MORE ON KNOTS IN ROBINSON'S ATTRACTOR

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Abstract. In an earlier paper the second author made a study of the knotted periodic orbits in a strange attractor for a set of differential equations in a paper by Clark Robinson. The attractor is modeled by a Lorenz-like template. It was shown that the knots and links are positive but need not be positive braids. Here we show that they are fibered, have positive signature, and that each knot-type appears infinitely often. We then construct a zeta type function that counts periodic orbits by the twisting of the local stable manifolds.

1. Introduction and background

For us a *knot* is a smooth oriented embedding of S^1 into $\mathbb{R}^3$ and a *link* is a finite set of knots with disjoint images. We are interested in knotted periodic orbits in solution sets of 3×3 ODEs and how they are linked. A knot or link is *positive* if it or its mirror image has a knot diagram with only positive crossings. A knot or link is a *positive braid* if it or its mirror image can be presented in braid form with only positive crossings. Not all positive knots are positive braids. The 5_2 knot is an example. [2]

A knot or link is *fibered* if the complement of a tubular neighborhood in S^3, taken as the one-point compactification of $\mathbb{R}^3$, can be fibered over S^1 with fiber an orientable surface known as a *Seifert surface*. Positive braids are fibered [15] but this is not always the case for positive knots. Again, 5_2 is an example. [13] The *genus* of a knot or link is the minimal genus of over all Seifert surfaces.

A *template* is a branched 2-manifold with a semi-flow. Templates are used to model chaotic invariant sets of certain flows in 3-manifolds. The classic example is the *Lorenz template* that contains the periodic orbits arising in the Lorenz equations. [1] Figure 1 shows the Lorenz template with a periodic orbit that forms a trefoil knot. A *Lorenz-like template* is a Lorenz template where some number of half twists have been added to each of the two bands. These are denoted by $L(m, n)$; see Figure 2. The template $L(0, 1)$ has been used to study a suspension of

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Smale horseshoe map, thus knots on $L(0,1)$ have been called *horseshoe knots*. [9]

Two templates are regarded as equivalent if one can be transformed it the other by a finite series of the two *template moves* shown in Figure 3 and ambient or smooth isotopies. Note that because of the second template move the usual invariants from algebraic topology do not apply, but see [10].

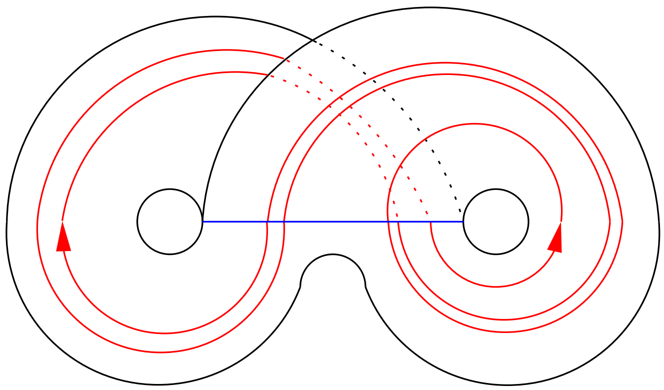

FIGURE 1. Lorenz Template with Trefoil Orbit

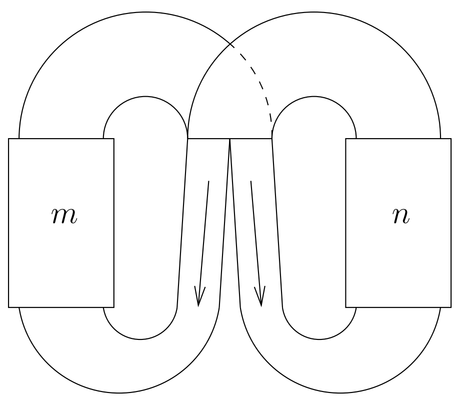

FIGURE 2. Lorenz-like Template

In [12] Clark Robinson studied the 3×3 system of ODEs below.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y + yz \\ \dot{z} &= -\gamma z + \delta x^2 \end{aligned} \tag{1}$$

where $\alpha = -0.71$, $\beta = 1.8690262$, $\delta = 0.1$, and $\gamma = 0.6$.

