L_2 \circ u \in \mathcal{W}$, where u (here and below) is an unknot in S^3 indexed by 1.

W2: If $L_1, L_2 \in \mathcal{W}$ and K_2 is a component of L_2 indexed by 0 or 2, then $L_1 \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W3: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 with indices 0 and 2 (resp.), then $(L_1 - K_1) \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W4: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 (resp.) each with index 0 or 2, then

$$((L_1, K_1)\#(L_2, K_2)) \cup m \in \mathcal{W},$$

where $K_1\#K_2$ shares the index of either K_1 or K_2 and m is a meridian of $K_1\#K_2$ indexed by 1.

W5: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L replacing a tubular neighborhood of K with a solid torus with three closed orbits, K_1, K_2, and K_3. K_1 is the core and so has the same knot type as K. K_2 and K_3 are parallel (p, q) cables of K_1. The index of K_2 is 1. The indices of K_1 and K_3 may be either 0 or 2, but at least one of them must be equal to the index of K.

W6: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L by changing the index of K to 1 and placing a $(2, q)$-cable of K in a tubular neighborhood of K, indexed by i.

W7: $\mathcal{W}$ is minimal. That is, $\mathcal{W} \subset \mathcal{W}'$ for any collection, $\mathcal{W}'$, satisfying W0-W6.

Remark 2.5. The last condition, W7, means that $\mathcal{W}$ is generated from the indexed Hopf link in S^3 by applying operations W1-W6.

Theorem 2.6 (Wada [6]**).** *The set of indexed links which can be realized as the collection of periodic orbits of a nonsingular Morse-Smale flow on S^3 is $\mathcal{W}$.*

Theorem 2.7. *The set of indexed links that can be realized as the set of periodic orbits of nonsingular Morse-Smale flows on S^3 that have transverse foliations is the subset of $\mathcal{W}$ generated by W0, W4, W5, W6.*

Proof. The Hopf link satisfies the Linking Property. If an indexed link or pair of indexed links satisfy the Linking Property, then the indexed link formed by applying W4, W5 or W6 does too. This is obvious for W5 and W6. We discuss the case for W4. The new orbit m links an attractor or repeller since $K_1\#K_2$ has index 0 or 2. Any component of $L_i - K_i$ that had nonzero linking number with K_i has the same linking number with $K_1\#K_2$.

If one applies W1, W2, or W3 to any indexed link, a unknotted periodic index 1 orbit is formed that does not link any other periodic orbit. Thus, the Linking Property fails. If an indexed link has a periodic index 1 orbit that does not link any other periodic orbit, then none of W1-W6 will remove it. □

3. Connection to Hamiltonian systems

Our discusion follows [1]. For more details see [2]. Let M^4 be a compact, smooth, 4-manifold with a symplectic structure. Denote local coordinates by (p_1, p_2, q_1, q_2).

Let $h : M^4 \to \mathbb{R}$ be a smooth function. The *skew-symmetric gradient* of h yields the vector field $\operatorname{sgrad} h = \left\langle -\frac{\partial h}{\partial q_1}, -\frac{\partial h}{\partial q_2}, \frac{\partial h}{\partial p_1}, \frac{\partial h}{\partial p_2} \right\rangle$. This vector field is called the *Hamiltonian vector field* and the associated flow is called the *Hamiltonian system* for h.

Let $r \in \mathbb{R}$. A Hamiltonian system induced by h is said to be *Bolt-integrable* on $Q = h^{-1}(r)$ if there exists a smooth real valued function f on some neighborhood U of Q in M^4 such that conditions a-d hold.

(a) The functions f and h are *independent* – meaning their gradients are linearly independent at each point.

(b) There exists a function $\lambda : \mathbb{R} \to \mathbb{R}$ such that the Poisson bracket $\{f, h\} \equiv \sum_{i=1,2} f_{p_i} h_{q_i} - h_{p_i} f_{q_i}$ can be written as $\{f, h\} = \lambda \circ h$. (The Poisson bracket depends only on the "energy level".)

(c) At $r = h(Q)$ we have $\lambda(r) = \lambda'(r) = 0$.

(d) The set of critial points of f on Q is the union of disjoint non-degenerate submanifolds. In fact it is made up of circles and tori if we assume f is orientable.

If f only satisfies (a-c) then f is merely an *integral* of h. The non-degeneracy requiment allows us to index the circles of critical points as attracting, repelling or saddle-like.

Lemma 11 of [1] gives a construction of a nonsignular Morse-Smale flow on Q whose indexed periodic orbits coincide with the indexed critial cricles of f. In the case where $Q = S^3$, Theorem 17 of [1] establishes that the subset of nonsignular Morse-Smale flows that can be realized in this way are just those whose invariant set consists of indexed links generated by W0, W4, W5, W6 – the same as in our Theorem 2.7.

Thus we are lead to ask, is the set of indexed links of fixed points realizable by Bott-integrable flows on a closed orientable 3-manifold Q, the same as the set of indexed links that can arise as the periodic orbits of the subset nonsingular Morse-Smale flows on Q that have transverse foliations?

Rob Ghrist has brought to our attention a paper by himself and John Etnyre [4] in which they study indexed links of fixed points of flows arising from certain contact structures on 3-manifolds. In general, the indexed links realizable on a 3-manifold Q in [4] are a subset of the set of indexed links that can arise as the invariant set of nonsingular Morse-Smale flows on Q. On S^3 they get the same set of links as in Theorem 2.7. Their proof involves deriving a contact flow from a Hamiltonian flow by a rotation of the gradient vector field.

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