TOKYO TECH LECTURE NOTES

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Abstract. These are rough working lecture notes. Expect typos!
Tuesdays 15:05 – 16:35
Fridays 13:20 – 14:50
January 8, 11, 15, 22, 25, 29, February 1, 5, 8.

1. Lecture 1 (January 8) : Introduction to Topology

1.1. **Continuity.** Topology is the abstract study of continuity. In analysis students learn that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = c$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in (c-\delta, c+\delta) \Rightarrow f(x) \in (f(c)-\epsilon, f(c)+\epsilon)$.

Exercise 1. Show that this is equivalent to the definition of continuity in most calculus courses, that f is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

Exercise 2. What would "go wrong" if we said ...$\exists \delta \geq 0$... instead of ...$\exists \delta > 0$... in the defintion of continuity? Which functions would be continuous?

We say f is continuous on a subset $D \subset \mathbb{R}$ if it is continuous at each point in D. Does anything go wrong if D is not an open set?

Let X and Y be sets and let $f : X \to Y$ be a function. The question we ask is, what are the simplest structures we need to impose on X and Y for the statement "$f : X \to Y$ is continuous" to be meaningful? Notice we could modify the definition of continuity as follows. Let $f : \mathbb{R} \to \mathbb{R}$. Then we say f is continuous at $x = c$ if for all open intervals I containing $f(c)$ there is an open interval J containing c such that $f(J) \subset I$.

Remark 1. We have not actually said what an open subset of the real line is. By definition a set $U \subset \mathbb{R}$ is open if and only if it is the union of open intervals. The empty set ϕ is regarded as open since it is a vacuous union. A subset $C \subset \mathbb{R}$ is closed if $\mathbb{R} - C$ is open. Thus $\mathbb{R}$ and ϕ are both open and closed.

Date: January 6, 2013.

Exercise 3. Show that this really is equivalent to the first definition we gave of continuity. These open intervals need not be symmetric about c or $f(c)$. Does that really matter? What would go wrong if we used closed sets, including single point sets, instead of open intervals? In fact we don't really need open intervals. We could let I and J be more general open sets. Right?

If we have a notion of which subsets of X and Y are "open" then we can generalize the definition of continuity. Let $\mathcal{O}$ be a collection of subsets of X. What properties should it have to earn the title of "the open subsets of X"? On $\mathbb{R}$ the open sets where defined as unions of open intervals. Thus any union of open sets is an open set. We shall require $\mathcal{O}$ to have this property. Another property of open sets in the real line is that finite intersections of open sets are open. We leave the proof to you. But notice infinite intersections of open sets need not be open:

$$\cap_{i=1}^{\infty}(-1/n, 1/n) = \{0\}$$

which is not open. But the finite intersection property is important and we shall require $\mathcal{O}$ have it. It is also useful to require that ϕ and X be in $\mathcal{O}$. This turns out to be all we need. These lead us to the following definitions.

Definition 1.1. A set X together with a collection of subsets $\mathcal{O}$ is a *topological space* if the following hold.

(1) X and ϕ are in $\mathcal{O}$.
(2) The union of any subcollection of $\mathcal{O}$ is in $\mathcal{O}$.
(3) The intersection of any finite subcollection of $\mathcal{O}$ is in $\mathcal{O}$.

Definition 1.2. Let $(X, \mathcal{O})$ and $(Y, \mathcal{U})$ be topological spaces. Let $f : X \to Y$. Then f is continuous at $x \in X$ if for every open V set of Y containing $f(x)$ there is an open set U of X containing x with $f(U) \subset V$.

Here is an alternative characterization of continuity.

Lemma 1.3. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous on X if and only if the inverse image of every open subset of Y is an open subset of X.*

Proof. Suppose f^{-1} takes open subsets to open subsets. Let $x \in X$ and let V be any open subset of Y containing $f(x)$. If $U = f^{-1}(V)$ is open we have that U is an open subset containing x that is mapped into V. Thus f is continuous at x and since x was arbitrary f is continuous on X.