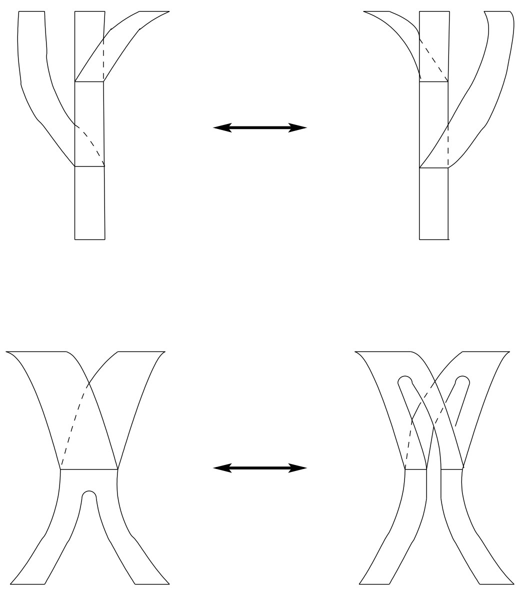

FIGURE 3. Template Moves

He showed that the system had a strange attractor similar to the Lorenz attractor except that the bands had half twists. In [17] the second author determined that $L(-1,-1)$ was a template model for this system and that the non-trivial knots in $L(-1,-1)$ are positive, but need not be positive braids. The proof involves surgering the template $L(-1,-1)$, without disturbing the periodic orbits, into a new template H in which all crosses are negative. See Figure 4. Earlier, Ghrist had shown [7] that two component links in $L(-1,-1)$ had to have negative or zero linking number.

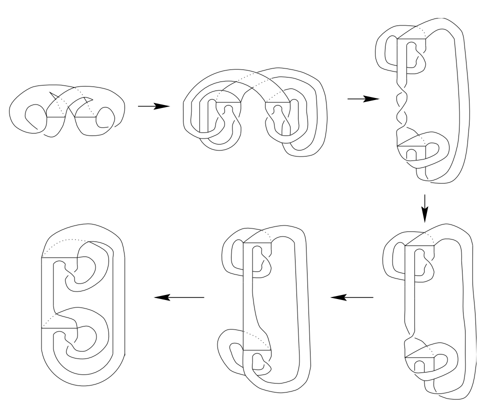

FIGURE 4

It is known that positive knots have positive signature (or negative depending on the sign convention used). [3] Thus, the non-trivial knots on $L(-1,-1)$ are not amphicheiral as these knots have signature zero.

It is known that the nontrivial knots in $L(0,n)$, for $n \geq 0$, are prime, positive braids; positive braids are fibered and nontrivial positive braids have positive signature and hence are non-amphicheiral. [1] For $n < 0$ the templates contain all knots and links. [6, 7] For $m, n > 0$ templates have composite knots with at most two components. [18] Knots in $L(-1,-1)$ have at most three prime factors. [18]

2. Knots and links in $L(-1,-1)$.

Given a starting point on the branch line an orbit on $L(m,n)$ gives unique biinfinite sequence of 0's and 1's using 0 for each pass through the left half of the branch line and 1 for each pass through the right half. [1] We can specify a periodic orbit with a finite sequence that is to be repeated. Thus 0 is the periodic orbit that loops just once along the left band. (If m is even this orbit will be in the boundary of the template.) The trefoil orbit shown in Figure 1 is given by 01011.

Theorem 2.1. *For orbits in $L(-1,-1)$ we have the following.*

a. *The orbit for* 01 *is unlinked with all other closed orbits.*
b. *The orbit for* 0 *is unlinked to orbits of the form* 01^n *and the orbit for* 1 *is unlinked to orbits of the form* 0^n1*,*
c. *Any pair of closed orbits not covered by (a) or (b) are linked.*

Proof. Claim (a) is apparent from Figure 5 where the 01 orbit is shown in purple. For claim (b) we refer to the same figure. The 0 orbit is shown in red. Orbits of the form 01^n only meet the upper branch line once and the must be to the left of the point where orbit for 0 meets this branch line. Therefore, these orbits will never cross. The second case is similar.

