TREFOIL SURGERY

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Abstract. This is an expository account of a theorem of Louise Moser that describes the types of manifolds that can be constructed via Dehn surgery along a trefoil in the 3-sphere. These include lens spaces, connected sums of two lens spaces, and certain Seifert fibered spaces with three exceptional fibers. Various concepts from the topological theory of three dimensional manifolds are developed as needed.

1. Introduction

Imagine you live in some sort of three-dimensional universe. But now suppose someone (or something - maybe some sort of space worm) has carved out a tunnel in space that forms a loop. Annoyed at this you grab a loop of "space tubing" to fill in the tunnel, gluing the outside of the tube to the wall of the tunnel. Now you go back to whatever it was you were doing before, but strange things start happening. Because you weren't careful in how you did the gluing it turns out the structure of your three dimensional world has changed. Indeed, it might not even be prime[1]!

This operation, called **Dehn surgery**, is a fundamental tool in the study of three-dimensional manifolds (spaces). Our goal here is to develop this theory, somewhat informally, and then use it to prove a classical theorem due to Louise Moser that describes the types of manifolds that can be derived by performing Dehn surgery on a trefoil shaped tunnel in a standard space called the three-dimensional sphere. The background material needed would normally take a couple of years of graduate level topology to master, yet the basic ideas are intuitive and visual.

2. Topological Manifolds

The material in this section can be found in the textbooks [7, 9]. We assume a basic familiarity with topology: open and closed sets, compactness, continuity and path connectedness. Two topological spaces

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[1] *Prime manifolds* will be defined later

X and Y are **homeomorphic** or **topologically equivalent** if there is a bicontinuous bijection $h : X \to Y$; this is denoted by $X \cong Y$. Such a function is called a **homeomorphism**. We say X can be **embedded** into Y if there is a continuous map $f : X \to Y$ such that $f : X \to f(X)$ is a homeomorphism.

An **n-dimensional manifold without boundary** M is a topological space such that for each point $x \in M$ there exists an open set containing x that is homeomorphic to an open ball in $\mathbb{R}^n$. (We may assume the homeomorphism is to the open unit ball centered at the origin and takes x to the origin.) If there are points y in M for which this fails but where there is a subset H of M containing y and a homeomorphism

$$h : H \to \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 < 1 \,\text{and}\, x_1 \geq 0\}$$

taking y to the origin, then M is a **n-dimensional manifold with boundary**. Such points y form the **boundary** of M which is denoted ∂M. The **interior** of M is $\text{int}(M) = M - \partial M$. [2]

Examples. An n-sphere, S^n, for $n \geq 0$, is a any space homeomorphic to the unit sphere in $\mathbb{R}^{n+1}$. We will be working with S^1, S^2 and S^3. These have empty boundary. A closed n-ball, B^n, for $n \geq 1$, is any space homeomorphic to the closed unit ball in $\mathbb{R}^n$. Notice $\partial B^{n+1} \cong S^n$. For $n = 2$ we call a 2-ball a disk and denote it by D^2. Let $I = [0, 1]$. Notice $B^1 \cong I$. A space homeomorphic to $I \times S^1$ is called an **annulus** or a **cylinder**. A space homeomorphic to $S^1 \times S^1$ is called a **torus**, denoted T^2, while any space homeomorphic to $D^2 \times S^1$ is called a **solid torus**. We will use V to denote a solid torus although there is no standard convention. A **core** of a solid torus is a circle that maps to $(0, 0) \times S^1$ by some homeomorphism $h : V \to D^2 \times S^1$. The spaces $I \times I \times I$ and $D^2 \times I$ are 3-balls despite not being round. All of these manifolds are compact.

3. GLUING, CONNECTED SUMS AND COMPACTIFICATION

We won't be precise in our definitions here but will proceed by examples. If we "identify" the end points of the unit interval we get a new manifold that is homeomorphic to the circle. If we take the square $I \times I$ and identify each point on the bottom edge with the point on the top edge that is above it we get a new manifold that is homeomorphic to a cylinder. We say that we have glued the top and bottom edges. If instead of gluing $(x, 0)$ to $(x, 1)$ we glued $(x, 0)$ to $(1 - x, 1)$ the result

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[2] Manifolds are also assumed to be Hausdorff and second countable.

would be a **Möbius band**. If we glue $(x, 0)$ to $(x, 1)$ and $(0, y)$ to $(1, y)$, for $x, y \in I$, the result is a torus.

Exercise 1. Explain why a Möbius band is not homeomorphic to an annulus but a strip with a full (360^o) twist is.

If we take two closed disks and identify their boundaries the result is a 2-sphere. If we take two closed 3-dimensional balls, B_1 and B_2, and identify points on their boundary 2-spheres the resulting 3-manifold without boundary is a 3-sphere. See Figure 1. The identification is achieved by choosing a homeomorphism $h : \partial B_1 \to \partial B_2$ and identifying x with $h(x)$ for each $x \in \partial B_1$. It can be proven that the topological type of the result is independent of the choice of h [7].

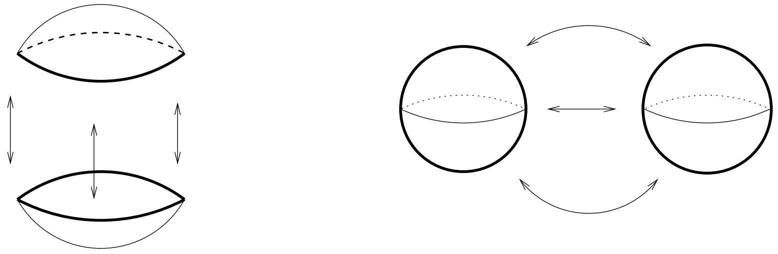

FIGURE 1. Gluing two disks gives a 2-sphere; gluing two balls gives a 3-sphere.

For any two path connected 3-manifolds M_i, $i = 1, 2$, we can form the **connected sum** as follows. Select a closed 3-ball in each that does not meet the boundary (if there is one) and remove their interiors. Now choose a homeomorphism from the new boundary 2-sphere of $M_1 - \text{int } B_1$ to the new boundary 2-sphere of $M_2 - \text{int } B_2$. Glue the two 2-spheres using this homeomorphism. The new manifold is denoted $M_1 \# M_2$ and its topological type is independent of choice of the 3-ball and the homeomorphism [5]. Figure 2 illustrates the result of forming the connected sum of two solid tori; it looks like a solid torus with a smaller solid torus carved out of it; the dashed circle represents the 2-sphere where the gluing occurred.

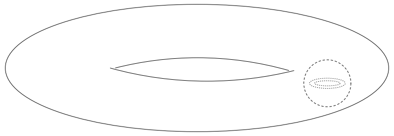

FIGURE 2. The connected sum of two solid tori

If the only way a manifold M can be written as a connected sum is $M \cong M \# S^3$ then we say M is **prime**. Every compact path connected 3-manifold can be written uniquely as a connected sum of prime 3-manifolds! [5]

The 3-sphere can be constructed by gluing two solid tori together. Figure 3 shows how to see this starting from gluing two 3-balls together. You decompose one of the 3-balls into a solid torus and a solid cylinder (in the donut hole). We do the gluing in two steps. First glue the top and bottom disks on the cylinder to the other 3-ball. This forms a solid torus. Now glue the two solid tori together and *voila*, we have realized S^3 as the union of two solid tori.

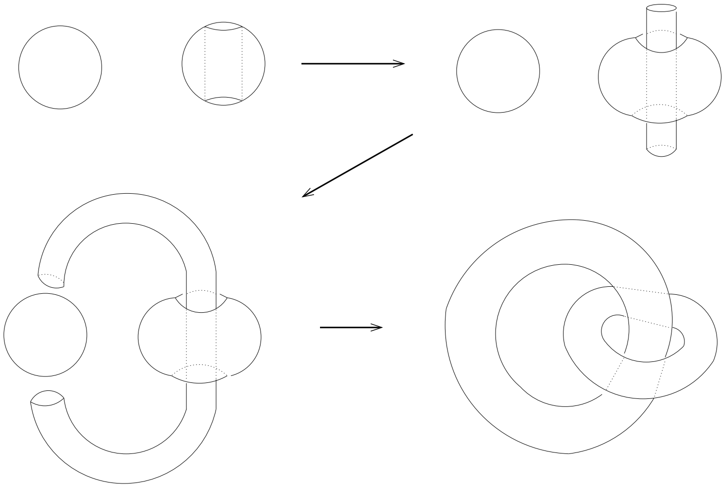

FIGURE 3. Realizing S^3 as the union of two solid tori

There is another way to construct spheres that will be useful for us. Consider the union of the real line $\mathbb{R}$ with a new point called ∞. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Topologize $\overline{\mathbb{R}}$ as follows. Let the open sets be all the open subsets of $\mathbb{R}$ together with sets of the form $\{\infty\} \cup O$ where $\mathbb{R} - O$ is compact. With this topology $\overline{\mathbb{R}}$ is homeomorphic to S^1. This is called the **one point compactification**. The same process can be applied to make $\mathbb{R}^2 \cup \{\infty\}$ homeomorphic to S^2 and $\mathbb{R}^3 \cup \{\infty\}$ homeomorphic to S^3.

