A Summary of Franks' work on Nonsingular Smale Flows on S^3.

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In the late 70s and early 80s John Franks wrote a series of papers describing the structure of nonsingular Smale flows in S^3. Here we summarize the major results.

Suppose there is a single attracting closed orbit and a single repelling closed orbit. All other basic sets are saddle sets. Then we can compute the absolute value of the linking number of the attracting repelling pair as follows. Suppose there are n saddle sets and that the full Parry-Sullivan invariant of the i-th one is given by $a_i + tb_i$. Then the absolute value of the linking number is the product $|a_1 - b_1| \cdots |a_n - b_n|$. [**F77**]

For example, the template L(1,1) gives linking number 3.

If we know how the saddle sets link the attracting and repelling collection of closed orbits we can say more, namely we can compute the Alexander polynomial of this link. The information about how the i-th saddle links the attracting and repelling orbits in recorded in a linking matrix L_i. It is defined by multiplying the entries of a signed incidence matrix, called the structure matrix, by powers of variables $t_1, \ldots, t_m$, one variable for each attractor and one for each repeller. The powers reflect the linking of a loop using a standard base point and a segment connecting a pair of elements of the Markov partition of a cross section of the saddle set. Then the Alexander polynomial is given by $\det(I - L_1) \cdots \det(I - L_n)$. [**F81**]

APPLICATION: For the Lorenz template $|a - b| = 1$. Indeed we can realize an embedding where the attractor repeller pair is a Hopf link. However, it can be shown that if both loops are essential, then the Alexander polynomial is not that of a Hopf link. If such an embedding is realizable, the attractor and or the repeller must be knotted.

In [**F83**] the following are proved. If S is any structure matrix of a basic set, then there exist a nonsingular Smale flow f_t on some 3-manifold with a basic set Λ whose structure matrix is A and every other basic set of f_t consists of a single attracting or repelling closed orbit (Theorem 1). If there exists a nonsingular Smale follow on S^3 with basic set Λ with structure matrix S then there exits a nonsingular Smale flow of S^3 with a twist-wise flow equivalent basic set with all other basic sets being attracting or repelling closed orbits (Proposition 3.2). Furthermore, if $\det(I-S) \neq 0$ then the group $Z^n/(I-S)Z^n$ is cyclic (Theorem 3.3).

Thus,

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

cannot realized as the structure matrix of a basic set in a nonsingular Smale flow on S^3.

Finally, in [**F85**] we have an *abstract* classification of nonsingular Smale flows on S^3. The major new tool is the *Lyapunov graph*. Given a Smale flow on a manifold there exist a smooth function from the manifold to the reals which is non-increasing with respect to the flow parameter. Thus, each basic set goes to a point. This is called a Lyapunov function. The Lyapunov graph is defined by identifying connected components of the inverse images of points in the real line. Each vertex of the graph is a point whose connected component contained a basic set. A vertex is labeled by the basic set it is associated with. Edges are oriented by the flow direction.

Suppose Γ is an abstract Lyapunov graph whose sinks and sources are each labeled with a single attracting or repelling periodic orbit and suppose each remaining vertex is labeled with the suspension of a subshift of finite type. Then Γ is associated with a nonsingular Smale flow on S^3, if and only if the following are satisfied. (1) The graph Γ is a tree with one edge attached to each source and each sink vertex. (2) If v is a saddle vertex whose basic set has incidence matrix M and with e_v^+ entering edges and e_v^- exiting edges then

$$e_v^+ \leq Z_M + 1,$$

$$e_v^- \leq Z_M + 1,$$
$$Z_S + 1 \leq e_v^+ + e_v^-.$$

Here Z_M is a the *Zeeman number* defined by $\dim \ker((I - M_2) : Z_2^n \to Z_2^n)$, where M_2 is the mod 2 reduction of M, Z_2 is the integers mod 2, and n is the size of M.

Thus, if there is a single attracting closed orbit and a single repelling closed orbit $Z_M = 0$ or 1. The converse holds as well. Further, if $|a - b| = 1$ we know that the linking number is 1. But, we do not know whether or not they can or must form a Hopf link.

References

[**F77**] J. Franks, *Non-singular flows on S^3 with hyperbolic chain-recurrent set,* Rocky Mountain J. Math., Vol. 7, No. 3, Summer 1977, 539-546.

[**F81**] J. Franks, *Knots, links and symbolic dynamics*, Ann. of Math. **116** (1981), 279-294.

[**F83**] J. Franks, *Symbolic dynamics in flows on three-manifolds*, Trans. Amer. Math. Soc. **279** (1983), 231-236.

[**F85**] J. Franks. *Nonsingular Smale flows on S^3*, Topology, **24** (1985), 265-282.