For claim (c), if two closed orbits traverse the same half twisted band, they will have positive linking number. Let α and β be two closed orbits that are not 01, 0 or 1. Suppose that α traverses the upper half twisted band and never the lower and that β traverses the lower half twisted band an never the upper. (Thus both are horseshoe knots since they can be presented in $L(0,1)$.) Consider the right most point r of the intersection of α and the lower branch line. If every point where β meets the lower branch line is to the right of r then either β never meets the upper branch line (thus β is the orbit y, which is ruled out), or it will be forced to traverse the upper half twisted band since the forward flow line from r must next meet the upper branch line to the right of its midpoint. Thus β meets the lower branch line to the left of r and hence crosses under α. ☐

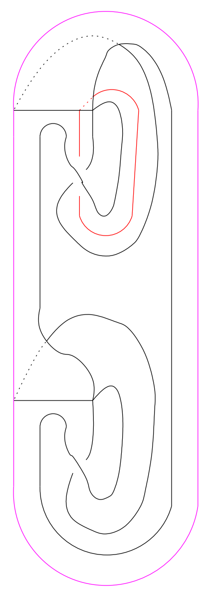

FIGURE 5. Orbits 01 and 1

Since links in $L(-1,-1)$ are positive, we know that the figure-8 knot, the Whitehead link and the Borromean rings are excluded. The next result shows that the five-knot is also excluded.

Theorem 2.2. *Knots and links in $L(-1,-1)$ are fibered.*

We review some facts about fibered knots and links. Let L be a link in S^3 and let F is a Seifert surface of L. We define a *push off map*, $\mu : F \to S^3 - F$ as follows. Regard S^3 and $\mathbb{R}^3 \cup \{\infty\}$ and assume F is away from ∞ so we can use vector addition in $\mathbb{R}^3$. We can assume that F is smooth and choose a unit normal vector $N(p)$ for each $p \in F$. We can assume $N(p)$ is smooth and there exists an $\epsilon > 0$ such that $\{\epsilon N(p)+p \,|\, p \in F\}$ is a diffeomorphic copy of F that does not intersect F. We define $\mu(p) = \epsilon N(p) + p$.

A push off map $\mu : F \to S^3 - F$ induces a homomorphism of the fundamental groups,

$$\mu_* : \pi_1(F, b) \to \pi_1(S^3 - F, \mu(b)).$$

The following three theorems are well known.

Theorem 2.3. [4] *The Seifert surface produced by Seifert's algorithm on a positive link diagram has minimal genus.*

Theorem 2.4. [1] *If F has minimal genus then μ_* is injective.*

Theorem 2.5. [15] *If μ_* is an isomorphism, then L is fibered.*

Proof of Theorem 2.2. The proof is similar to Stalling's original proof that positive braids are fibered. [15, 1] We consider a link L in H. Following Figure 7 we can construct a Seifert surface F by gluing together a collection of disks and bands with half twists. The only difference with Figure 5.2 of [1] is that one disk will have bands connecting it with two other disks. This surface will have minimal genus by Theorem 2.3. By Theorems 2.5 and 2.4 it only remains to show that μ_* is onto.

Referring to Figure 6 we define three numbers n_1, n_2 and n_3 as follows. Let n_1 be the number of stands of L coming from upper branch line that go directly to the lower branch line; let n_2 be the number of strands of L that wrap around the upper half twisted band; and let n_3 be the number of strands of L that wrap around the lower half twisted band. For the example in Figure 6 we have $n_1 = 5$, $n_2 = 3$ and $n_3 = 4$.

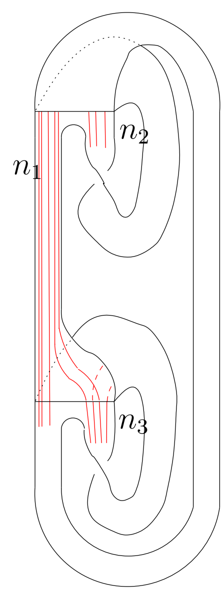

FIGURE 6. n_i, $i = 1, 2, 3$, defined

We switch now and work with the set of links that can be constructed with disks and strips with half twist in the manner of Figure 7. This is larger than the set of links supported by H. Call this set of links DS.

We deal with some trivial cases. If $n_1+n_2+n_3 = 1$ the L is an unknot and is thus fibered. If either n_2 or n_3 is zero, then L is a horseshoe knot and hence fibered. If n_1 is zero, then n_2 and n_3 can only be zero or one. Thus, L is either an unknot or an unlink of two components. Thus, L is fibered.