4. KNOTS

A **knot** is a circle embedded in the interior of a 3-manifold, that is there is a homeomorphism $h : S^1 \to K \subset \operatorname{Int} M$. A knot is said to be an **unknot** if it forms the boundary of a disk in M. Thus the unit circle U in the xy-plane in $\mathbb{R}^3$ is unknotted. Two knots K_1 and K_2 in

M are regarded as equivalent or as having the same **knot type** if they are **ambiently isotopic**, which we define next.

Definition 4.1. Two knots K_1 and K_2 in M are **ambiently isotopic** if there is a continuous function $S : M \times I \to M$ such that $S(x, 0)$ is the identity (hence $S(K_1, 0) = K_1$), $S(K_1, 1) = K_2$ and for each $t \in I$ $S(x, t) : M \to M$ is a homeomorphism.

Given a knot K in a 3-manifold M a **tubular neighborhood** of K is a solid torus that misses ∂M and has K as its core [13]. It is denoted by $N(K)$. A solid torus whose core is unknotted is said to be **standardly embedded**; likewise for the boundary of such a solid torus.

Let V be a standardly embedded solid torus in S^3 and let $T = \partial V$. A knot in T that does not bound a disk in V is called a **torus knot**. The simplest torus knot, besides the unknot, is the **trefoil**; see Figure 4.

Although only unknots bound disks every knot in S^3 is the boundary of some orientable (*i.e.*, two sided) surface. Such a surface is called a **Seifert surface**. We won't prove this fact here (see [3]) but Figure 4(upper left) shows a Seifert surface for the trefoil.

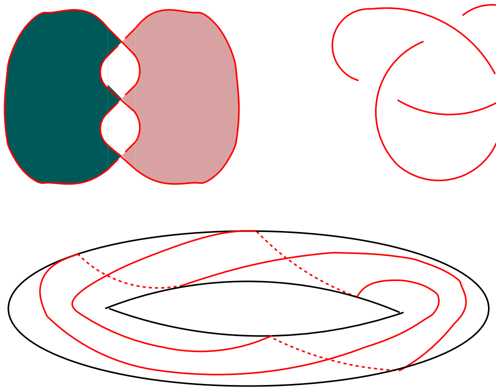

Figure 4. Three views of the trefoil knot

Exercise 2. Convince yourself that the three curves in Figure 4 are ambiently isotopic. Convince yourself that the Seifert surface shown is homeomorphic to a torus with the interior of a disk removed.

5. Homology Groups

How do we know the 2-sphere and the torus are not homeomorphic? Questions like this led to the creation of the field of **algebraic topology**. This is a very large subject. The special tool will shall use is the first homology group. The basic idea is for each topological space X there is an associated Abelian group called the **first homology group**, $H_1(X)$. If two groups are isomorphic we write $G_1 \approx G_2$.

The idea is to study loops. A **loop** in M is the continuous image of S^1 in M. Thus a loop can have self crossings or even be a point.

Definition 5.1. Two loops, λ_0 and λ_1 in M are **homotopic** if there is a continuous function $H : S^1 \times I \to M$ such that $H(S^1, 0) = \lambda_0$ and $H(S^1, 1) = \lambda_1$. Such a function H is called a **homotopy**.

We shall illustrate the idea behind the first homology group by looking at loops in a disk with two holes. Let $\ddot{D}$ be a closed disk in $\mathbb{R}^2$ with two smaller open disks removed; we require the closures of these disks to be disjoint so $\ddot{D}$ is a 2-manifold. See Figure 5.

Now for any loop λ in the plane and a point P not on λ there is an integer called the **winding number** that counts how many times λ wraps around P. To do this λ is given an orientation and we count counterclockwise as positive and clockwise as negative. This definition is done formally in some topology texts [6] and in is covered in most complex analysis courses. For a loop in $\ddot{D}$ we can define winding numbers with respect to each hole. We can define winding numbers for finite unions of loops by just adding. This determines a function $W : \{\text{loops in } \ddot{D}\} \to \mathbb{Z}^2$ given by

$$W(\lambda) = (w_1(\lambda), w_2(\lambda))$$

where w_1 and w_2 are winding numbers with respect each hole. Figure 5 includes some examples. The winding numbers for the four curves shown are

$$\begin{aligned} W(c_1) &= (1,0), \\ W(c_2) &= (0,1), \\ W(c_3) &= (1,1), \\ W(c_4) &= (-2,1). \end{aligned}$$

It can be shown that homotopic loops have equal winding numbers.

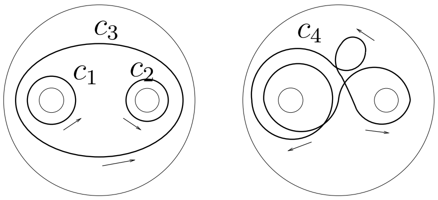

Figure 5. Disk with two holes

We shall declare that two finite collections of loops in $\ddot{D}$ are **homologous** if they give the same pair of winding numbers. This is an equivalence relation. We use $\lambda_1 \sim \lambda_2$ to mean λ_1 is homologous to λ_2. Let $H_1(\ddot{D})$ be the set of these equivalence classes. We can make $H_1(\ddot{D})$ into a group as follows. Let A and B be in $H_1(\ddot{D})$. These are actually equivalence classes. Let $\alpha \in A$ and $\beta \in B$. Let γ be the union of the curves making up α and β and let Γ be the equivalence class of γ. Then define $A+B = \Gamma$. The class corresponding to (0,0) gives us an identity element and the existence of inverses is clear by reversing orientations. In fact we can see, intuitively at least, that $H_1(\ddot{D}) \approx \mathbb{Z}^2$. Referring back to Figure 5 we can compute $c_3 \sim c_1 + c_2$ and $c_4 \sim -2c_1 + c_2$. The equivalence classes of c_1 and c_2 generate $H_1(\ddot{D})$.

Now suppose we glue one of the disks back in. We get an annulus $A \cong S^1 \times I$. Since there is only one hole we should have $H_1(A) \approx \mathbb{Z}$ and this is indeed the case. Intuitively we imagine that gluing in a disk kills off a generator of the group. It should not be too surprising to learn that $H_1(D) = 0$, the trivial group, and that $H_1(S^1) \approx \mathbb{Z}$. Any space where all loops are homotopic to a point will have trivial first homology group. Thus $H_1(S^2) = 0$.

It can be shown that $H_1(S^1 \times D) \approx \mathbb{Z}$. Here the winding number model we used cannot be applied directly since we cannot fit a solid torus into the plane but it is not too hard to imagine building a group that tracks loops rapping about an axis through the “donut hole”. It will always be the case that homotopic loops are homologous. (Say that three times fast!)

Less obvious is the fact that $H_1(T^2) \approx \mathbb{Z}^2$. The basic idea is that T^2 has two holes! It has the “donut hole” that the solid torus has and the “tunnel” inside of it is also a hole. As with the solid tours the homology class of a loop going once around the long way acts as a generator but now the homology class of a loop going around the short way becomes a second generator. This is even more obvious if we recall that $T^2 = S^1 \times S^1$! Looking at T^2 this way let $(*, *)$ be the point where the two S^1 factors meet an then let $M = S^1 \times \{*\}$ and $L = \{*\} \times S^1$. To simplify notation we will denote their respective homology classes by M and L as well. (We shall do this from now on when working with homology groups.) In Figure 6 we show several loops representing different homology classes on T^2. A simple closed curve on T^2 that represents the class $pL + qM$ is called a (p, q)-curve. If T^2 is standardly embedded in S^3 then a (p, q)-curve is called a (p, q)-knot. The trefoil knot is a (3,2)-knot.

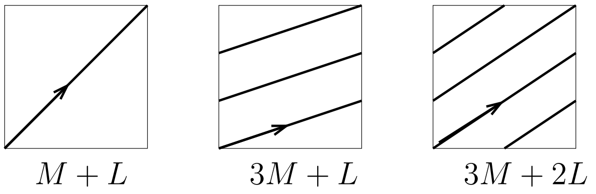

Figure 6. Torus math!

