1 Vector Spaces in $\mathbb{R}^n$

Definition 1.1. A nonempty subset V of some $\mathbb{R}^n$ is a **Vector Space** if these two properties hold:

1. If $\mathbf{v}$ and $\mathbf{w}$ are both in V then their vector sum $\mathbf{v}+\mathbf{w}$ is in V.
2. If $\mathbf{v} \in V$ and $r \in \mathbb{R}$ then their scalar product $r\mathbf{v}$ is in V.

These are called **closure axioms**. The idea is that when we do these operations in V we stay within V. Thus, we can think of V as a closed universe, a *space* onto itself.

Example 1. The solution set P of $y = x^2$ is not a vector space. *Proof:* $(1,1) \in P$ and $2 \in \mathbb{R}$ but $2(1,1) = (2,2)$ is not in P.

Example 2. The unit circle in $\mathbb{R}^2$, denoted U, is not a vector space. *Proof:* The points $(1,0)$ and $(-1,0)$ are both in U, but $(1,0)+(-1,0) = (0,0)$ is not in U.

Example 3. Any line in $\mathbb{R}^2$ that goes through the origin is a vector space.

Proof 1. Let L be the points of a line passing through $(0,0)$. We can let $Ax + By = 0$ be an equation whose solution set is L. Suppose (a,b) and (c,d) are on L and let $r \in \mathbb{R}$. We must show that $(a+c, b+d)$ and (ra, rb) are on L. We do this by checking that they satisfy the equation $Ax+By = 0$.

$$A(a+c) + B(b+d) = Aa + Ac + Bb + Bd =$$

$$(Aa + Bb) + (Ac + Bd) = 0 + 0 = 0.$$

And,

$$A(ra) + B(rb) = r(Aa + Bb) = r0 = 0.$$

Thus, the vector space axioms are satisfied. □

Proof 2. Let L be the points of a line passing through $(0,0)$. Then L has a parametric equation of the form

$$\langle x(t), y(t)\rangle = \mathbf{v}t,$$

where $\mathbf{v}$ is a vector in $\mathbb{R}^2$. Let $r \in \mathbb{R}$ and let (x_1, y_1) and (x_2, y_2) be any two distinct points in L. Thus, there exist real numbers, t_1 and t_2, such that $\langle x_1, y_1\rangle = \mathbf{v}t_1$ and $\langle x_2, y_2\rangle = \mathbf{v}t_2$. Then $r\langle x_1, y_1\rangle = \mathbf{v}rt_1$ and $\langle x_1 + x_2, y_1 + y_2\rangle = \mathbf{v}(t_1 + t_2)$. Hence, L is closed under vector addition and scalar multiplication. □

Remark. Vector spaces are sometimes called *linear spaces*.

Problem 1. Prove that the solution set of $y = 2x + 1$ fails to be a vector space.

Problem 2. Prove that any line passing through the origin of $\mathbb{R}^3$ is a vector space.

Problem 3. Prove that any plane passing through the origin of $\mathbb{R}^3$ is a vector space.

Problem 4. Prove that the plane with equation $x+y+z = 1$ is not a vector space. (Do not use the Fact below.)

Fact. Every vector space contains the origin. *Proof:* Let V be a vector space. Since a vector space is nonempty we can pick a $\mathbf{v} \in V$. Then $0\mathbf{v} = \mathbf{0}$, so the origin, $\mathbf{0}$, is in V.

Problem 5. Prove that the subset of $\mathbb{R}^n$ containing just the origin, $\{\mathbf{0}\} = \{(0, \ldots, 0)\}$, is a vector space.

Problem 6. Prove that the solution set of $A\mathbf{x} = \mathbf{0}$ is a vector space for any $m \times n$ matrix A, but that the solution set of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \neq \mathbf{0}$ is not. (Remember, the empty set is not a vector space.) Note: a system of the form $A\mathbf{x} = \mathbf{0}$ is called **homogeneous** and its solution set is called the **null space** of the matrix A.

Problem 7 (Subtle)**.** Show that Problems 2 and 3 are can be viewed as special cases of problem 6. What about Problems 1 and 4?

Example 4 (For class discussion)**.** What are all of the vector spaces in $\mathbb{R}^3$? What about $\mathbb{R}^n$?

Problem 8. Let V and W be vector spaces in some $\mathbb{R}^n$.
a) Prove that $V \cap W$ is always a vector space.
b) Give an example showing that $V \cup W$ need not be a vector space.