From now on we assume each n_i is positive. If they are all equal to one, then L is an unknot and hence fibered. We proceed by induction. Suppose μ_* is onto for $n_i \in \{1, 2, 3, \ldots, N_i\}$, for $i = 1, 2, 3$ where each $N_i \geq 1$.

Let L be a link in DS for which $1 \leq n_1 \leq N_1$, $1 \leq n_3 \leq N_3$, and $n_2 = N_2+1$. Let L' and F' be a link and Seifert with $n_i = N_i$, $i = 1, 2, 3$. Then $\mu_* : \pi_1(F') \to \pi_1(S^3 - F')$ is onto. A Seifert surface F for L can be obtained from F' by attaching some twisted strips to D_{n_2} to a new disk D_{n_2+1} as shown in Figure 8. If the number of new twisted strips is exactly one, then the link has not changed. If $k > 1$ new strip are added $k-1$ new generators for F' and $S^3 - F'$ are created. We have

$$\pi_1(F) \cong \pi_1(F') * \mathbb{Z}^{k-1}.$$

Because each of the k twisted strips has the same twist each of the new generators of $\pi_1(F)$ is mapped onto a new generator of $\pi_1(S^3 - F)$ with no new relations. See Figure 8. Thus,

$$\pi_1(S^3 - F) \cong \pi_1(S^3 - F') * \mathbb{Z}^{k-1}$$

and μ_* is onto. The same argument works if we increased N_1 or N_3 instead of N_2. □

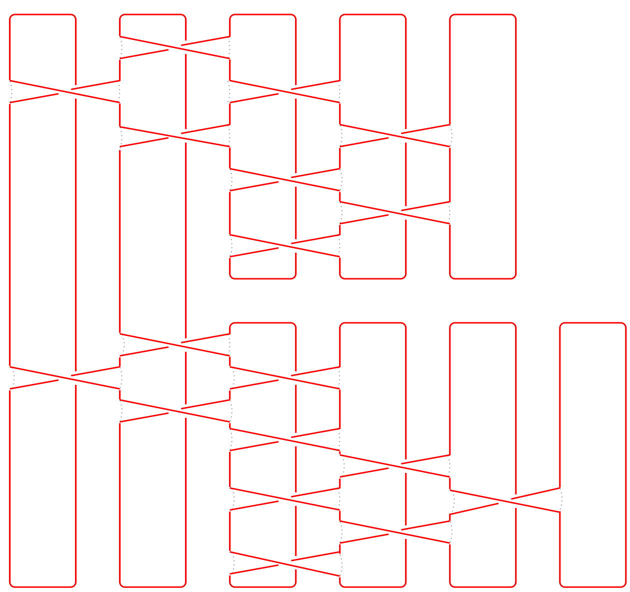

FIGURE 7. DS

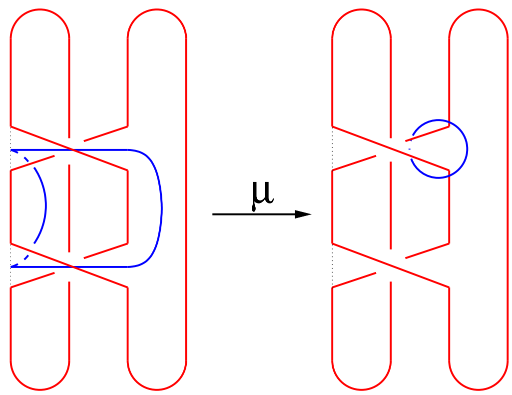

FIGURE 8. μ^* takes generators to generators

There are 2977 prime knots with crossing number 12 or less. There are 33 prime knots with crossing number 12 or less that are fibered and known to be positive. In addition, there are seven prime knots with crossing number 12 that are fibered and whose positivity statuses are unknown. We also note that no prime positive fibered knot with crossing number 12 is alternating. [2] Thus, while it is known that any template supports infinitely many distinct knot types [5] the collection of prime knots in $L(-1,-1)$ seems rather narrow.

The proof of Theorem 2.2 can be adapted to prove the following generalization.