Now suppose f is continuous for every $x \in X$. Let V be an open subset of Y and let $U = f^{-1}(V)$. We need to show U is open. Let $x \in U$. Then V is an open set containing $f(x)$. Thus there is a open subset $U_x \subset X$ containing x such that $f(U_x) \subset (V)$. We can see that $U_x \subset U$. Then U is the union of all the U_x for $x \in U$. Hence U is open. □

Example 1. Let $f(x) = x^2$ be a function from $\mathbb{R}$ to $\mathbb{R}$. Convince yourself f^{-1} takes open sets to open sets. But notice $f((-1,1)) = [0,1)$ is not open.

Example 2. We put four different topologies on the real line $\mathbb{R}$ and look at which functions are continuous. Let $\mathcal{U}$ be the usual open sets of $\mathbb{R}$. Let $\mathcal{T} = \{\phi, \mathbb{R}\}$. it is called the trivial topology. Let $\mathcal{D}$ = all subsets of $\mathbb{R}$. It is called the discrete topology. Finally let $\mathcal{F} = \{U \subset \mathbb{R} \,|\, \mathbb{R} - \mathcal{U} \text{ is finite}\}$. It is called the finite complement topology. The reader should check that each is a valid topology. Now consider the following.

- Any function from $(\mathbb{R}, \mathcal{D})$ to any topological space is continuous.
- Any constant function from $(\mathbb{R}, \mathcal{U})$ to $(\mathbb{R}, \mathcal{D})$ is continuous but $f(x) = x$ is not.
- Any function from any topological space to $(\mathbb{R}, \mathcal{T})$ is continuous.
- Suppose $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is continuous. Then $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{F})$ is continuous too, but $f : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{U})$ need not be.
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1.2. Connectedness.

Definition 1.4. A subset C of a topological space X is **disconnected** if there exists a pair of open sets U and V such that $U \cap C \neq \phi$, $V \cap C \neq \phi$, $U \cap V = \phi$ and $C \subset U \cup V$. If no such pair exists then C is **connected**.

Theorem 1.5. *It can be shown that the intervals of the real line are connected.*

Exercise 4. Prove that the continuous image of a connected space is connected.

Definition 1.6. Let $Q \subset X$. A **component** of Q is a connected subset C such that if we add one more point to C from anywhere in $X - C$ the result is disconnected. The number of components can be finite or infinite.

Definition 1.7. A topological space X is **path connected** if for any two points x and y in X there is a continuous function f (a **path**) from $[0,1]$ into X such that $f(0) = x$ and $f(1) = y$.

Exercise 5. Prove that a path connected space is connected. (The converse is false. Google "topologist's sine curve.")

Exercise 6. The space $\{1, 2, 3\}$ with the trivial topology is connected and even path connected. Prove this.

1.3. **Compactness.** A closed bounded interval $[a, b]$ of the real line has some important properties. Any continuous function $f : [a, b] \to \mathbb{R}$ has a maximum. That is there must be a number $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. There are several ways this can be generalized to other topological spaces. We will present the most common one which involves the study of open covers of a topological space. Here is an example.

Let $X = [0, 1)$. Then the collection $\mathcal{C} = \{[0, 1 - \frac{1}{n}) \,|\, n \geq 1\}$ is an **open covering** of X. Notice that no finite subcollection could cover X. Now consider $\mathbb{R}$ with open covering $\{(-n, n) \,|\, n \geq 1\}$. Again there is no finite subcover. But for I, or any closed bounded subset of $\mathbb{R}$, any open covering has a finite subcover. We won't prove this here, but try some examples on I. The converse also holds: if a subset of $\mathbb{R}$ is not both closed and bounded then there exists an open covering that does not have a finite subcover. This leads to the following definition.

Definition 1.8. A topological space X is **compact** if every open cover has a finite subcover.

Theorem 1.9. *Let $f : X \to Y$ be continuous. If X is compact so is its image in Y.*

Outline of proof. Take an open cover $\mathcal{V}$ of Y. Pull it back with f^{-1} to get an open cover of X. It has a finite subcover. Then use the corresponding subcover of $\mathcal{V}$ to get a finite open covering of Y. □

Exercise 7. Give an example of a function that takes a closed set to one that is not closed.

1.4. **Subspaces and Products.** Let $(X, \mathcal{O})$ be a topological space. Let Q be some subset of X. We define a topology on Q as follows. Let $\mathcal{Q} = \{U \cap Q \,|\, U \in \mathcal{O}\}$. In words, a subset of Q is open if it can formed as the intersection of an open subset of X with Q. It is easy to prove that this does give a topology on Q. It is called the **subspace topology**.