Let $\widetilde{M}$ be the Möbius band. It can be shown that $H_1(\widetilde{M}) \approx \mathbb{Z}$. But now we are going to introduce a new space. The boundary of a Möbius band is S^1. Thus we can glue a disk D to $\widetilde{M}$. This gives a 2-manifold without boundary. It is called the **projective plane** and is denoted P^2. You cannot visualize it in $\mathbb{R}^3$ but is exists as a mathematical object. What is $H_1(P^2)$? The boundary of the Möbius band now bounds a disk. So it must die. That is it is in the identity equivalence class. However, the loop C that is the core of the Möbius band is still non trivial. Yet twice C is trivial; that is a loop that travels around C twice is homotopic to $\partial\widetilde{M}$. Thus, in $H_1(P^2)$ we have $2C \sim 0$. This can be used to prove that $H_1(P^2) \approx \mathbb{Z}/2$.

There is one more basic fact about homology groups that will shall need. If we have two finite disjoint collections of simple closed curves in a manifold and together they form the boundary of an orientable surface then they, with suitable orientations, represent the same homology class. The reason for this is at the hart of the formal definition of homology groups but is a little too technical to present here. As an example consider three curves c_1, c_2 and c_3 of $\ddot{D}$ and partition them into two collections anyway you like. A further consequence of this is that if a single simple closed curve in a manifold forms the boundary of an orientable surface then it represents the identity element.

The main significance of homology groups is the following. Let M and N be manifolds. If M is homeomorphic to N then $H_1(M) \approx H_1(N)$. The converse is clearly not true; however it can be shown that any time one space can be “continuously deformed” into another the two spaces will have isomorphic homology groups. See [7, 9] for the formal definition of a deformation.

6. Finitely Generated Abelian Groups

The first homology group of a compact manifold is a finitely generated Abelian group. We take a brief algebraic detour to review these. The set $\mathbb{Z}^n$ is an Abelian group under vector addition. It can be generated by a finite number of elements. Let A be an $n \times n$ matrix with integer entries. It induces a function from $\mathbb{Z}^n$ to $\mathbb{Z}^n$ via matrix multiplication. We will denote this function by A so $A : \mathbb{Z}^n \to \mathbb{Z}^n$. By standard facts from linear algebra it is a homomorphism. The image of the homomorphism is a subgroup of $\mathbb{Z}^n$ which we denote by $A\mathbb{Z}^n$. For example multiplication by 2 from $\mathbb{Z}$ into $\mathbb{Z}$ has the even integers as its image. In our notation $2\mathbb{Z} = \{\dots, -2, 0, 2, \dots\}$. (We are writing the 1×1 matrix [2] as 2.)

Given an Abelian group G and a subgroup H one can form the quotient group G/H. The members of G/H are subsets of G that differ by a member of H. For example, $\mathbb{Z}/2\mathbb{Z}$ has two elements, the even integers and the odd integers. The induced addition operation is that even plus even is even, odd plus odd is even and even plus odd is odd. We tend to be sloppy and write $\mathbb{Z}/2\mathbb{Z}$ as $\{0, 1\}$, taking addition to be addition mod 2, but really the 0 stands for the set of even integers and the 1 stands for the set of odd integers.

Let $G = \mathbb{Z}/n\mathbb{Z}$. If $n = \pm 1$ then $n\mathbb{Z} = \mathbb{Z}$ and G is the trivial group, which has one element. In general the number of elements in G, that is the **order** of G, is $|n|$, unless $n = 0$. In this case $0\mathbb{Z} = \{0\}$ and $G \approx \mathbb{Z}$ and so has infinite order.

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $G = \mathbb{Z}^2/A\mathbb{Z}^2$. The reader should work out that $G \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ which has six elements. If instead $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ the reader should check that $G \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, which has infinitely many elements. In general we have the following theorem which we shall not prove.

Theorem 6.1. *Let A be an $n \times n$ integer matrix. Then the order of $\mathbb{Z}^n/A\mathbb{Z}^n$ is $|\det A|$ if $\det A$ is not zero and is infinite if $\det A = 0$.*

Quite a bit more is true. Any finitely generated Abelian group is isomorphic to $\mathbb{Z}^n/A\mathbb{Z}^n$ for some n and integer $n \times n$ matrix A. While it can happen that different matrices yield isomorphic groups there is a simple algorithm involving row and column operations that determines when this happens. [4]

Exercise 3. a. Verify the claims made just before Theorem 6.1.
b. Show that the groups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ are isomorphic.
c. Show that the groups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ are not isomorphic.
d. Show that $\dfrac{\mathbb{Z}^2}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

e. Show that $\dfrac{\mathbb{Z}^2}{\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \mathbb{Z}^2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

7. DEHN SURGERY

Now we come to **Dehn surgery**. Let M be a 3-manifold and let K be a knot in M with tubular neighborhood $N(K)$. Now remove the interior of $N(K)$ from M; let $M' = M - \text{int } N(K)$. Let V be a solid torus disjoint from M. Let $h : \partial V \to \partial N(K) \subset M'$ be a homeomorphism. Now glue V to M' by using h to identify ∂V with $\partial N(K)$. Let $M_{K,h} = M' \cup_h V$ denote the new manifold. Its topological type depends on both K and h but not on the choice of the tubular neighborhood. Here is another major theorem on 3-manifold topology: every 3-manifold without boundary can be constructed via Dehn surgeries on S^3 [13].

The simplest Dehn surgeries are those done on the unknot in S^3. Since removing an unknotted solid torus from S^3 results in a second unknotted solid torus, these Dehn surgeries are equivalent to gluing two solid tori together. We might just recover S^3, but this need not be the case. It depends on the homeomorphism. Manifolds that are formed in this manner are called **lens spaces**. To study them we need to know more about self homeomorphisms of the torus.

7.1. Torus Maps.

Think of the torus T^2 as given by $I \times I$ with the opposite edges identified. A 2×2 integer matrix A induces a map from $\mathbb{R}^2$ to itself that preserves the integer lattice. If we use arithmetic modulo 1 then A determines a map from $[0,1) \times [0,1)$ to itself. From this we can get a map from T^2 to T^2.

Exercise 4. Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -4 \end{bmatrix}$. Then define $A(x,y) = (5x+y, 3x-4y)$ mod 1. Thus, $A\left(\frac{1}{2}, \frac{1}{6}\right) = \left(\frac{2}{3}, \frac{5}{6}\right)$. However, this map is not one-to-one. Find another point $(x,y) \in [0,1) \times [0,1)$ such that $A(x,y) = (\frac{2}{3}, \frac{5}{6})$. In fact this map is 23-to-1! Find 23 points $(x,y) \in [0,1) \times [0,1)$ such that $A(x,y) = (0,0)$.

Fact 7.1. *A* 2×2 *integer matrix* A *induces a homeomorphism on* T^2 *if and only if* $\det(A) = \pm 1$.

Exercise 5. Prove this.

Exercise 6. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. In Figure 7 we show the image of $I \times I$ under the action of A in $\mathbb{R}^2$ and how it wraps around itself with mod

1 arithmetic. Redo this for $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}$. Try to draw each of these on a donut!

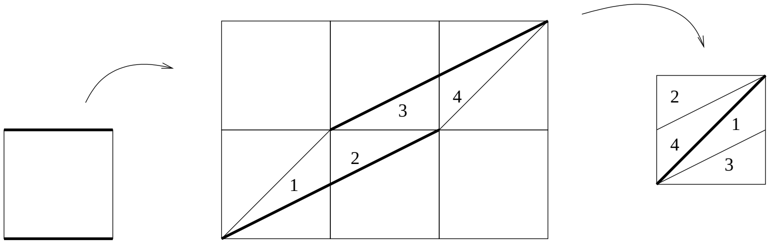

Figure 7. A linear torus homeomorphism

These maps are linear. Of course there are nonlinear self homeomorphisms of the torus. For our proposes, they don't matter since we shall see that any self homeomorphism of the torus is essentially equivalent to a linear one in the sense defined next.

Definition 7.2. Two homeomorphisms $f_i : M \to M$ for $i = 0, 1$ are **isotopic** if there exists a continuous function $H : M \times I \to M$ such that

1. $H(x, 0) = f_0(x)$,
2. $H(x, 1) = f_1(x)$, and
3. $H(x, t) : M \to M$ is a homeomorphism for each fixed t.

We state two theorems without proof, but they should be intuitively plausible. Proofs can be found in [1] and [15] respectively.

Theorem 7.3. *Every homeomorphism from a torus to a torus is isotopic to one induced by a* 2×2 *integer matrix with determinant* ± 1.

Theorem 7.4. *Let* M *be a 3-manifold and let* T *be a torus component of* ∂M. *Let* V *be a solid torus and let* $f_i : \partial V \to T$ *for* $i = 0, 1$ *be homeomorphisms. Let* $M_i = M \cup_{f_i} V$ *be the result of gluing* V *to* M *via* f_i. *If* f_0 *is isotopic to* f_1, *then* M_0 *is homeomorphic to* M_1.