Example 5. Let $\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^n$. Let $W = \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \,|\, r_1, r_2 \in \mathbb{R}\}$. Then W is a vector space. (W is called the **span** of $\{\mathbf{v}_1, \mathbf{v}_2\}$.)

Proof. Every element of W is a vector in R^n. Let $\mathbf{u}$ and $\mathbf{v}$ be in W and let r be any real number. We can write $u = a\mathbf{v}_1 + b\mathbf{v}_2$ and $v = c\mathbf{v}_1 + d\mathbf{v}_2$. Then $r\mathbf{u} = ra\mathbf{v}_1 + rb\mathbf{v}_2$ which is in W since ra and rb are real numbers. Also, $\mathbf{u} + \mathbf{v} = (a + c)\mathbf{v}_1 + (b + d)\mathbf{v}_2$ is in W. Thus, W is a vector space. ☐

Example 6. A set of matrices can sometimes be thought of as a vector space. For example the set of 3×3 matrices is really just $\mathbb{R}^9$, since matrix addition behaves like vector addition and multiplying a scalar and a matrix is just like multiplying a scalar and a vector. However, when thinking of matrices in this way we are ignoring matrix multiplication; it is not defined for vector spaces.

Example 7. Consider the set of 3×3 symmetric matrices. The sum of two symmetric matrices is still symmetric and scalar multiplication also preserves the symmetry. We can think of this set as a 6 dimensional vector space sitting inside of $\mathbb{R}^9$. Why?

2 Abstract Vector Spaces

It often happens that a set that is not a subset of any $\mathbb{R}^n$ still has properties very much like those of a vector space.

Example 1. Adding polynomials is a lot like adding vectors and we can think of multiplying a polynomial by a real constant as an analog of scalar multiplication. Let C be the set of cubic polynomials and let P_3 be the set of all polynomials of degree 3 or less. If $p(x) = x^3 + 2$ and $q(x) = x - x^3$ then $p(x) + q(x)$ is not a cubic polynomial. Thus, C does not behave like a vector space. But P_3 does act very much like a vector space. It is "closed" under polynomial addition and multiplication by scalars. (Although, P_3 is not "closed" when we multiply one polynomial by another.)

Example 2. The solution set S of the differential equation $y'' = -y$, that is the set of all real functions $f(x)$ such that $f''(x) = -f(x)$, is a like a vector space. If f and g are both in S and $r \in \mathbb{R}$ then the reader should be able to check that $f(x)+g(x)$ and $rf(x)$ are also in S. Thus, the algebraic structure of S is much like that of a vector space. We challenge the reader to figure out what the set S is.

Example 3. The set of convergent infinite series can be thought as vector space. Suppose $r \in \mathbb{R}$ and that $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$ both converge to finite limits. Then, it is shown in calculus that $\Sigma_{i=1}^{\infty} a_i + b_i$ and $\Sigma_{i=1}^{\infty} r a_i$ both converge to finite limits.

Definition 2.1. An **Abstract Vector Space** is a nonempty set V together with two binary operations called **vector addition** ($V \times V \to V$) and **scalar multiplication** ($\mathbb{R} \times V \to V$) that obey the following axioms.

I. Closure axioms[1]:

   a. Vector addition is closed: $\mathbf{v} + \mathbf{w} \in V$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.

   b. Scalar multiplication is closed: $r\mathbf{v} \in V$ for all $\mathbf{v} \in V$ and $r \in \mathbb{R}$.

II. Algebraic axioms:

   a. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, for all $\mathbf{v}, \mathbf{w} \in V$. (commutativity)

   b. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$, for all $\mathbf{v}, \mathbf{w}, \mathbf{u} \in V$. (associativity)

   c. There exists $\mathbf{z} \in V$, such that $\mathbf{z} + \mathbf{v} = \mathbf{v}$, for all $\mathbf{v} \in V$. (additive identity)

   d. For each $\mathbf{v} \in V$, there exists $\bar{\mathbf{v}} \in V$ such that $\bar{\mathbf{v}} + \mathbf{v} = \mathbf{z}$. (additive inverses)

   e. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$, for all $r \in \mathbb{R}, \mathbf{v}, \mathbf{w} \in V$. (distributivity)

   f. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$, for all $r, s \in \mathbb{R}, \mathbf{v} \in V$. (distributivity)

   g. $r(s\mathbf{v}) = (rs)\mathbf{v}$, for all $r, s \in \mathbb{R}$, and $\mathbf{v} \in V$. (associativity of scalar multiplication)

   h. $1\mathbf{v} = \mathbf{v}$, for all $\mathbf{v} \in V$ (scalar multiplicative identity)

Abstract vector spaces are often referred to simply as vector spaces when no confusion can arise. The list of algebraic axioms may seem quite cumbersome. However, they are very natural and you should soon get used to them.

