Get Ready for Math Grad School

UNIT ONE
Calculus

Purpose. This unit is intended as a review of the material covered at SIUC in MATH 150, 250, and some of 251 and 305. Here are their catalog descriptions.

> **MATH 150.** *Calculus I.* Major concepts and techniques of single variable calculus with careful statements but few proofs. Differential and integral calculus of the elementary functions; analytic geometry.
> **MATH 250.** *Calculus II.* Develops the techniques of single-variable calculus begun in Calculus I and extends the concepts of function, limit, derivative and integral to functions of more than one variable. The treatment is intuitive, as in Calculus I. Techniques of integration, introduction to multivariate calculus, elements of infinite series.
> **MATH 251.** *Calculus III.* Further topics in calculus. Definite integrals over solid regions, applications of partial derivatives, vectors and vector operations, derivatives of vector functions, line integrals, Green's Theorem.
> **MATH 305.** *Introduction to Differential Equations.* First-order equations (including initial value problems, basic numerical methods, existence and uniqueness of solutions, separable equations, linear equations, exact equations, substitution methods and applications). Higher-order equations (including the general solution to homogeneous linear equations, linear independence, method of undetermined coefficients, the general solution to linear non-homogeneous equations, variation of parameters, and applications). Power series solutions. Partial differential equations and Fourier series.

Instructions. There are two sets of problems. The first are calculations. The answers are given so you can check your work as you go. These are not to be turned in.

The problems in the second set are to be written out in complete and correct English sentences. Type or write neatly. Each problem should be on a separate sheet of paper with your name on it. State the problem and then have your solution. Turn in you work to the Get Ready instructor by email or conventional mail. You do not need to send them all in together. If you have done some and want feedback send those in. If you are stuck on a problem for more than two days, contact the instructor for help.

You will likely need to have a calculus textbook book on hand. Some recommendations are below. But, whatever textbook you had for comparable courses as an undergrad should be fine.

Recommended Textbooks.

- *Calculus with Early Transcendental Functions*, by Stewart, any edition.
- *Calculus with Analytic Geometry*, by Edwards & Penny, any edition.
- *Calculus of One and Several Variables*, by Salas, Hille, & Anderson, any edition.
- *Elementary Differential Equations and Boundary Value Problems*, by Boyce & DiPrima, any edition.

Problem Set I: Not to be turned in.

(1) $\displaystyle\lim_{t\to-\frac{\pi}{2}^+}\frac{1+\sin t}{\cos t}$

(2) $\displaystyle\lim_{\theta\to 0^-}\cot\theta.$

(3) $\displaystyle\lim_{x\to\infty}\cos(\arctan x).$

(4) $\displaystyle\lim_{x\to 0}\frac{x+4}{x^2-16}.$

(5) $\displaystyle\lim_{x\to-\infty}\frac{4e^x+8}{3e^x+7}.$

(6) $\displaystyle\lim_{x\to\infty}\sqrt{16x^2+x}-4x.$

(7) $\displaystyle\lim_{x\to 2}e^{\frac{1}{x-2}}.$

(8) $\displaystyle\lim_{x\to 0^+}x\ln x.$

(9) $(x\csc x^3)'.$

(10) $(\ln(x^2+1))'.$

(11) $(x^{\cos x})'.$

(12) $(\arcsin x)'.$

(13) $\displaystyle\frac{\partial(x^2y^3+\sin(xy))}{\partial x}.$

(14) Let $f(x,y) = x^3y+\ln(xy)+y^2\cos(xy)$. Find f_{xy} and f_{yx}. Compare them.

(15) $\displaystyle\int\frac{1}{x(x^2+1)}+\frac{1}{x\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}}\,dx.$

(16) $\displaystyle\int\frac{\rho}{\rho^2-1}+\frac{1}{\rho^2-1}\,d\rho.$

(17) $\displaystyle\int\sec q+\ln q+q\tanh q^2\,dq.$

(18) $\displaystyle\int\frac{1}{1+\theta^2}+\frac{\theta}{1+\theta^2}\,d\theta.$

(19) $\displaystyle\int e^{2x}+xe^{x^2}+x^2e^x\,dx.$

(20) $\displaystyle\int\sec^3 x\,dx$

(21) Evaluate $\int_C y^3\,ds$ where the curve C in the xy-plane is given by $\mathbf{r}(t)=\langle t^3,t\rangle$ for $0\le t\le 2$.

(22) Find the surface area of the portion of the graph of $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.
(23) Find $f(x)$ such that $f'(x) = \cos^2 x$ and $f(0) = 3$.
(24) Find $y(x)$ such that $y' + 2xy = x$ and $y(0) = 2$.
(25) Find the solution set of $y'' + y = 0$.
(26) Find the solution set of $y'' + 4y' + 4y = 0$.
(27) Determine whether each series below converges or diverges; if it converges find its limit.

a. $\displaystyle\sum_{p=2}^{\infty} \frac{1}{p \ln p}$ b. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n(n+1)}$

c. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n$ d. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

(28) Determine whether each series below converges absolutely, conditionally or diverges.

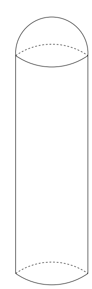
a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

c. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}$ d. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

Problem Set II: To be turned in.