These two theorems tell us that the topological type of a Dehn surgery is determined by the 2×2 matrix of a linearization of the attaching map. However, it turns out that only the first column of the matrix is needed. To see this we need to know a little more about what goes on inside a solid torus.

Exercise 7. Compare and contrast the definitions of ambient isotopy, isotopy and homotopy. How are they similar how are they different? Look their definitions up in several textbooks and compare.

7.2. **Solid Tori.** On a solid torus V a **meridian** is a simple closed curve in the boundary that bounds a disk inside the solid torus but does not bound a disk in the boundary. A **longitude** is a simple closed curve in the boundary that meets a meridian in one point, passing through it. Any two meridians of V are ambient isotopic within ∂V. This is not true for longitudes, but any two longitudes are ambient isotopic within V. This subtle point is essential. Once we choose a meridian and a longitude that meet at one point, we assign them orientations which for us will just means we put arrows on them. See Figure 8.

Theorem 7.5. *Let V be a solid torus. Let $h : \partial V \to \partial V$ be a self homeomorphism. Then h can be extended to a self homeomorphism of V to V if and only if h takes meridians to meridians. This means that the linearization of h is of the form $\begin{bmatrix} \pm 1 & q \\ 0 & \pm 1 \end{bmatrix}$ (the signs are unlinked).*

Outline of a proof. One direction is easy, the other is quite involved. First, if $h^* : V \to V$ is a homeomorphism its restriction to ∂V is a homeomorphism and we suppose this is equal to h. Thus h takes any meridian, say M, to a simple closed curve in ∂V. Let D be a disk in V with $\partial D = M$. Then $h^*(D)$ is also a disk whose boundary is in ∂V. Either this boundary is a meridian as claimed or a curve in ∂V that bounds a disk in ∂V. If the latter holds then $V - h^*(D)$ is not path connected, even though it is homeomorphic to $V - D$ which is. This contradiction shows h must take meridians to meridians.

If h takes meridians to meridians then we need to show it can be extended to a self homeomorphism on V. Before proceeding we prove the following. Let $h : \partial B \to \partial B$ be a self homeomorphism of the surface of a 3-ball. Then h can be extended to a self homeomorphism of B. First use the unit 3-ball centered at the origin of $\mathbb{R}^3$. For each $u \in B - \{(0,0,0)\}$ there is a unique point $w \in \partial B$ and number $r \in (0,1]$ such that $u = rw$. Define $h^* : B \to B$ by $h^*(0,0,0) = 0$ and $h^*(u) = rh(w)$. The reader can check that this gives a continuous extension of h. Suppose B' is some other topological 3-ball with homeomorphism $h : \partial B' \to \partial B'$ and $g : B' \to B$. Define $h^* : B' \to B'$ by $h^* = g^{-1}(ghg^{-1})^* g$. This all works for any n dimensional ball.

Now back to our solid torus. Suppose $h : \partial V \to \partial V$ takes the meridian M_1 to the meridian M_2. Then if D_1 is a disk in V with boundary M_1 its image will be a disk D_2 in V with boundary M_2. Remove D_1 from V, push the gap open and take the closure. Call the resulting 3-ball B_1. Do the same with D_2 to create B_2. Since h since maps D_1 to ∂D_2 homeomorphically we can extend it to a homeomorphism of D_1 to D_2. Using two copies of this extension we can create a homeomorphism from ∂B_1 to ∂B_2. We can extend this to a homeomorphism from B_1 to B_2 and then use it to determine a homeomorphism from V to V that is an extension of h. ☐

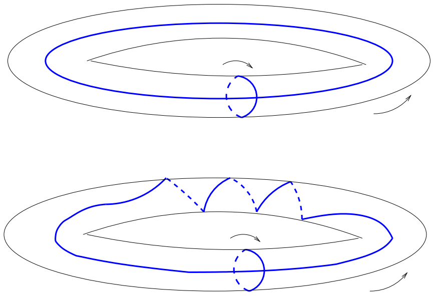

FIGURE 8. Meridian-longitude pairs

Now we return to the topic of Dehn surgery.

Theorem 7.6. *Let M be a 3-manifold that has a torus T as a component of its boundary. Let V be a solid torus and let $h : \partial V \to T$ be a homeomorphism. By using suitable coordinates on T and ∂V we may assume that h is linear and represented by an integer matrix*

$$H = \begin{bmatrix} p & r \\ q & s \end{bmatrix}.$$

The the homeomorphism type of $M \cup_h V$ is completely determined by p and q.

We give an example that illustrates the main ideas in the proof.

Example 1. Suppose $p = 3$ and $q = 2$. Since $ps - rq = \pm 1$ it follows that $r = \pm(1 + np)$ and $s = \pm(1 + nq)$ for some integer n where the signs must agree. The choice of n can be thought of as choosing different longitudes and this does not affect the topological type of the gluing. The choice of signs is equivalent to reversing the choices for the orientations of the meridian or longitude. Again this does not affect the outcome.

For lens spaces the notation $L(p, q)$ is used to denote the lens space determined by a homeomorphism that takes a meridian to a (p, q)-curve on the other torus; p and q must be coprime. You can check that $L(1, q) \cong S^3$ for any q. It can also be shown that $L(0, 1) \cong S^2 \times S^1$. Sometimes S^3 and $S^2 \times S^1$ are considered degenerate or trivial lens spaces. It is also easy to show that $L(p, q) \cong L(p, -q) \cong L(-p, q) \cong L(p, q + np)$ for any integer n. Therefore, every nontrivial lens space can be represented by $L(p, q)$ where $0 < q < p$ and p and q are coprime. However this representation is not unique. It is known that $L(p, q) \cong L(p', q')$ if and only if $p = \pm p'$ and ($q = \pm q'$ or $qq' = \pm 1 \mod p$) [13].

Exercise 8. Classify the lens spaces $L(p, q)$ for $1 \leq p \leq 7$, $0 < q < p$, and p and q coprime.

We close this section with a final note about longitudes. The definition of a longitude of a solid torus gave a lot of options for choosing one. However, if a solid torus is given as being inside S^3 then we can define a **preferred longitude**. A preferred longitude of a solid torus V in S^3 is a longitude that could be the boundary of a Seifert surface of the core of V minus the interior of V. It can be shown that up to isotopy there is only one choice for the preferred longitude [3, 13].

If the torus is standardly embedded in S^3 then a preferred longitude will bound a disk in $S^3 - \text{Int } V$ and is easy to visualize. Determining a preferred longitude of knotted solid torus is not visually obvious. Figure 9 shows the preferred longitude for a trefoil. We will use this later. (A preferred longitude has *linking number* zero with the core [13].)

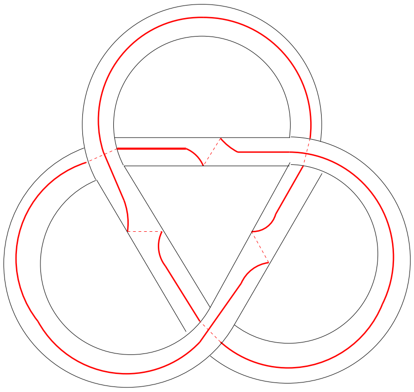

FIGURE 9. A preferred longitude of the trefoil knot based on a figure from [13].

8. SEIFERT FIBERED MANIFOLDS

A **Seifert fibered manifold** is a 3-manifold M together with a fiber structure F that is a decomposition of M into a union of disjoint copies of S^1, called the fibers, such that each S^1 fiber not in ∂M has a closed tubular neighborhood that is equivalent to a Seifert fibered solid torus which we describe below. Any boundary components are tori that are fibered in the manner described below. The two references we have drawn on for this material are Seifert's original paper (in translation) [15] and the course notes of Brin [2].

Solid Torus Fibrations. Consider the solid cylinder $C = D^2 \times I$. Let $F_c = \{\{(r,\theta)\} \times I : (r,\theta) \in D^2\}$. This gives a fibration of C by closed intervals. Let $D_i = D^2 \times \{i\}$ for $i = 0, 1$ be the bottom and top disks of C respectively. For any real number ψ let $R_\psi : D_0 \to D_1$ be given by $R_\psi(r,\theta,0) = (r,\theta+\psi,1)$. Identify D_0 and D_1 using R_ψ as the homeomorphism to form a solid torus V. If ψ is a rational multiple of π the fibers on C become joined at their end points to form circles. The core circle will contain just one copy of I. If $\psi = p2\pi/q$ for coprime integers p and q then the other circles will be formed from q copies of I. Such an object is called a (p,q) **fibration of the solid torus**. The core is called the **exceptional fiber** unless $p = 0$ or $q = 1$ in which case we say the fibration is trivial. Non-exceptional fibers are called **ordinary fibers**. If V is standardly embedded in $\mathbb{R}^3$ then it is fibered by (p,q) torus knots and its core.

