Get Ready for Math Grad School

UNIT THREE
Real Analysis

Purpose. This unit is intended as a review of the material covered in MATH 352 at SIUC. Here is the catalog description:

> **MATH 352.** *Theory of Calculus.* An introduction to understanding and writing proofs in mathematical analysis, through a careful study of limits, continuity, the derivative, and the integral.

You may have taken a similar course at another institution. If you have never had such a course, this unit will be over your head.

Instructions. Work the problems below. All answers are to be written out in complete and correct English sentences. Type or write neatly. Each problem should be on a separate sheet of paper with your name on it. State the problem and then give your solution.

You will likely need to have a calculus textbook and one or two introductory real analysis books on hand. Some recommendations are below. But, whatever textbook you had for a comparable course as an undergraduate should be fine.

Turn in you work to the Get Ready instructor by email or conventional mail. You do not need to send them all in together. If you have done some and want feedback send those in. If you are stuck on a problem for more than two days, contact the instructor for help.

Recommended Textbooks.

- *Calculus with Early Transcendental Functions*, by Stewart, any edition.
- *Calculus with Analytic Geometry*, by Edwards & Penny, any edition.
- *Calculus of One and Several Variables*, by Salas, Hille, & Anderson, any edition.
- *Understanding Analysis*, by Abbott, any edition.
- *Elementary Analysis: Theory of Calculus*, by Ross, any edition.
- *Limits: A New Approach to Real Analysis*, by Beardon, any edition.

The Problems.

(1) Prove that $\sqrt{3}$ is not a rational number.
(2) Prove that the set of real numbers is uncountable.
(3) Give a formal ϵ-δ style proof that $\lim_{x\to 2} 3x + 5 = 11$.

(4) Give a formal ϵ-δ style proof that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

(5) Define $f(x)$ to be $x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and 0 for $x = 0$. Is f differentiable at $x = 0$? If so, is its derivative continuous at $x = 0$? Prove your claims.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that if f has a local maximum or minimum at c then $f'(c) = 0$.

(7) Use the Intermediate Value Theorem, the Mean Value Theorem, and other facts, to prove that the function

$$f(x) = 4x^5 + x^3 + 2x + 1$$

has one and only one real zero.

(8) Prove that the sum of the first n positive odd integers is equal to n^2.

(9) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \, \Delta x$, where $x_i = 1 + i/n$ and $\Delta x = 1/n$. Do this directly without using integration.

(10) Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin(x_i))\Delta x$, where $x_i = i\pi/n$ and $\Delta x = \pi/n$, as a definite integral, then evaluate it.

(11) Prove that $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ when $|r| < 1$.

(12) Find the intervals of convergence for the two series below.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$

(13) Evaluate $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. Hint: Taylor series.

(14) Let $f_n(x) = x^n$ on the interval $[0, 1]$. What is the limit of f_n as $n \to \infty$? Is the convergence uniform? Explain.

(15) Let

$$q(x) = \begin{cases} 0 & \text{for } x \text{ rational} \\ 1 & \text{for } x \text{ irrational} \end{cases}$$

Is $q(x)$ Riemann integrable on $[0, 1]$? Explain.