Example 3. Consider $I = [0, 1] \subset \mathbb{R}$. Then in the subspace topology the set $[0, 0.2)$ is open since $(-1, 0.2) \cap [0, 1] = [0, 0.2)$. Now we can talk about a function on I being continuous or not at the end points. In calculus one typically use limits from the left and right to define continuity at end points of closed intervals. For example $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Suppose we wish to study continuous functions on $X \times Y$. We need a topology for $X \times Y$. At first we might try $\mathcal{B} = \{U \times V \,|\, U \in \mathcal{X} \,\&\, V \in \mathcal{Y}\}$. But this does not quite work. For example consider $\mathbb{R} \times \mathbb{R}$ and the open unit disk $\{(x, y) \,|\, x^2 + y^2 < 1\}$. Surely we want the open disk to be open. Yet it cannot be written in the form $U \times V$ for open subsets of $\mathbb{R}$. Also $\mathcal{B}$ is not closed under even finite unions:

$$(0, 2)^2 \cup (1, 3)^2 \notin \mathcal{B}.$$

(Although $\mathcal{B}$ is closed under finite intersections.) The right definition is this: let $\mathcal{Z}$ be all the possible unions of members of $\mathcal{B}$. Then one can prove that $\mathcal{Z}$ does give a topological structure for $X \times Y$. It is called the **product topology**.

Exercise 8. Let $Q = I \times I$ be the unit square in $\mathbb{R}^2$. We can define a topology for Q is two ways. First, use the product topology of the subspace topologies each of I. Second, use the subspace topology on Q as a subspace of $\mathbb{R} \times \mathbb{R}$ with the product topology. Convince yourself these are the same.

1.5. **Homeomorphisms.** Now we use the idea of continuity to talk about when two topological spaces are "essentially the same". This is similar to the isomorphism problem in algebra. Let X and Y be topological spaces. Suppose $h : X \to Y$ has the following properties: it is one-to-one, onto, continuous and h^{-1} is continuous. Then h is called a **homeomorphism** and we say X and Y are **homeomorphic** or **topologically equivalent**. If X is homeomorphic to Y we may write $X \approx Y$.

The major problem in topology is given two topological spaces how can we determine whether or not they are topologically equivalent. First we consider finite sets. If X has m elements and Y has n elements then there cannot be a bijection between them unless $m = n$. This is just counting.

Let $X = \{1, 2, 3\}$. We will put three different topologies on X.

- $\mathcal{T}_1 = \{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$.
- $\mathcal{T}_2 = \{\phi, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$.
- $\mathcal{T}_3 = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$.

Check that these are topologies. If we define $h : X \to X$ by $f(1) = 3$, $f(2) = 1$ and $f(3) = 2$ then you can check that h is a homeomorphism from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$. But there is no such homeomorphism from

$(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_3)$. Here is a proof. Suppose $k : X \to X$ was such a homeomorphism. Then $k^{-1}(\{1\})$ must be a one element member of $\mathcal{T}_1$. Therefore $k^{-1}(1) = 1$. But by the same reasoning we must have $k^{-1}(2) = 1$. Thus k cannot exist.

Notice $\mathcal{T}_1$ has 4 members while $\mathcal{T}_3$ has 5. In general, if two topologies on a finite set are homeomorphic then the number of open sets must be the same in each.

Exercise 9. Let $X_n = \{1, 2, 3, \ldots, n\}$. How many topological structures can X_n have? How many are topologically distinct? That is how many topological equivalence classes are there for X_n? This is probably hard. Try working it out for $n = 3$, 4 and 5.

Exercise 10. Let X and Y be topological spaces. Suppose m is the number of components of X and n is the number of components of Y. If $X \approx Y$ show that $m = n$.

1.6. **Cut points.** Is $[0, 1)$ homeomorphic to $(0, 1)$? Suppose $h : [0, 1) \to (0, 1)$ is a homeomorphism. If we restrict the domain of h to $(0, 1)$ and call this k then $k : (0, 1) \to (0, 1) - \{\{h(0)\}\}$ is a homeomorphism. Check this. But $(0, 1)$ has just one component and its image has two. Contradiction.

Definition 1.10. Let X be a topological space. Let $p \in X$ and let C be the component that contains p; of course it could be that $C = X$. If $C - p$ in the subspace topology is not connected we say that p is a **cut point** of C (or X).