Two fibrations of a manifold M are **fiber equivalent** if there is a homeomorphism $h : M \to M$ that takes fibers to fibers. Let F_0 and F_1 be two fibrations of a manifold M. They are **fiber isotopic** if there is a continuous function $S : M \times I \to M$ such that

1. $S(\cdot,0)$ is the identity on M.
2. For any $t \in I$ the $S(\cdot,t)$ induces a fibration on M. That is for each fiber $F \in F_0$ the image $S(F,t)$ is a circle and for each t all these circles fit together to form a fibration.
3. Finally, for $t = 1$ the induced fibration is F_1.

It is obvious that isotopic fibrations are fiber equivalent.

It will be helpful to also be able to talk about fibrations of two dimensional tori derived in the same manner as for solid tori. Usually these will arise when a torus is the boundary of a fibered solid torus or some other Seifert fibered manifold. If we have two tori, one with a (p,q) fibration and the other with a (r,s) fibration, then there is a fiber preserving homeomorphism from one to the other.

Seifert fibered manifolds have been completely classified up to fiber equivalence by Seifert [15]. We will not need the full classification theorem, but we do need to understand the classification of fibrations of the solid torus. By gluing two solid tori together with a homeomorphism that takes fibers to fibers on the boundary tori, we will be able to construct many fibrations of the 3-sphere and the lens spaces.

Classifying solid torus fibrations. When are two fibrations of the solid torus equivalent? Clearly we only need to know p modulo q.

Also, changing the sign of either is equivalent to changing the choice of orientations for the coordinates systems. A homeomorphism is not required to preserve orientation so sign changes won't affect the fiber equivalence class. Thus we can assume $0 \leq p < q$ where p and q are still coprime. It can be shown that subject to these restrictions p and q determine a unique fiber equivalence class. The proof goes roughly like this. Suppose we had a (p, q) fibration of a solid torus V_1 and a (r, s) fibration of a solid torus V_2. Let $h : V_1 \to V_2$ be a fiber preserving homeomorphism. Let M_1 be a meridian on V_1 and suppose D_1 is a disk in V_1 whose boundary is M_1. Then the image of D_1 must be a disk $D_2 \subset V_2$ whose boundary is a curve in ∂V_2. But then this curve is a meridian. Assume h can be represented by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and let (M_i, L_i) be the meridian-longitude pair of ∂V_i, $i = 1, 2$. Then $h(M_1) \sim aM_2 + cL_2$. Hence $a = \pm 1$ and $c = 0$. Since the determinant is ± 1 then $d = \pm 1$. By changing the orientations of the meridians and longitudes if necessary we may assume $a = d = 1$. Now $h(l_1) = bM_2 + L_2$. In other words h just adds some full twists to V_1. All this can do to the fibration is add multiples of q to p. Thus p and q determine a unique equivalence class.

Crossing Curves. Given a solid torus V a choice of an oriented meridian-longitude pair establishes a coordinate system of the torus ∂V. But for a fibered torus it useful to have another coordinate system. Let V be a fibered solid torus and let F be a fiber in ∂V. A **crossing curve** on ∂V is any simple closed curve that meets every fiber once and only once. Given a crossing curve Q there is a simple closed curve represented by $Q' = \pm Q + nF$ that is also a crossing curve. A choice of an oriented fiber/crossing curve pair determines a coordinate system on ∂V.

Example 2. Figure 10 gives two views of a (3,1) fibration of a torus. On the right the fibers are vertical and the top circle is attached to the bottom circle via a 120^o rotation. On the left the fibers are slanted and the edges of the square form a meridian-longitude pair. On each we have selected a fiber for F and two crossing curves. One, denoted Q is also longitude while the other Q' is homologous to $Q + F$. Notice that $F \sim 3Q + M$, where M is a meridian.

Fibrations of S^3. Recall our construction of S^3 as the union of two solid tori glued along their boundaries. The gluing we used identified a meridian of one torus with a longitude of the other. Taking the homeomorphism to be linear its matrix would be any of $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$ since we haven't specified orientations. But are these the only possibilities? No.

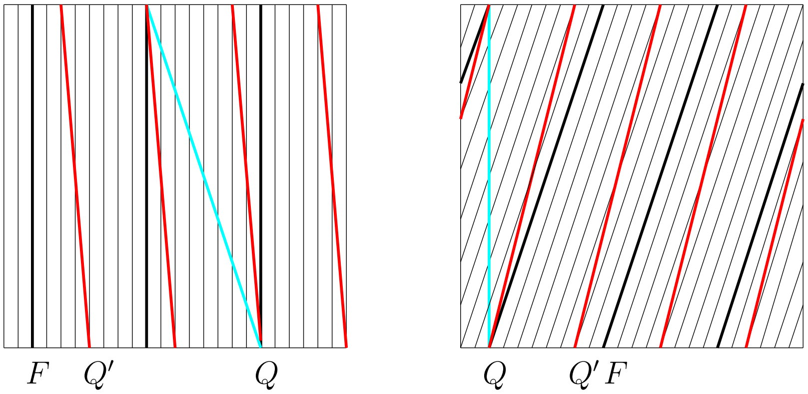

FIGURE 10. Crossing curves for (3,1) fibration

Remember the flexibility we had in choosing longitudes? A self homeomorphism with matrix of the form $\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}$ merely effects the choice of longitude and not the topology. We can apply such maps to each torus before gluing them together without changing the topological type of the result; a formal proof of the this claim can be found in [2, 15]. So if we take the composition of these three maps we get for the allowed gluings,

$$\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \pm q & \pm q \pm 1 \\ \pm 1 & \pm s \end{bmatrix}.$$

Now we endow each solid tori with a fibration. Because the gluing homomorphism must take fibers to fibers there are some restrictions. Let V_1 have a given (m, n) fibration. Suppose the fibration of V_2 is (u, v). What can we say about the possible values of u and v? It is necessary that

$$\begin{bmatrix} \pm q & \pm q \pm 1 \\ \pm 1 & \pm s \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Thus $u = \pm qm + \pm qn \pm n$ and $v = \pm m \pm sn$.

It can be proven that these are the only fibrations of S^3 [15]. We remark that the fibrations of S^3 have been used as a tool to study the twisting of molecular structures in "softly condensed matter" [14].

Fibrations of lens spaces. The construction of fibered lens spaces is similar to what we did with S^3. Let $T_i = \partial V_i$ for $i = 1, 2$. Let $h : T_1 \to T_2$ have matrix $\begin{bmatrix} q & r \\ p & s \end{bmatrix}$. Let V_1 has a (m, n) fibering. What then are the allowed fibrations of T_2? A fiber of T_1 can be represented by $\begin{bmatrix} m \\ n \end{bmatrix}$. It's image under h is represented by

$$\begin{bmatrix} q & r \\ p & s \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} qm + rn \\ pm + sn \end{bmatrix} = \begin{bmatrix} m' \\ n' \end{bmatrix}.$$

Thus we have formed a fibration of $L(p, q)$ with up to two exceptional fibers of indices n and n'. If $n = 1$ then $m = 0$ and $n' = s$ and there is at most one exceptional fiber. Since $4qs - rp = \pm 1$ we know $qs = \pm 1 \bmod p$. For example, if $p = 5$ and $q = 2$ then

$$s \in \{\cdots, -7, -2, 3, 8, 13, \cdots\} \cup \{\cdots, -13, -8, -23, 2, 7, \cdots\}$$

so the index of the exceptional fiber is in $\{2, 3, 7, 8, 12, 13, \cdots, 2+5k, 3+5k, \cdots\}$. It can happen that $n = n' = 1$ in which case there are no exceptional fibers.

Exercise 9. Which lens spaces have fibrations with no exceptional fibers?

The Orbit Surface. Take $D^2 \times S^1$ as a trivially fibered solid torus. If we identify each circle fiber to a point we get D^2 back again. This process is called taking the **quotient** of $D^2 \times S^1$ over S^1. It is sort of like division or an inverse to taking the product. Most topology textbooks have sections devoted to **quotient spaces** [7, 9]. In fact the gluing operations we did earlier are also examples of forming quotient spaces. It is a very large subject.

Consider the fibered solid torus V determined by a rotation of $2\pi/3$. If V is standardly embedded in S^3 the non-exceptional fibers are trefoil knots wrapping around the unknotted exceptional fiber at the core. What happens when we identify the fibers to points? Lets back up and look at just a circle where each point is to be identified with the two points that are 120^o and 240^o degrees from it. This would induce a three to one map from the circle to another circle. (Use a rubber band or hair band to see this.) Similarly if we identify each point of an annulus with the two points that are 120^o and 240^o degrees from it we would get another annulus. If we do this to a disk, the center point is only identified to itself and the quotient is another disk. There is an induced map from the original to this new disk that is 3-to-1 everywhere except for the center point which is mapped to only one point, the center of the new disk.