Theorem 2.6. *Consider a tree with N vertices and K edges embedded in $\mathbb{R}^2$. Each edge is labeled with a nonzero integer $M_1, M_2, \ldots, M_K$. Replace each vertex with a disk in $\mathbb{R}^2$ that are disjoint. Now in $\mathbb{R}^2 \times [-\epsilon, \epsilon]$ replace each edge, I, with $|M_I|$ half twisted bands, with the crossings the same sign as M_I, between the disks corresponding to the vertices of edge I. This is to be done without the bands crossing themselves or each other; that is the projection back into $\mathbb{R}^2$ will have one crossing for each half twist and no others. The boundary of the resulting complex is a link whose only crossings are at the half twists in the bands. Such a link is fibered.*

Theorem 2.7. *Let L be a link in H. Let μ be the number of components, c be the number of crossings and let n_1, n_2 and n_3 be defined as earlier. The genus is denoted by g and we let r be the rank of $\pi_1(S^2-L)$. Then*

$$g = \frac{c - n_1 - n_2 - n_3 - \mu + 2}{2}, \tag{2}$$

$$r = c - n_1 - n_2 - n_3 + 1. \tag{3}$$

Proof. The second formula follows from the first since $r = 2g + \mu - 1$.

Let F be the minimal Seifert surface for L as constructed before. We can put F in to a *disk with strips model surface* M [11]; see Figure 9. Then M is the union of a central disk D and $2g + \mu - 1$ strips. A homeomorphism takes F to M. The disk D is the image of

$$D_1 \cup \cdots D_{n_1} \cup D'_1 \cup \cdots \cup D_{n_2} \cup D''_1 \cup \cdots D''_{n_3} \cup$$

$$\{n_1 + n_2 - 1 \text{ twisted bands along the top of Figure 7}\} \cup$$

$$\{n_3 \text{ twisted bands along the lower right of Figure 7.}\}$$

The remaining $c - (n_1 + n_2 + n_3 - 1)$ twisted bands in F will be mapped to the strips of M. Thus,

$$2g + (\mu - 1) = c - (n_1 + n_2 + n_3 - 1).$$

The first equation follows. □

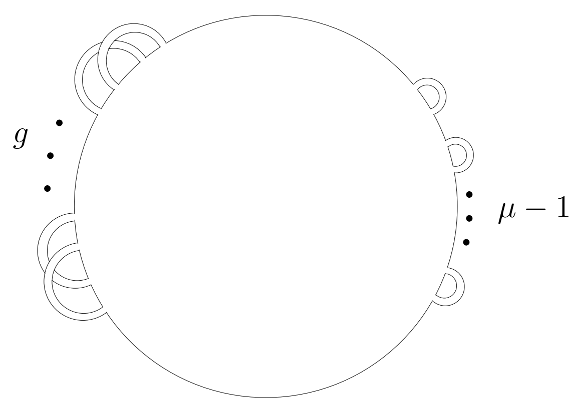

FIGURE 9. Disk with strips model of a Seifert surface

Theorem 2.8. *Let K be the knot type of a closed orbit in $L(-1,-1)$. There are infinitely many closed orbits in $L(-1,-1)$ of the same knot type.*

Proof. We work with H. Let w be a minimal word in 0 and 1 for a closed orbit O_w in with knot type K. Cyclic permutations of w do not change the orbit it represents. We can thus assume that the first letter of w represents the left most intersection point of O_w with the top branch line or the bottom branch line if O_w misses the top branch line. We also assume w is not 01. This insures there is a positive distance between this point and the left edge.

Then for any $n > 0$ the word $w' = (01)^n w$ represents an orbit $O_{w'}$ in H. By inspection we see in Figure 10 that after a series of R_1 Reidemeister moves the knot type of $O_{w'}$ is also K.

The exclusion of $w = 01$ does not cause a problem because there are other closed orbits that are unknotted, for example $w = 0$. Hence, there are infinitely many copies of the unknot in H. □

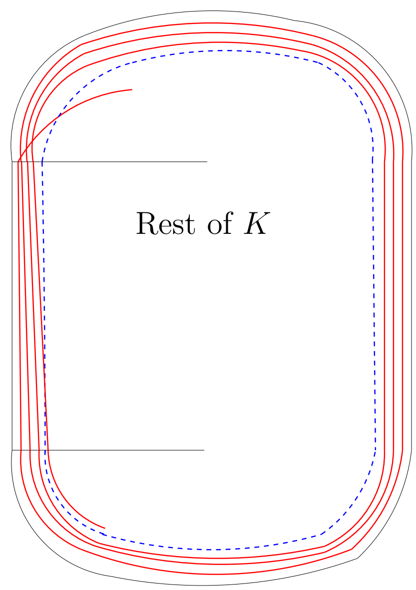

FIGURE 10. A knot K with some redundant strands

This results also holds for the template $L(0, m)$ - use 0^n instead of $(01)^n$.

