Get Ready for Math Grad School

UNIT TWO
Algebra

Purpose. This unit is intended as a review of the material covered of MATH 221, 302 and 319. Here are their catalog descriptions.

> **MATH 221.** *Introduction to Linear Algebra.* Vector spaces, linear functions, systems of equations, dimensions, determinants, eigenvalues, quadratic forms.
> **MATH 302.** *Mathematical Communication and the Transition to Higher Mathematics.* A course in communicating mathematical ideas with a special emphasis on reading, writing, and critiquing mathematical proofs. Topics covered include logic, proofs, set theory, relations, functions. Additional illustratory topics will be drawn from linear algebra, number theory, complex variables, and geometry.
> **MATH 319.** *Introduction to Abstract Algebra I.* Basic properties of groups and rings: Binary operations, groups, subgroups, permutations, cyclic groups, isomorphisms, Cayley's theorem, direct products, cosets, normal subgroups, factor groups, homomorphisms, rings, integral domains.

Instructions. Work the problems below. All answers are to be written out in complete and correct English sentences. Type or write neatly. Each problem should be on a separate sheet of paper with your name on it. State the problem and then give your solution.

You will likely need to have a linear algebra and an abstract algebra textbook on hand. If you did not save your undergraduate textbooks, we have some recommendations of below. Also listed below are links to "handouts" you may find helpful.

Turn in your work to the Get Ready instructor by email or conventional mail. You do not need to send them in all together. If you have done some and want feedback send those in. If you are stuck on a problem for more than two days, contact the instructor for help.

Recommended Textbooks.

- *Linear Algebra*, by Jim Hefferon.
  Free at: `http://joshua.smcvt.edu/linearalgebra/`
- *Elementary Linear Algebra*, by Ron Larson, any edition.
- *Linear Algebra*, by Bronson, any edition.

- *A First Course in Abstract Algebra*, by John B. Fraleigh, any edition.
- *Groups and Symmetry*, by M. A. Armstrong.
- *Abstract Algebra: Theory and Applications*, by Judson and Beezer. Free at: `http://abstract.ups.edu/download.html`
- *Chapter Zero: Fundamental Notions of Abstract Mathematics*, by Carol Schumacher, any edition.
- *Reading, Writing, and Proving: A Closer Look at Mathematics*, by Daepp & Gorkin, any edition.

Handouts. (See links on course webpage.)

- Sets and Functions
- Algebraic Systems
- Vector Spaces

The Problems.

(1) Let S be a nonempty set and let $\sim$ be an equivalence relation on S. Prove that the equivalence classes are disjoint.

(2) Let B and A_i, for $i = 1, 2, 3, ...$, be sets. Prove that

$$B - \left(\bigcap_{i=1}^{\infty} A_i \right) = \bigcup_{i=1}^{\infty} (B - A_i).$$

(3) Use the Product Rule and Mathematical Induction to prove that $(x^n)' = nx^{n-1}$, for $n = 1, 2, 3, ...$

(4) Consider a list of n numbered objects: $\{O_1, O_2, \dots, O_n\}$. We are allowed to switch adjacent entrees. Suppose we wish to switch entries i and j, $i \neq j$. Prove that an odd number of adjacent switches is required. (An application of this is that when we switch two rows or columns of a square matrix of real numbers the determinant changes sign.)

(5) Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ be a polynomial with real coefficients and leading coefficient equal to one. Let $r_1, r_2, \dots, r_n$ be the n roots of $p(x) = 0$, counting repeated roots. Prove that $|a_0| = |r_1 r_2 \cdots r_n|$.

(6) The integers $\mathbb{Z}$ form a group under addition. What are all its subgroups?

(7) Let $U(1)$ be a circle of radius one and center the origin in the complex plane. Prove that $U(1)$ is a group under complex multiplication.

(8) Let $SL(2)$ be the set of 2×2 matrices of real numbers that have determinant equal to one. Prove this a group under matrix multiplication.

(9) Let $G_1 = (\mathbb{R}, +)$ and $G_2 = ((0, \infty), \times)$. Show these are groups. Let $\exp : G_1 \to G_2$ be given by $\exp(x) = e^x$. Show that this is a group isomorphism.

(10) Prove that every permutation of a finite set is the product of its cycles.

(11) Let G be a group of order 9. Prove it is Abelian.

(12) Let G be a group of order 15. Prove it is cyclic.

(13) (a) Let $h : G_1 \to G_2$ be group homomorphism. Prove that h is injective (one-to-one) if and only if the kernel of h is the trivial group. (b) Let M be an $n \times n$ matrix of real numbers. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation induced by M, that is $T(\mathbf{v}) = M\mathbf{v}$. Show that T is injective if and only if the nullspace of M consists of just the origin.

(14) Let H and K be normal subgroups of the group G, with K a subgroup of H. Prove that H/K is a normal subgroup of G/K.

(15) The ring $\mathbb{Z}/8\mathbb{Z}$ is the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ with addition and multiplication modulo 8. Does the equation $6 = 4 \times X$ have a solution?

(16) Let $R = \{a + bt \,|\, a, b \in \mathbb{R}\}$. Define addition by $(a_1 + b_1 t) + (a_2 + b_2 t) = (a_1 + a_2) + (b_1 + b_2)t$ and multiplication by $(a_1 + b_1 t) \times (a_2 + b_2 t) = (a_1 a_2 + b_1 b_2) + (a_1 b_2 + a_2 b_1)t$. [This is just standard multiplication mod $t^2 = 1$.] Then R is a ring, but is it an integral domain?

(17) Prove that every field is an integral domain.

(18) Let V be a vector space and let U and W be subspaces. Prove that $U \cap W$ is a vector space. Give an example showing $U \cup W$ need not be a vector space.

(19) What are all the vector subspaces of $\mathbb{R}^3$?

(20) Show that the solution set of the differential equation

$$y'' + y = 0$$

is a vector space.

(21) Find the eigenvalues and a basis for each eigenspace of $\begin{bmatrix} 3 & 5 & 3 \\ 1 & 7 & 3 \\ 1 & 2 & 8 \end{bmatrix}$.

(22) Diagonalize $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

(23) Apply the Gram-Schmidt orthonormalization process to

$$\{[0,1,0,1],[2,2,1,1],[0,1,1,0]\}.$$