In our example the point $0 \in [0, 1)$ is not a cut point but since every point of $(0, 1)$ is a cut point we derived a contraction. We will do another example.

Example 4. we define three sets in $\mathbb{R}^2$ and give each the subspace topology. Let $A = [-1, 1] \times \{0\}$, $B = \{(x, y) \,|\, x^2 + y^2 = 1$ and $C = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$. Show no two of these are homeomorphic.

Solution. Suppose $h : A \to B$ is a homeomorphism. Let $A' = A - \{(0, 0)\}$, $B' = B - \{h(0, 0)\}$ and let h' be the restriction of h to A'. But this is impossible since A' has two components and B' has only one no matter where $h(0, 0)$ is. A similar argument shows A is not homeomorphic to C. (How would you prove that B or C with one point deleted is still connected?)

Now suppose $g : B \to C$ is a homeomorphism. Let $B' = B - \{(1, 0), (-1, 0)\}$, let $C' = C - \{g(1, 0), g(-1, 0)\}$ and let g' be the restriction of g to B'. But now g' would be a homeomorphism from a space with two components to a connected space. ☐

There are limits to this method. We can distinguish between $\mathbb{R}$ and $\mathbb{R}^n$ for any $n > 1$ but not between $\mathbb{R}^2$ and $\mathbb{R}^3$ since both remain connected when a finite number of points are deleted.

1.7. **Manifolds.** An n**-dimensional manifold without boundary** M is a topological space such that for each point $x \in M$ there exists an open set containing x that is homeomorphic to an open ball in $\mathbb{R}^n$. (We may assume the homeomorphism is to the open unit ball centered at the origin and takes x to the origin.) If there are points y in M for which this fails but where there is a subset H of M containing y and a homeomorphism

$$h : H \to \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1^2 + x_2^2 + \cdots + x_n^2 < 1 \,\text{and}\, x_1 \geq 0\}$$

taking y to the origin, then M is a n**-dimensional manifold with boundary**. Such points y form the **boundary** of M which is denoted ∂M. The **interior** of M is $\text{int}(M) = M - \partial M$. [1]

Here are some standard examples of manifolds. The unit interval $I = [0, 1]$. A **circle** or $1 - sphere$, also denoted S^1 is any sphere homeomorphic to $\{(x, y) \,|\, x^2 + y^2 = 1\}$. A **2-disk**, D^2, is a any space homeomorphic to the closed unit disk $\{(x, y) \,|\, x^2 + y^2 \geq 1$. A **2-sphere**, S^2, is a any space homeomorphic to the unit sphere in $\mathbb{R}^3$, $\{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\}$. A **closed 3-ball**, B^3, is a any space homeomorphic to the closed unit ball in $\mathbb{R}^3$, $\{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 1\}$. A **3-sphere**, S^3, is any space homeomorphic to $\{(w, x, y, z) \,|\, w^2 + x^2 + y^2 + z^2 = 1\}$ as a subspace of $\mathbb{R}^4$. The torus T^2 is $S^1 \times S^1$. The spaces $I = [0, 1]$, D^2 and B^3 have boundary, $\partial I = \{0, 1\}$, $\partial B^3 = S^2$ and $\partial D^2 = S^1$. The n-spheres and the torus do not have boundary. All of these spaces are path connected.

1.8. **Gluing, Connected Sums and Compactification.** See Section 4 of TS paper.

1.9. **Knots.** See Section 3 of paper.

1.10. **Homology.** Let $O = \{(0, 0)\}$, $I = [0, 1] \times \{0\}$ and let T be the triangle shaped disk in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$. Given a manifold M we define three group called **chain groups**. Consider first the set S_0 of all maps from O into M. These are called 0-dimensional simplices and can be thought of as just the points of M. Then let C_0 be all formal sums of the form $n_1 p_1 + \cdots + n_k p_k$ where each $p_i \in S_0$

[1] Manifolds are also assumed to be Hausdorff and second countable.

and each $n_i \in \mathbb{Z}$. We add elements of C_0 is the obvious way and let 0 stand for the null symbol. For example,

$$(2p_1+3p_2-p_3)+(p_1-3p_2+2p_5) = 3p_1+0-p_3+2p_5 = 3p_1-p_3+2p_5.$$

With this C_0 becomes a group. We also declare that $P+Q=Q+P$ for any two members of C_0 so that C_0 is Abelian. We call C_0 the group of 0-dimensional chains or the **0-chain group**.