For an attractor for the Lorenz equations, the template model is a subset of what we are calling $L(0, 0)$. In particular the orbits for 0 and 1 are not realized. It is also the case for the attractor in Robinson's equations that they are a subset of the orbits on the $L(-1, -1)$ template. However, the 01 orbit is always realized as a periodic orbit so Theorem 2.8 holds for the attractor in Robinson's equations.

3. A ZETA FUNCTION

For a map $f : X \to X$ one can define a formal power series

$$\sum_{n=1}^{\infty} \frac{F_n s^n}{n},$$

where F_n is the cardinally of the fix point set of the n-th iterate of f. When f is a diffeomorphism with a hyperbolic chain recurrent set then each F_n is finite, the series has a positive radius of conference and the exponential of the limit is a rational function called the *zeta function* of f.

$$\zeta(s) = \exp\left(\sum_{n=1}^{\infty} \frac{F_n s^n}{n}\right).$$

See, for example, [14].

For a topological flow the period of a closed orbit is not invariant. We can study first return maps on cross sections, but the period of a closed orbit will in general depend on the choice of the cross section. One might attempt to circumvent this by tracking closed orbits according to knot theoretic invariants. Williams has constructed a type of zeta function to distinguish among different Lorenz attractors (that arise for different parameter values) - in certain cases. [19] But as we have seen in some templates each realized knot-type has infinitely many realizations. In [16] this is circumvented by tracking the twisting in the local stable manifolds of closed orbits - we visualize these as ribbons. However, this only worked for templates that could be presented as positive braids. The template in Figure 11, where the crossings near the upper branch line are negative and while the crossings near the lower branch line are positive, contains infinitely many untwisted unknots. Here we show that although $L(-1,-1)$ cannot be presented as a positive braid it is still possible to define a rational zeta function tracking closed orbit according to their twisting.

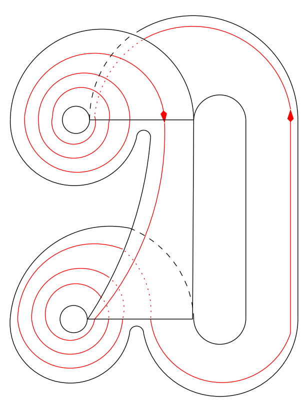

FIGURE 11. Template with infinitely many untwisted unknots

The notion of twisting we will use is not the standard one. A ribbon which has a braid presentation such that each crossing of one strand over another is positive and each twist in each strand is positive, will be called a *positive ribbon.* The core and boundary of a positive ribbon are positive braids. We will use the following notation. If R is a braided ribbon, let c be the sum of the crossing numbers of the core of R, using $+1$ for positive crossings and -1 for negative ones. Let t be the sum of the half twists in the strands of R and let n be the number of strands of the core. Then as in [16] we define the *usual twist*, the *modified twist* and the *computed twist*, respectively, by

$$\tau_u = c + t/2, \qquad \tau_m = n - 1 + t/2, \qquad \tau_c = 2n + t.$$

Notice $\tau_c = 2\tau_m + 2$. It is clear that τ_u is an isotopy invariant since it is just the linking number of the boundary components when the ribbon is an annuls and half the linking number of the boundary and the core of the ribbon when it is a Möbius band. However, τ_c and τ_m are not isotopy invariants of ribbons in general but are only invariant if the final presentation is also a positive braid. [16] Since our orbits are not presented as positive braids τ_m and τ_c must be defined differently than in [16].

Definition 3.1. Let w_1 and w_2 be braids (open) with $n_1 + n_2$ and $n_1 + n_3$ strands respectively. Place them so their regular projection into the xy-plane is disjoint and aligned as in Figure 12. Form a link by attaching the n_1 lower left most strands of w_1 to the n_1 upper left most strands of w_2, the n_1 lower left most strands of w_2 to the n_1 upper left most strands of w_1, attach the n_2 lower right most strands of w_1 to the n_2 upper right most strands of w_1, and the n_3 lower right most stands of w_2 to the n_3 upper right most strands of w_2. All this is to be done without creating any new crossings as in Figure 12. A link or a knot that has such a projection is said to be *double braided*. Likewise, we can define a ribbon to be double braided.