Some Quick Facts: The following properties will be derived for vector spaces from the above axioms and the axioms of arithmetic:

a. A vector space has only one zero element. *Proof:* Suppose $\mathbf{z}$ and $\mathbf{u}$ are both zero elements of a vector space. Then $\mathbf{u} = \mathbf{z} + \mathbf{u} = \mathbf{u} + \mathbf{z} = \mathbf{z}$. So, $\mathbf{u}$ and $\mathbf{z}$ must be the same.

[1]Strictly speaking the closure axioms are implicit in the definition of a binary operation. However, we have chosen that state them explicitly because of their importance in the theory of vector spaces.

- $\diamond$ From now on we shall denote the zero element by $\mathbf{0}$ and refer to it as the unique **zero element**.

b. For any $\mathbf{v}$, $0\mathbf{v}$ is the zero element. *Proof:* $\mathbf{v} = 1\mathbf{v} = (1+0)\mathbf{v} = 1\mathbf{v} + 0\mathbf{v} = \mathbf{v} + 0\mathbf{v}$. Now $\mathbf{v}$ has an additive inverse $\bar{\mathbf{v}}$. Add this to both sides: $\mathbf{v}+\bar{\mathbf{v}} = \mathbf{v}+\bar{\mathbf{v}}+0\mathbf{v}$. This gives $\mathbf{0} = \mathbf{0}+0\mathbf{v}$, which implies $\mathbf{0} = 0\mathbf{v}$.

c. For any $r \in \mathbb{R}$, $r\mathbf{0} = \mathbf{0}$. *Proof:* **Problem 1**.

d. If $r\mathbf{v} = \mathbf{0}$, then either $r = 0$ or $\mathbf{v} = \mathbf{0}$. *Proof:* **Problem 2**.

e. Additive inverses are unique. *Proof:* Let $\mathbf{v}$ be in a vector space. Suppose that $\bar{\mathbf{v}}$ and $\tilde{\mathbf{v}}$ are both inverses of $\mathbf{v}$. Then $\bar{\mathbf{v}} = \bar{\mathbf{v}}+\mathbf{0} = \bar{\mathbf{v}}+\mathbf{v}+\tilde{\mathbf{v}} = \mathbf{v}+\bar{\mathbf{v}}+\tilde{\mathbf{v}} = \mathbf{0}+\tilde{\mathbf{v}} = \tilde{\mathbf{v}}$. So, $\bar{\mathbf{v}}$ and $\tilde{\mathbf{v}}$ are the same.

f. For any $\mathbf{v}$, $-1\mathbf{v} = \bar{\mathbf{v}}$. *Proof:* $\mathbf{v}+-1\mathbf{v} = 1\mathbf{v}+-1\mathbf{v} = (1-1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$. Thus, $-1\mathbf{v} = \bar{\mathbf{v}}$ by uniqueness.

- $\diamond$ From now on we shall denote the additive inverse of $\mathbf{v}$ by $-\mathbf{v}$.

The forgoing may strike the reader as unnecessary. Are not these results obvious? The answer is that they are not. Further, the mental discipline needed to construct the proofs is of value in itself. Thus, you may be tested on these proofs. It may also seem that the last axiom, IIh, is obvious. But in fact it does not follow from the previous axioms as you will show in Problem 5 below.

Example 4. The set P_3 of polynomials of degree three or less, is a vector space. We shall check each axiom of Definition 2. We do this in excruciating detail for the record; normally one would combine many of the obvious steps. Do not be shocked if you find a couple of typos.