Next let S_1 be the set of all continuous maps of I into M. These are called 1-simplices. We define C_1, the **1-chain group**, just as we did the 0-chain group but using linear combinations of 1-simplices. Finally define S_2 by, you guessed it, the set of all continuous maps from T into M and the define C_2 like we did before. It is called the **2-chain group**.

The three chains groups, by themselves do not tell us much about the manifold. But the interactions between them will. We define two group homomorphisms $\partial_1 : C_1 \to C_0$ and $\partial_2 : C_2 \to C_1$, called the **boundary maps**, as follows. Let $s \in S_1$. Define $\partial_1(s) = s(1) - s(0)$, that is the two end points thought of as a 0-chain. We can linearly extend this set map from S_1 into C_0 to a homomorphism from C_1 into C_0. For example, let s and t be in S_1. Then

$$\partial(2s+3t) = 2s(1) - 2s(0) + 3t(1) - 3t(0).$$

Why the minus sign? Suppose s and t are in S_1 and that $s(1) = t(0)$. Then $\partial(s+t) = t(1) - s(0)$ which seems "natural".

Defining ∂_2 is a little harder. The disk T has three line segments that make up its boundary. Each can be thought of as a copy of I. Let $z \in S_2$. We can think of each restriction of z to a boundary segment of T as a 1-simplex. Our definition of $\partial_2 z$ will be some linear combination of three boundary 1-simplices. The coefficients will be ± 1. To get the signs "right" will take a little effort. First we represent z by the image of T's vertices in order. Suppose $z(0,0) = a$, $z(1,0) = b$ and $z(0,1) = c$. Then we would write $[abc]$ for z. Obviously many other other 2-simplices would have the same symbol. This will not hurt anything. Now for the 1-simplex given by z restricted to the edge from $(0,0)$ to $(1,0)$ write $[ab]$. Define $[bc]$ and $[ac]$ similarly. Then we define $\partial_2 z = \partial_2[abc] = [bc] - [ac] + [ab]$. Now watch!

$$\partial_1\partial_2(z) = \partial_1([bc] - [ac] + [ab]) = (c-b) - (c-a) + (b-a) = 0.$$

So, the boundary of the boundary is empty. Now extend ∂_2 linearly to get a homomorphism from C_2 into C_1.

Since $\partial_1\partial_2 Z = 0$ for any $Z \in C_2$ (convince yourself of this) we know the image of ∂_2 is contained in the kernel of ∂_1; in fact it is a subgroup. The kernel of ∂_1 is called the subgroup of **1-cycles**. The image of ∂_2 is called the subgroup of **1-boundries**, that is they are 1-cycles that are the boundary of some 2-chain. The first homology group of a manifold M is defined to be

$$H_1(M) = \frac{\text{kernel}\,\partial_1}{\text{image}\,\partial_2}.$$

It contains a wealth of information about M. One can go on to define the n-chain group for any n, the boundary maps $\partial_n : C_n \to C_{n-1}$ and then define the n-th homology groups by $H_n(M) = \text{kernel}\,\partial_n/\text{image}\,\partial_{n+1}$. We will only need $H_1(M)$.

Next we find the first homology groups of some of the manifolds we have looked at. We won't be rigorous here. I want to focus on the intuition for simplicity and because it is the basic intuitive idea that led to the definition of the homology groups. Start with the disk D^2. Draw some segments that form a cycle with no self crossings. Notice they are the boundary for a smaller disk. Thus in $H_1(D^2)$ that 1-cycle is in the same equivalence class as the identity. Cycles in D^2 can be very complicated and have many self crossings. But the algebra works out such that they are all the boundary of some 2-chain. Thus, $H_1(D^2) = 0$, the trivial group with only the identity element. For the 2-sphere we get the same result. So far homology is not too interesting!

Now consider an annulus A. It is easy to draw a 1-cycle that goes around the hole and thus does not bound any 2-chain. Figure 1.