All knots on H are double braided. We now redefine τ_c and τ_m for positive double braided ribbons by using $n = n_1 + n_2 + n_3$ in the previous formulas. Now we can define our zeta function.

Definition 3.2. For a positive double braided template let $T_{q'}$ be the number of closed orbits with computed twist $\tau_c = q'$. Let $\mathcal{T}_q = \Sigma_{q'|q} q' T_{q'}$. Then

$$\zeta(t) = \exp\left(\sum_{q=2}^{\infty} \frac{\mathcal{T}_q t^q}{q}\right).$$

The $\mathcal{T}_q$ are finite by the same argument as Lemma 4.2 of [16]. Clearly $\zeta(t)$, as a formal expression, is invariant under isotopy and the two template moves. We will show that it is a rational function, actually the reciprocal of a polynomial in t, that is easy to compute.

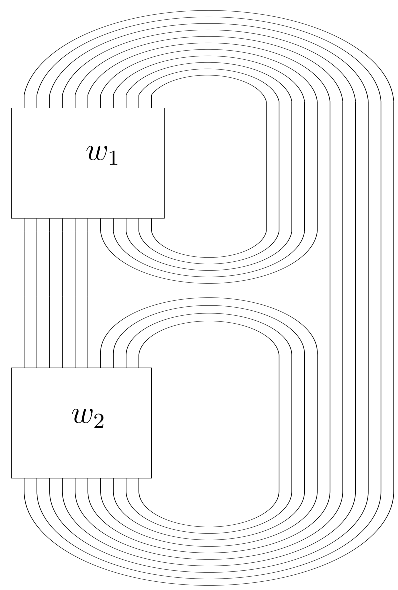

FIGURE 12. A double braid

We define a *twist matrix* of a positive double braided template with respect to a given Markov partition as follows. First we select a Markov partition consisting of line segments transfer to the template's flow each cutting completely across a branch. If the is no forward flowline from segment i to segment j we let $A_{ij} = 0$. If there is a flowline from i to j we let $A_{ij} = t^q$ where q is the number of half twists in the flowline's local unstable manifold. The template can always be isotoped so that each q is an integer. Each closed orbit has a period with respect to the given Markov partition. The diagonal entries of A^n represent closed orbits of period n with the powers of t giving their twist. A different choice of the Markov partition changes the periods of the orbits, but not their twists.

Figure 13 gives a Markov partition for H. Then, we have $A_H(t) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ t^3 & t^3 & 0 & 0 \\ t^2 & t^2 & 0 & 0 \\ 0 & 0 & t^3 & t^3 \end{bmatrix}$.

Theorem 3.3. $\zeta(t) = \dfrac{1}{\det(I - A(t))}$.

The proof is exactly the same as the proof of Theorem 5.2 in [16]. Thus,

$$\zeta_H(t) = t^6 - 2t^3 - t^2 + 1.$$

Suppose we add to H an extra half twist to the right band coming from the lower branch line and call this template H'. Then

$$A_{H'}(t) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ t^3 & t^3 & 0 & 0 \\ t^2 & t^2 & 0 & 0 \\ 0 & 0 & t^4 & t^4 \end{bmatrix}$$

and

$$\zeta_{H'}(t) = t^7 - t^4 - t^3 - t^2 + 1.$$

Thus, they are not equivalent. For $L(m,n)$ with m and n nonnegative we get

$$\zeta_{mn}(t) = -t^{m+2} - t^{n+2} + 1.$$

Thus, neither H or H' is equivalent to $L(m,n)$ nonnegative m and n. We knew this already for H because it positive knots that are not positive braids.

We believe the methods here can be extended to all positive templates. It would be interesting to know if one can determine if a given positive template is or is not equivalent to some positive braid template using this zeta function.

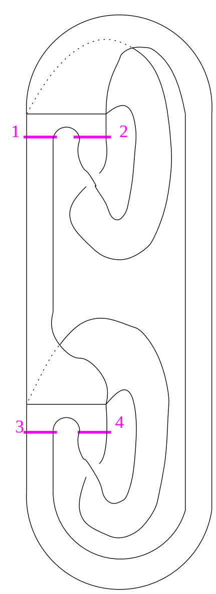

FIGURE 13. Markov partition for H

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