If we take V and remove the core fiber the quotient map sending each circle fiber to a point would give an annulus. This annulus cannot be realized as a cross section. Indeed there is a natural 3-to-1 map from a cross sectional annulus, one that meets each fiber three times, to this annulus. If we include the exceptional fiber the quotient space is a disk. This is called the **orbit surface** for the fibration of V. It cannot be realized as a cross section, but there is a natural 3-to-1 (almost everywhere) map from a cross section to the orbit surface disk.

Given any Seifert fibered manifold M one can do the same operation of identifying the fibers to points. Since each interior fiber has a tubular fibered neighborhood each point in the quotient will have a disk neighborhood. If $\partial M = \phi$ then the orbit surface will be a *bona fide* 2-manifold, a surface. If $\partial M \neq \phi$ then the orbit surface will be a surface with boundary. If two fibered manifolds are fiber equivalent, then their orbit surfaces are homeomorphic. (If you have studied quotient spaces you might try to prove this claim.) In this paper the orbit surface will always be a disk or S^2 but any surface is realizable in general.

Drilling and Filling. There is a special type of Dehn surgery for fibered manifolds often called **drilling and filling**. Let M be a fibered manifold and let F be an interior fiber. Let V be a tubular neighborhood of F and let V' be some other fibered solid torus with core F'. Remove the interior of V from M and let h be a fiber preserving homeomorphism from $\partial V'$ to $\partial V \subset M$. Then the new manifold $M' = (M - \text{int}V) \cup_h V'$ is a fibered manifold. It maybe that F was an exceptional fiber or an ordinary fiber and that F' is an ordinary fiber or an exceptional fiber with a different index. Since V and V' both have a disk for their orbit surface it can be shown that the orbit surface of M and M' are the same. In any case when performing this procedure some crossing curve of $\partial V'$ is identified with a crossing curve of ∂V.

If the fiber F is an ordinary one then it can also be shown that the topological type of M' is independent of the which ordinary fiber is chosen but it does depend on the map h and the fiber type of V'. Drilling and filling plays an important role in classifying fibered manifolds.

Given a compact fibered manifold M it can be shown that there is a finite number of exceptional fibers. Label them $F_1, \dots, F_n$ and suppose they are characterized by the invariants (α_i, β_i), respectively for $i = 1, ..., n$. Drill each of them out and fill the "worm holes" with trivially fibered solid tori. Call the new space M'. Its orbit surface it the same as M's and we shall assume that it is S^2. It could be that M' is $S^2 \times S^1$ but it does not have to be.

Select a fiber F_0 from M' and remove the interior of a tubular neighborhood V_0 of F_0. Call the resulting manifold M'_0. How many ways can be glue V_0 back in? Let $h : \partial V_0 \to \partial M'_0$ be a fiber preserving homeomorphism. Let $M'' = M'_0 \cup_h V_0$.

Since V_0 is trivially fibered h must take a meridian to a crossing curve. Since all (unoriented) meridians are homologous in ∂V_0 the choice will not affect the topological choice of M''. However the choice of the crossing curve can. Now M_0' is $D^2 \times S^1$. (Why?) If h takes a meridian to $\partial D^2 \times *$ then M'' will just be $S^2 \times S^1$. But if h takes a meridian to another crossing curve, say $\partial D + \beta_0 F$, where F is a fiber, then M'' will be distinct from $S^2 \times S^1$. We call β_0 the **obstruction term**.

Theorem 8.1 (Seifert [15]). *Let M be a compact Seifert fibered manifold. Assume the orbit surface is S^2 and that M has n exceptional fibers. Then M is completely characterized by the invariants of the exceptional fibers (α_i, β_i), for $i = 1, \ldots, n$ and the obstruction term β_0.*

The correspondence is not unique. Certain symmetries apply. For example $(\beta_0, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ is equivalent to $(-\beta_0 - 3, (\alpha_1, \alpha_1 - \beta_1), (\alpha_2, \alpha_2 - \beta_2), (\alpha_3, \alpha_3 - \beta_3))$.

9. Homology Calculations

We give some homology calculations that will be helpful.

It should be no surprise to learn that $H_1(S^3) = 0$. Let K be a knot in S^3 and let $N(K)$ be a tubular neighborhood of K. Now let $M_K = S^3 - \operatorname{int}(N(K))$. Then M_K is the **knot complement space** of K. Then $H_1(M_K) \approx \mathbb{Z}$.

We will only sketch the proof. The homology group of the boundary of $N(K)$ has two generators. We take these to be a meridian M of $N(K)$ and the preferred longitude L. Now L bounds a Seifert surface and hence is null homotopic $L \sim 0$. That leaves only M and it can be shown no power of M is homologous to 0. Thus $H(M_K) \approx \mathbb{Z}$.

Take any knot complement manifold and glue in a solid torus using (p, q) surgery. Call the manifold M. Now $|q|$ times the meridian of the knot is homologous to 0. The solid torus core is homologous to a preferred longitude which in turn is homologous to 0. These two facts can be used to show that $H_1(M) \approx \mathbb{Z}/|q|\mathbb{Z}$.

A special case is the lens spaces, where the role of p and q got switched giving $H_1(L(p, q)) \approx \mathbb{Z}/p\mathbb{Z}$.

GET P's and Q's straightened out!!

The next result is a bit more involved.

Theorem 9.1. *Let M be a Seifert fibered space with orbit surface S^2 and n exceptional fibers with crossing numbers (α_i, β_i) for $i = 1, \ldots, n$ and obstruction term β_0. Then $H_1(M)$ is the Abelian group with $n + 1$ generators, which we denote $F, M_1, \cdots, M_n$, and $n + 1$ relations,*

$$\beta_0 F - M_1 - \cdots - M_n = 0 \text{ and } \beta_i F + \alpha_i M_i = 0$$

for $i = 1, ..., n$. The F generator can be represented by any ordinary fiber and the M_i's can be represented by meridians of the tubular neighborhoods of the exceptional fibers.

We shall give only a rough justification. Recall that M_0' is a disk with n holes cross S^1. The homology of the disk with n holes is generated by $M_1, \cdots, M - n$ and hence is isomorphic to $\mathbb{Z}^n$. Taking the cross product with S^1 creates a new generator which we shall call F. Thus, $H_1(M_)' \approx \mathbb{Z}^{n+1}$.

Each time we glue a solid torus to a boundary component of M_0' we we get a new relation since some nontrivial curve is identified to a meridian which of course bounds a disk. Let V_i be the solid torus whose core will be the i^{th} exceptional fiber. The ordinary fiber F can be isotoped to a fiber near ∂V_i. Then the gluing prescription means that $\beta_i F + \alpha_i M_i$ now bounds a disk inside V_i and hence we have the relation $\beta_i F + \alpha_i M_i = 0$, for $i = 1, \ldots, n$. The sum $M_1 + \cdots + M_n$ is homologous to the curve that that we called $\partial D^2 \times *$ earlier. Let $E = M_1 + \cdots + M_n$. Then the crossing curve that is glued to a meridian of V_0 is $\beta_0 F - E$ which gives the other relation.

We will make use of the following corollary.

Corollary 9.2. *For $n = 3$ the order of $H_1(M)$ is*

$$|\, \beta_0\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \beta_2\alpha_1\alpha_3 + \beta_3\alpha_1\alpha_2 \,|.$$

Proof. The relations can presented in matrix from as

$$\begin{bmatrix} \beta_0 & -1 & -1 & -1 \\ \beta_1 & \alpha_1 & 0 & 0 \\ \beta_2 & 0 & \alpha_2 & 0 \\ \beta_3 & 0 & 0 & \alpha_3 \end{bmatrix}.$$

By Theorem 6.1 we just have to compute its determinant. □

Exercise 10. If there are no exceptional fibers and the orbit surface is S^2 then M is a lens space. If the obstruction term is $\beta_0 \in \mathbb{Z}$ then which lens space do we get?

10. Surgery along a Torus Knot

Theorem 10.1. *Let K be an (r, s) torus knot in S^3 and let M be the manifold the results from preforming a (p, q) Dehn surgery along K. Set $\sigma = rsp - q$.*

1. *If $|\sigma| > 1$ then M is a Seifert manifold over S^2 with three exceptional fibers of multiplicities $\alpha_1 = s$, $\alpha_2 = r$ and $\alpha_3 = |\sigma|$. The proof will show how to compute the obstruction term and the three β_i terms.*

(2) *If* $\sigma = \pm 1$ *then* M *is the lens space* $L(|q|, ps^2)$.
(3) *If* $\sigma = 0$ *then* M *is* $L(r, s)\#L(s, r)$.