(1) A 13 ft ladder is leaning against a house when the base starts to slide away. At the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. (a) How fast is the top of the ladder sliding down the wall then? (b) At what rate is the area of the triangle formed by the ladder, wall and ground changing then?

(2) A silo (base not included) is placed on an existing concrete pad with a hemispherical top. It costs \$10/ft^2 to construct the hemisphere portion and \$5/ft^2 to construct the cylindrical walls. Determine the dimensions if the total volume is 9π ft^3 to keep the cost to a minimum.



(3) Find a formula for the volume inside the ellipsoid determined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(4) Suppose $\sqrt{x^2 + y^2} = x + \cos y$. Regard y as an implicit function of x and find dy/dx.

(5) A student is asked to find $\displaystyle\int_0^1 \sin(n\pi x)\sin(5\pi x)\,dx$, where $n = 1, 2, 3, \ldots$. He mindlessly types the integral into a computer and

reports that the answer is

$$-\frac{\sin(n\pi)}{\pi(n^2-25)}.$$

Help him.

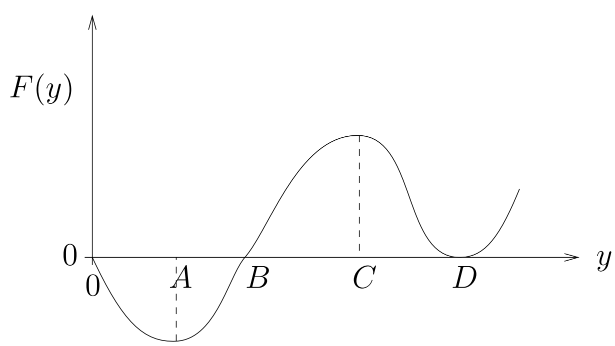
(6) Find the Taylor series of e^{x^2}, centered about 0. What is the radius of convergence?

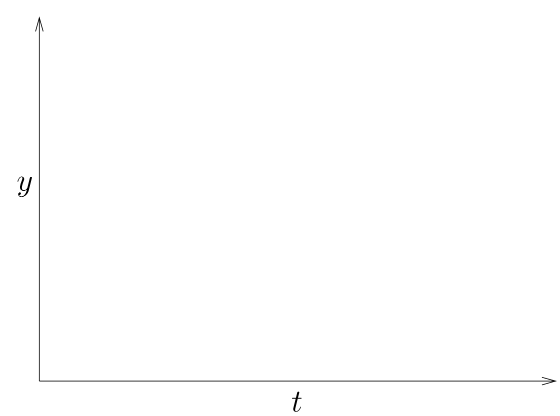
(7) Find the Taylor series of $\arctan x^3$, centered about 0. What is the radius of convergence?

(8) Prove that $\lim_{x\to\infty} \frac{x}{(\ln x)^n} = \infty$ for all integers $n > 0$.

(9) Let S be a smooth surface with boundary curve C in $\mathbb{R}^3$. Assume that f and g are functions from $\mathbb{R}^3$ to $\mathbb{R}$ that have continuous second-order partial derivatives. Prove that the work done by the vector field $(f \bigtriangledown g)$ in going once around C is equal to the flux through S of the vector field $\bigtriangledown f \times \bigtriangledown g$. Assume the orientation of S is consistent with the direction along C.

(10) Suppose $y'(t) = F(y(t))$, where the graph of $F(y)$ is given below. Carefully draw several solution curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $y(t)$ and t are non-negative.





Answers to Problem Set I.

1. 0; 2. $-\infty$; 3. 0; 4. $\frac{1}{4}$; 5. $\frac{8}{7}$; 6. $\frac{1}{8}$; 7. 1; 8. 0; 9. $\csc x^3 - 3x^3 \csc x^3 \cot x^3$; 10. $\frac{2x}{x^2+1}$; 11. $(x^{\cos x})(\cos x - x \sin x)$; 12. $1/(1-x^2)$; 13. $2xy^3 + y\cos(xy)$; 14. Both are $3x^2 - 3y^2 \sin(xy) - y^3 x \cos(xy)$; 15. $(-1/2)\ln(1+1/x^2) - \sqrt{1+1/x^2} + \sqrt{x^2+1} + C$; 16. $\ln|\rho - 1| + C$; 17. $\ln|\sec q + \tan q| - q + q\ln q + \frac{1}{2}\ln(\cosh(q^2)) + C$; 18. $\arctan\theta + \frac{1}{2}\ln(1+\theta^2) + C$; 19. $\frac{1}{2}e^{2x} + \frac{1}{2}e^{x^2} + (x^2 - 2x + 2)e^x + C$; 20. $(\sec x + \tan x + \ln|\sec x + \tan x|)/2 + C$; 21. $(145^{\frac{3}{2}} - 1)/54 \approx 32.315$; 22. $2\pi(2\sqrt{2}-1)/3 \approx 3.829$; 23. $x/2 + \sin(2x)/4 + 3$; 24. $(1+3e^{-x^2})/2$; 25. $y(x) = C_1 \cos x + C_2 \sin x$; 26. $y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$; 27. a. div, b. 1/4, c. 1/6, d. div; 28. a. cc, b. ac, c. ac, d. div.