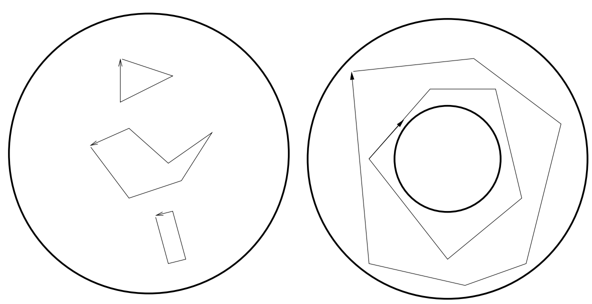

FIGURE 1. 1-cycles in a disk and an annulus

Therefore $H_1(A)$ is not the trivial group. If we draw two non-intersecting 1-cycles around the hole then together they form the boundary of a smaller annulus which is the image of some 2-chain. Thus these two 1-cycles are in the same homology class and are said to be **homologous**. Consider the 1-cycle formed by taking one of the 1-cycles in Figure 1 and assigning a weight of 5 to each of the 1-simplices. We can think of it as representing walking around the annulus five times. It still has boundary 0 and there is no 2-chain whose boundary it could be. It turns out the homology class of 1-cycles that "go around once clockwise" generate $H_1(A)$. Thus it can be shown that it is isomorphic to $\mathbb{Z}$.

If we take a disk and cut out two disjoint smaller disks then $H_1 \approx \mathbb{Z}^2$, and so on. Thus H_1 is a kind of hole counter. Now consider the torus T^2. How many holes does it have? It might seem like it has only one, but in a sense it has two. The donut hole in the middle but also the "hollowed out" interior. It turns out we can generate the homology group with two classes: the equivalence class of a 1-cycle going around the "donut hole" is one and a 1-chain going around meridian of the tube is the other. Thus, $H_1(T^2) \approx \mathbb{Z}^2$.

Exercise 11. It turns out $H_1(S^1) \approx \mathbb{Z}$. Justify this.

Exercise 12. When we punched a hole in the disk the homology grouped changed from trivial to one isomorphic to $\mathbb{Z}$. This is because 1-cycles that where 1-boundaries in the disk where no longer 1-boundries in the annulus. Now, suppose we punch a hole in a torus. Will the homology group change? Are there 1-boundies that no longer bound a 2-chain? What if we punch out two holes?

Exercise 13. For a manifold M, $H_0(M)$ is defined to be $C_0(M)/\text{image } \partial C_1$. Convince yourself that if M has n path connected components then $H_0(M) \approx \mathbb{Z}^{n-1}$.

1.11. **Homotopy.**

Definition 1.11. Two continuous functions $f_i : M \to N$ for $i = 0, 1$ are **homotopic** if there exists a continuous function $H : M \times I \to N$ such that

(1) $H(x, 0) = f_0(x)$,
(2) $H(x, 1) = f_1(x)$.

Definition 1.12. Let X and Y be topological spaces. If there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to identity maps then we say X and Y are homotopic spaces. We may write this as $X \sim Y$.

Any two spaces that are homeomorphic are obviously homotopic. The classification of topological spaces up to homotopy equivalence is coarser than for topological equivalence.

Example 5. Let T be the subspace of $\mathbb{R}^2$ given by $[-1, 1] \times \{0\} \cup \{0\} \times [0, 1]$. We will show that T and $[-1, 1]$ are not homeomorphic but are homotopic.

Proof. We can use cut point theory to show they are not homeomorphic. To show they are homotopic let $H((x,y),t) = (x, y \cdot (1-t))$. $\square$

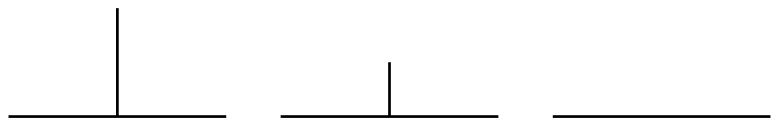

FIGURE 2. Homotopy for Example 5

Theorem 1.13. *Homotopic spaces have isomorphic homology groups.*

Thus I, S^1, and T^2 are not homotopic to each other. However, there are homotopies between an annulus and S^1, between T^2 minus a point and the wedge of two circles (a "figure 8"), between I, D^2 and a point. A space homotopic to a point is said to be **retractable**.

Exercise 14. Construct homotopies for the cases discussed above.

Now we should be ready to dive into the paper "Trefoil Surgery".

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