Proof. THE SET UP. We will use the $\mathbb{R}^3 \cup \infty$ model for S^3. Let U be the unit circle in the xy-plane and let Z be the z-axis union $\{\infty\}$. We first partition S^3 into two solid tori, V_1' and V_2', with common boundary, where the core of V_1' is U and the core of V_2' is Z. Let M_i' and L_i' be preferred meridian-longitude pairs for V_i', $i = 1, 2$, where $M_1' = L_2'$ and $L_1' = M_2'$.

Now let K be an (r, s) torus knot on $\partial V_1' = \partial V_2'$. Let $N(K)$ be a tubular neighborhood of K that is small enough that $V_i = V_i' - \text{int}N(K)$ are still solid tori, $i = 1, 2$. Thus $V_1 \cup V_2$ is the knot complement space of K. The V_i look like the V_i' but with a trough dug out along K.

The intersection $\partial V_i' \cap \partial N(K)$ consists of two curves parallel to K. Call them K_i, $i = 1, 2$. They partition the boundary of each V_i into two annuli. Let A be the annulus between the K_i that the V_i have in common, that is $A = V_1 \cap V_2$. Let A_1 be $\partial V_1 - \text{int}A$ and A_2 be $\partial V_2 - \text{int}A$, that is A_1 and A_2 are the "bottoms" of the troughs.

Let (M_i, L_i) be meridian-longitude pairs for V_i, $i = 1, 2$ chosen by retracting M_i' and and L_i' through $N(K)$ as shown in Figure 11.

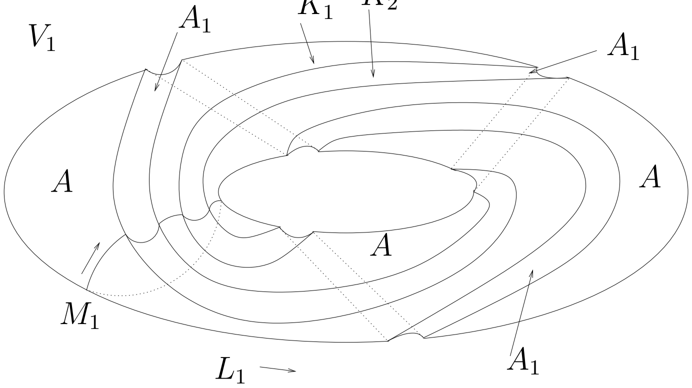

FIGURE 11. The Set Up: $\partial V_1 = A \cup A_1$; $\partial A = \partial A_1 = K_1 \cup K_2$

Let (M_3, L_3) be a preferred meridian-longitude pair for $N(K)$. Recall this means $L_3 \sim 0$ in $V_1 \cup V_2$. (See the end of Section 5.)

Next let V_4 be a new solid torus with meridian-longitude pair (M_4, L_4). This is the solid torus we shall glue to $V_1 \cup V_2$ via a homeomorphism

$$h : \partial V_4 \to \partial(V_1 \cup V_2) = \partial N(K).$$

Let $\begin{bmatrix} q & a \\ p & b \end{bmatrix}$ be the matrix representing h. Thus $h(M_4) = qM_3 + pL_3$.

The following homology calculation, done with respect to $V_1 \cup V_2$, will be used repeatedly. $K_1 \sim rZ$ and $Z \sim sM_3$ so $K_1 \sim rsM_3$. Thus $K_1 - rsM_3 \sim 0 \sim L_3$.

$$\begin{aligned} h(M_4) &= pL_3 + qM_3 \\ &\sim p(K_1 - rsM_3) + qM_3 \\ &= pK_1 - (rsp - q)M_3 \\ &= pK_1 - \sigma M_3. \qquad\qquad (*) \end{aligned}$$

Case 1. Suppose $|\sigma| \geq 2$. We augment our set up by using an (r, s) fibration of S^3 such that the knot K is now a fiber and the neighborhood of K is required to be a fibered neighborhood. In this fibration U and Z have multiplicities s and r respectively. We will need to figure out the fibration of V_4 such that h preserves fibers. The orbit surface will remain S^2 for the surgeried manifold.

Now M_3 is a crossing curve on $\partial N(K)$. Therefore, the fibration of V_4 will have a fiber of multiplicity $|\sigma|$ as its core. So we have a Seifert fibered space of the form $(S^2, \beta_0, (s, \beta_1), (r, \beta_2), (|\sigma|, \beta_3))$.

Example 3. Suppose K is a $(3, 2)$ torus knot and that the Dehn surgery is of type $(6, 31)$ Thus $r = 3$, $s = 2$, $p = 6$, $q = 31$ and $|\sigma| = 5$.

Recall that by Corollary 9.2 the order of $H_1(M)$ is $30\beta_0 + 15\beta_1 + 10\beta_2 + 6\beta_3$. But from Section 9 we have that the order of $H_1(M)$ is $|q|$. Thus we want to find any solutions to

$$30\beta_0 + 15\beta_1 + 10\beta_2 + 6\beta_3 = \pm 31.$$

First assume it is $+31$. Since $s = 2$ we know that $\beta_1 = 1$. Using this and dividing both sides by 2 gives

$$15\beta_0 + 5\beta_2 + 3\beta_3 = 8.$$

Since $r = 3$ and $|\sigma| = 5$ we know $\beta_2 \in \{1, 2\}$ and $\beta_3 \in \{1, 2, 3, 4\}$. Clearly then $\beta_0 \leq 0$. Suppose $\beta_0 = 0$. Then $\beta_2 = 1$ and $\beta_3 = 1$ are the only solutions. If $\beta_0 < 0$ you can check that there are no other solutions.

For -31 the result is $\beta_0 = -3$, $\beta_1 = 1$, β_2 and β_4, which is equivalent.

Equations of this type are called linear Diophantine equations. There is a general algorithm for finding their solutions. This is a fairly standard topic in number theory textbooks [11, 12]. Try some other choices for r, s, p and q. Is the solution always unique?

Case 2. Suppose $\sigma = \pm 1$. Recall $h = \begin{bmatrix} q & a \\ p & b \end{bmatrix}$. The topological type of the Dehn surgery is determined solely by q and p. We are free to chose a and b so long as $\det h = \pm 1$. If we chose $a = rs$ and $b = 1$ we get $\det h = qb - ap = q - rsp = -\sigma = \mp 1$.

From equation $(*)$ we have $h(M_4) \sim pK_1 \mp M_3$. We didn't need to study $h(L_4)$ in Case 1, but here we do.

$$h(L_4) = rsM_3 + L_3 \sim rsM_3 + K_1 - rsM_3 \sim K_1.$$

In other words the longitude on V_4 is going to a curve parallel to the knot K.

Now we glue V_4 to V_1. We claim that in this case the result must be a solid torus. Both V_4 and V_1 can be written as $S^1 \times D^2$. For specificness we write

$$V_1 = S_1 \times D_1 \quad \text{and} \quad V_4 = S_4 \times D_4.$$

Let α be an arc in ∂D_4 and let $A_4 = S_4 \times \alpha$ be the annulus in ∂V_4 that has core L_4 and will be identified to A_1 in ∂V_1. Each copy of the disk D_1 in V_1 meets A_1 in r arcs. We can choose the homeomorphism to take each $* \times \alpha$ arc to a component of $A_1 \cap (*' \times D_1)$. Then the union $V_1 \cup V_4$ can be realized as a product $S^1 \times DD$ where DD is a disk formed by gluing r copies of D_4 to $0 \times D_1$ along copies of α. See Figure 12.

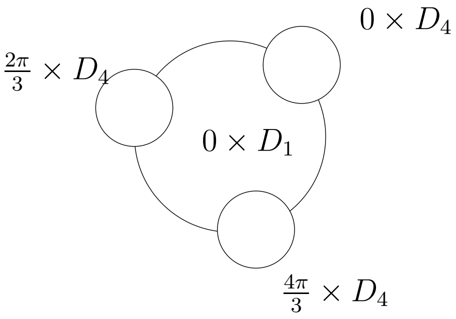

Figure 12. Cross section of $V_1 \cup V_4$

Now that we see that $V_1 \cup V_4$ is a solid torus it is immediate that $V_1 \cup V_4 \cup V_2$ is the gluing of two solid tori and hence a lens space. It remains to do some homology calculations to determine which lens space it is.

Remember we have four sets of meridional-longitudinal pairs. But now we need a fifth since $V_1 \cup V_4$ is a new solid torus. Call these

(M_5, L_5). We will compute M_5 in terms of (M_2, L_2), that is we shall solve

$$M_5 = xM_2 + yL_2;$$

we won't need to find L_5. Then we will have the $L(x, y)$ lens space.