Ia. $(ax^3+bx^2+cx+d)+(ex^3+fx^2+gx+h) = (a+e)x^3+(b+f)x^2+(c+g)x+(d+h) \in P_3$

Ib. $r(ax^3+bx^2+cx+d) = rax^3+rbx^2+rcx+rd \in P_3$

IIa. $(ax^3+bx^2+cx+d)+(ex^3+fx^2+gx+h) = (a+e)x^3+(b+f)x^2+(c+g)x+(d+h) = (e+a)x^3+(f+b)x^2+(g+c)x+(h+d) = (ex^3+fx^2+gx+h)+(ax^3+bx^2+cx+d)$

IIb. $[(ax^3+bx^2+cx+d)+(ex^3+fx^2+gx+h)]+(ix^3+jx^2+kx+l) = ([a+e]+i)x^3+([b+e]+j)x^2+([c+g]+k)x+([d+h]+l) = (a+[e+i])x^3+(b+[e+j])x^2+(c+[g+k])x+(d+[h+l]) = (ax^3+bx^2+cx+d)+[(ex^3+fx^2+gx+h)+(ix^3+jx^2+kx+l)]$

IIc. Use $0 = 0x^3+0x^2+0x+0 \in P_3$ as the zero element.

IId. Given $\mathbf{v} \in P_3$ use -1 times $\mathbf{v}$ as the inverse.

IIe. $r[(ax^3+bx^2+cx+d)+(ex^3+fx^2+gx+h)] = r[(a+e)x^3+(b+f)x^2+(c+g)x+(d+h)] = r(a+e)x^3+r(b+f)x^2+r(c+g)x+r(d+h) = (ra+re)x^3+(rb+rf)x^2+(rc+rg)x+(rd+rh) = (rax^3+rbx^2+rcx+rd)+(rex^3+rfx^2+rgx+rh) = r(ax^3+bx^2+cx+d)+r(ex^3+fx^2+gx+h)$

IIf. $(r+s)(ax^3+bx^2+cx+d) = (r+s)ax^3+(r+s)bx^2+(r+s)cx+(r+s)d = (ra+sa)x^3+(rb+sb)x^2+(rc+sc)x+(rd+sd) = (rax^3+rbx^2+rcx+rd)+(sax^3+sbx^2+scx+sd) = r(ax^3+bx^2+cx+d)+s(ax^3+bx^2+cx+d)$

IIg. $r(s(ax^3+bx^2+cx+d)) = r(sax^3+sbx^2+scx+sd) = r(sa)x^3+r(sb)x^2+r(sc)x+r(sd) = (rs)ax^3+(rs)bx^2+(rs)cx+(rs)d = (rs)(ax^3+bx^2+cx+d)$

IIh. $1(ax^3+bx^2+cx+d) = ax^3+bx^2+cx+d$

Problem 3. a) Prove that the set in Example 2 is vector space.
b) Prove that the set in Example 3 is vector space.

Problem 4. Let Z be the set of continuous functions that are zero at zero. Let T be the set of continuous functions that are two at zero. It each case we define vector addition and scalar multiplication in the obvious way. Convince yourself that Z is indeed a vector space. However, T is not a vector space. For example, T is not closed under vector addition or scalar multiplication. For each axiom of Definition 2 either give an example showing it fails for T or prove that it does hold for T. (Note: To prove T is not a vector space it is enough to find one example where one axiom fails.)

Problem 5. Consider the set $\mathbb{R}^2$ with the usual vector addition but with scalar multiplication defined as follows: $r(x, y) = (rx, 0)$. Show that axioms Ia-b and IIa-g hold but that IIh is false.

Example 5 (For class discussion)**.** Which of the following sets of functions do you think are vector spaces? Use the usual definition of the addition of functions and of multiplying a function by a number for vector addition and scalar multiplication respectively.

1. The set F of all functions from $\mathbb{R}$ to $\mathbb{R}$.
2. The set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
3. The set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$.
4. The set E of all even functions from $\mathbb{R}$ to $\mathbb{R}$.
5. The set O of all odd functions from $\mathbb{R}$ to $\mathbb{R}$.
6. What about $E \cup O$ and $E \cap O$?
7. Let $T > 0$. The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ with period T.
8. The set of all positive functions from $\mathbb{R}$ to $\mathbb{R}$.
9. Let $[a, b]$ be an interval on the real line. The set of all functions from $[a, b]$ into $\mathbb{R}$ with $\int_a^b f(x)\,dx = 0$.
10. Let $[a, b]$ be a interval on the real line. The set of all functions from $[a, b]$ into $\mathbb{R}$ with $\int_a^b f(x)\,dx = 1$.
11. The set of all functions $y = f(x)$ such that $2y' + y = 0$. This set is called the **solution set** of the **differential equation**, $2y' + y = 0$.
12. The solution set of $2y' + y = 7$.
13. The solution set of $y'' = x^2$. What is this solution set?