Looking at Figure 13 we see that

$$L_1 \sim M_2 + rM_3 \text{ and } M_1 \sim L_2 - sM_3.$$

These homology calculations are still in $V_1 \cup V_2$, the knot complement space. (Figure 13 attempts to show V_1 with a small tubular neighborhood of the core drilled out, thus creating a thick torus ($T^2 \times I$) with a trough dug out along K, which is presented as a cube with the top & bottom sides and the left & right sides respectively identified. Not shown is V_2 which would look like its mirror image. Recall V_1 and V_2 are glued along the annulus A. To the sides of V_1 are diagrams showing how the various meridians and longitudes are related if $r = 3$ and $s = 2$. It will likely take a good while for the reader to see this.)

Now we glue in V_4. Recall that K_1 is an (r, s) curve. Thus,

$$M_4 \quad \sim \quad pK_1 - \sigma M_3 \sim p(rM_1 + sL_1) - \sigma M_3 = prM_1 + psL_1 - \sigma M_3.$$

And finally,

$$\begin{aligned}
M_5 \quad &\sim \quad M_1 - \sigma s M_4 \\
&\sim \quad M_1 - \sigma s(prM_1 + psL_1 - \sigma M_3) \\
&\sim \quad (1 - \sigma rsp)M_1 - \sigma ps^2 L_1 + sM_3 \\
&\sim \quad (1 - \sigma)(L_2 - sM_3) - \sigma ps^2(M_2 + rM_3) + sM_3 \\
&= \quad (1 - \sigma rsp)L_2 - s(1 - \sigma rsp)M_3 - \sigma ps^2 M_2 - \sigma prs^2 M_3 + sM_3 \\
&= \quad (1 - \sigma rsp)L_2 - \sigma ps^2 M_2 + (-s + \sigma prs^2 - \sigma prs^2 + s)M_3 \\
&= \quad (1 - \sigma rsp)L_2 - \sigma ps^2 M_2,
\end{aligned}$$

where these homology calculations are in the new manifold M. If $\sigma = 1$ then $M_5 \sim -qL_2 - ps^2 M_2$; if $\sigma = -1$ then $M_5 \sim +qL_2 + ps^2 M_2$. Therefore $M \cong L(|q|, ps^2)$ as claimed.

CASE 3. Suppose $\sigma = 0$. Then $q = rsp$. Since q and p can only have 1 as a common divisor, and $p > 0$ by convention, it must be that $p = 1$. Thus by equation $(*)$ $h(M_4) \sim K_1$. That is the meridian M_4 of V_4 is identified with K_1. Another meridian, M_4', of V_4 will then be identified with K_2.

We construct the union $V_1 \cup V_4 \cup V_2$ in stages. Partition V_4 into two solid cylinders C and C' by choosing two disjoint meridional disks D and D' in V_4 with $\partial D = M_4$ and $\partial D' = M_4'$. The boundary of C minus the interiors of the two disks is an annulus, call it A_3. See Figure 14.

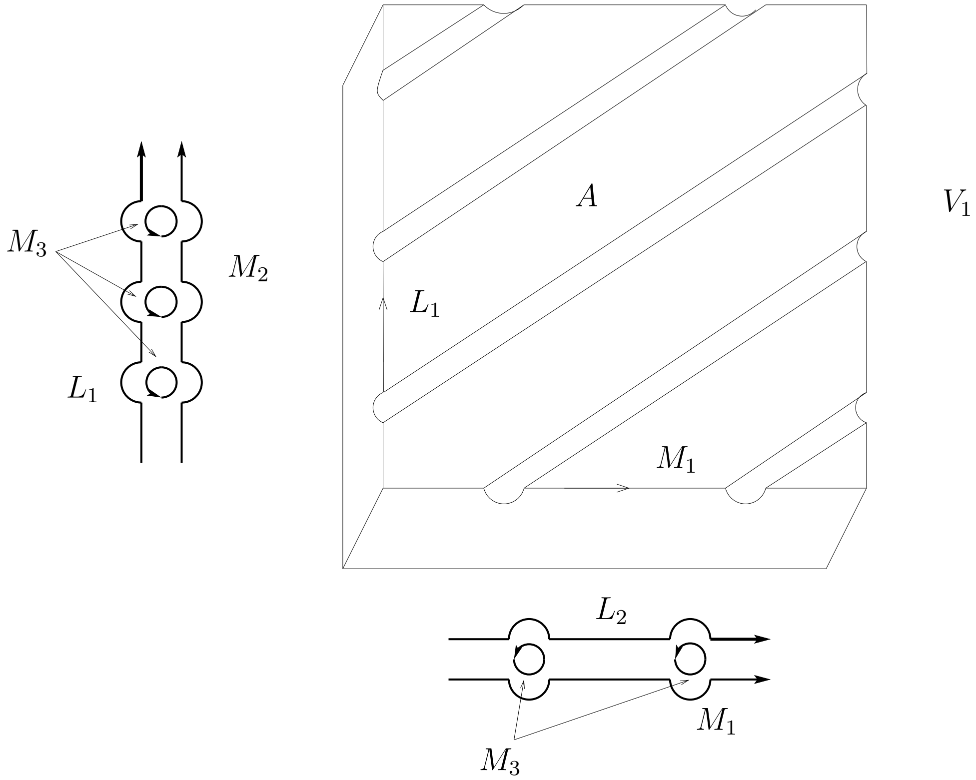

FIGURE 13. V_1 for $r = 3$, $s = 2$

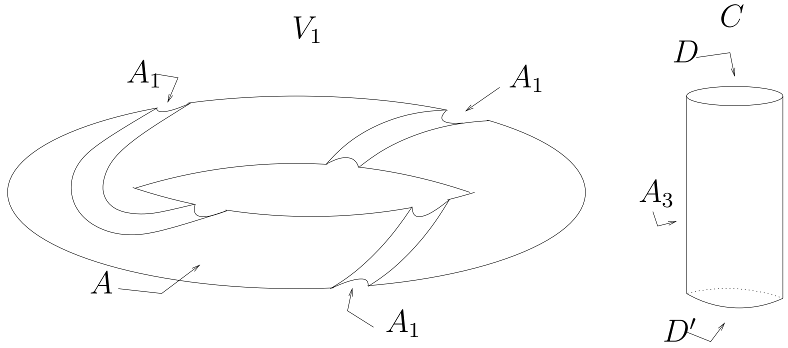

FIGURE 14. V_1 and C

We glue C to V_1 by attaching A_3 to A_1. This space has boundary $D \cup A \cup D'$, which must be a 2-sphere, call it S. Likewise $V_2 \cup C'$ is a manifold whose boundary is a 2-sphere, call it S'. Then M is formed from $V_1 \cup C$ and $V_2 \cup C'$ by identifying their boundary spheres. Thus

M is the connected some of two manifolds. We will show that $V_1 \cup C$ and $V_2 \cup C'$ are lens spaces with an open 3-ball removed.

In fact we claim $V_1 \cup C$ is homeomorphic to the lens space $L(r, s)$ minus an open 3-ball. To see this we glue a 3-ball B to $V_1 \cup C$ and show that this space is $L(r, s)$. We do this in two steps. Partition B into a solid torus V_B and a solid cylinder C_B as shown in Figure 15. Let D_B and D'_B be the disks composing $\partial B \cap C_B$. Let $A_B = \partial B - \operatorname{int}(D_B \cup D'_B)$. Attach C to C_B by identifying D_B to D_1 and D'_B to D_2. Then $C \cup C_B$ is a solid torus. Now attach V_1 to V_B by identifying the annulus A_1 with A_B. As before this forms a solid torus. Thus,

$$V_1 \cup C \cup B = (V_1 \cup V_B) \cup (C \cup C_B)$$

is the union of two solid tori and is a lens space. Since a meridian ∂D_1 of $C \cup C_B$ is identified to an (r, s) curve on $\partial(V_1 \cup V_B)$ the lens space is $L(r, s)$.

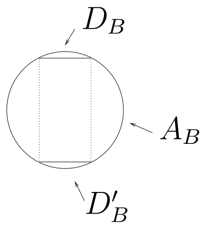

FIGURE 15. The 3-ball B partitioned

If we attach C' to V_2 we can show that this is homeomorphic to the lens space $L(s, r)$ minus an open ball. Thus M is formed by taking the connected sum of $L(r, s)$ and $L(s, r)$. Note: It is known that $L(r, s)\#L(s, r)$ cannot be given a Seifert fibration. ☐

This concludes the proof. The figure-8 knot has just four crossings and is not a torus knot. Surgery along the figure-8 knot is the next logical topic to pursue. This turns out to be much more involved than surgery along torus knots. See [16] as a place to start. There is a large and growing literature on this topic.

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