3 Subspaces

Definition 3.1. Let V be a vector space and let W be a nonempty subset of V. If W also has a vector space structure (using the same operations as on V) then we say W is a *subspace* of V.

Theorem 3.2. *If W is a nonempty subset of a vector space V then W is a subspace of V if the two closure axioms of Definition 2 hold.*

Before giving a proof we shall do an application.

Example 1. The set P_3 of polynomials of degree three or less is a subset of the set of functions, which is a vector space. Using Theorem 3.2 we only need to check axioms Ia and Ib to see the P_3 is a vector space.

Proof of Theorem 3.2. Let $W \subset V$, where V is a vector space. Assume axioms Ia and Ib hold for W and that W is not empty. We need to check axioms IIa-h.

IIa. Since IIa holds for all vectors in V, it holds for all vectors in W.

IIb. The same argument works.

IIc. Take any element $\mathbf{w} \in W$. Then by Ib $0\mathbf{w} \in W$. Now, $\mathbf{0} = 0\mathbf{w}$. Thus $\mathbf{0} \in W$. For any $\mathbf{v} \in V$ and $\mathbf{0} + \mathbf{v} = \mathbf{v}$ so $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in W$. Hence $\mathbf{0}$ serves as the zero element of A.

IId. Similar to IIc.

IIe. Same as IIa.

IIf. Same as IIa.

IIg. Same as IIa.

IIh. Same as IIa.

□

Problem 1. Prove whether or not each of the sets below is a vector space.

a. The closed interval $[-2, 5]$ in $\mathbb{R}$.

b. The closed ray $[0, \infty)$ in $\mathbb{R}$.

c. $\{(x, y) \in \mathbb{R}^2 \mid |x| = |y|\}$

d. $\{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$

e. $\{(x, y) \in \mathbb{R}^2 \mid x + 3y = 0\}$

f. $\{(x, y) \in \mathbb{R}^2 \mid x + 3y = 4\}$

g. $\{(x, y, z) \in \mathbb{R}^3 \,|\, 3x + 2y - z = 0\}$

h. $\{(x, y, z) \in \mathbb{R}^3 \,|\, xy + z^3 = 0\}$

i. $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2yz^7 = 1\}$

j. $\{(w, x, y, z) \in \mathbb{R}^4 \,|\, w + 3x + 2y - z = 0\}$

k. $\{(w, x, y, z) \in \mathbb{R}^4 \,|\, w + 3x + 2y - z = 2\}$

l. $\{(w, x, y, z) \in \mathbb{R}^4 \,|\, w + x + y + z^2 = 0\}$

m. $\{(a, b, c, d, e, f, g) \in \mathbb{R}^7 \,|\, d = 0\}$

n. $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 0\}$

o. $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 4\}$

p. The set of functions $\{f : \mathbb{R} \to \mathbb{R} \,|\, f(3) = f(5) = 0\}$.

q. The set of functions $\{f : \mathbb{R} \to \mathbb{R} \,|\, f(3) = f(5)\}$.

r. The set of functions $\{f : \mathbb{R} \to \mathbb{R} \,|\, f(3) = f(5) = 2\}$.

s. The set of functions $\{f : \mathbb{R} \to \mathbb{R} \,|\, \text{for all } a \text{ and } x \text{ in } \mathbb{R}, f(ax) = af(x)\}$.

t. $\{a \sin x + b \cos x^2 + ce^x + dx^5 \,|\, \text{ for all real values of } a, b, c, \&d\}$.

u. The solution set of $y' + x^3y = 0$.

v. The solution set of $y'' + 3xy' + 4y = 0$.

w. The solution set of $y' + x^3y^2 = 0$. Hint: $y = 4/x^4$ is a solution.

x. The solution set of $y' + x^3y = 5$.

y. All functions from $(0, \infty)$ into $\mathbb{R}$ that can be written in the form $\ln(x^a)$ for some real constant a

z. All functions from $\mathbb{R}$ into $\mathbb{R}$ that can be written in the form $\sin(ax)$ for some real constant a.

α. The set of all noninvertible 2×2 matrices.

β. The set of all 4×4 diagonal matrices.

γ. The set of all 3×3 matrices such that $A^T = A$.

Answers: n, n, n, n, y, n, y, n, n, y, n, n, y, n, n, n, y, n, y, y, n, y, y, y, y, n, n, y, n, n, y, y, respectively.