A zeta function for positive templates

Mike Sullivan
CUNY

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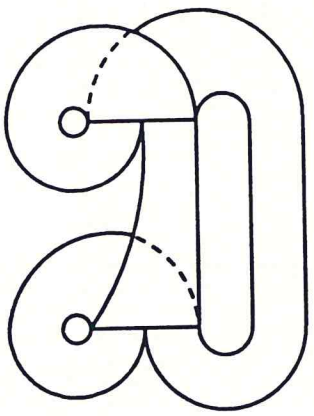
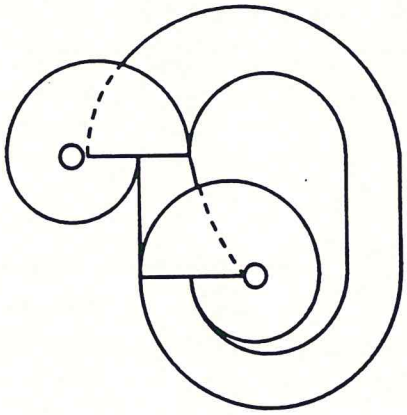
DEFINITION: A *template* is a compact branched 2-manifold with smooth expansive semi-flow.

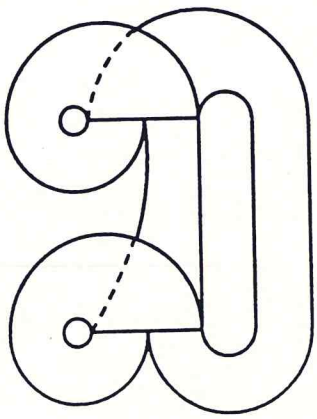
THEOREM: [Birman & Williams, 1983] Given a flow ϕ_t on a 3-manifold M having a hyperbolic structure on its chain recurrent set, e.g. a pseudo-Anosov flow, there is a template (T, ϕ'_t), $T \subset M$, such that with a finite number of specified exceptions (usually one or two) the knots and links in (M, ϕ_t) correspond one-to-one with those in (T, ϕ'_t) via an ambient isotopy. (The thm also holds for pseudo-Anosov.)



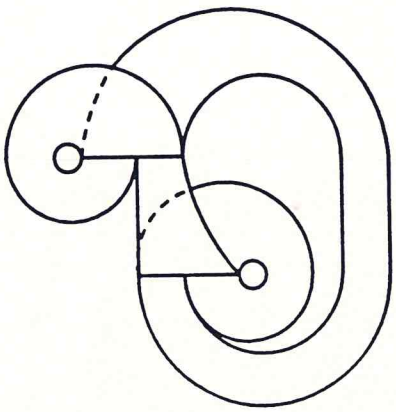
Template A



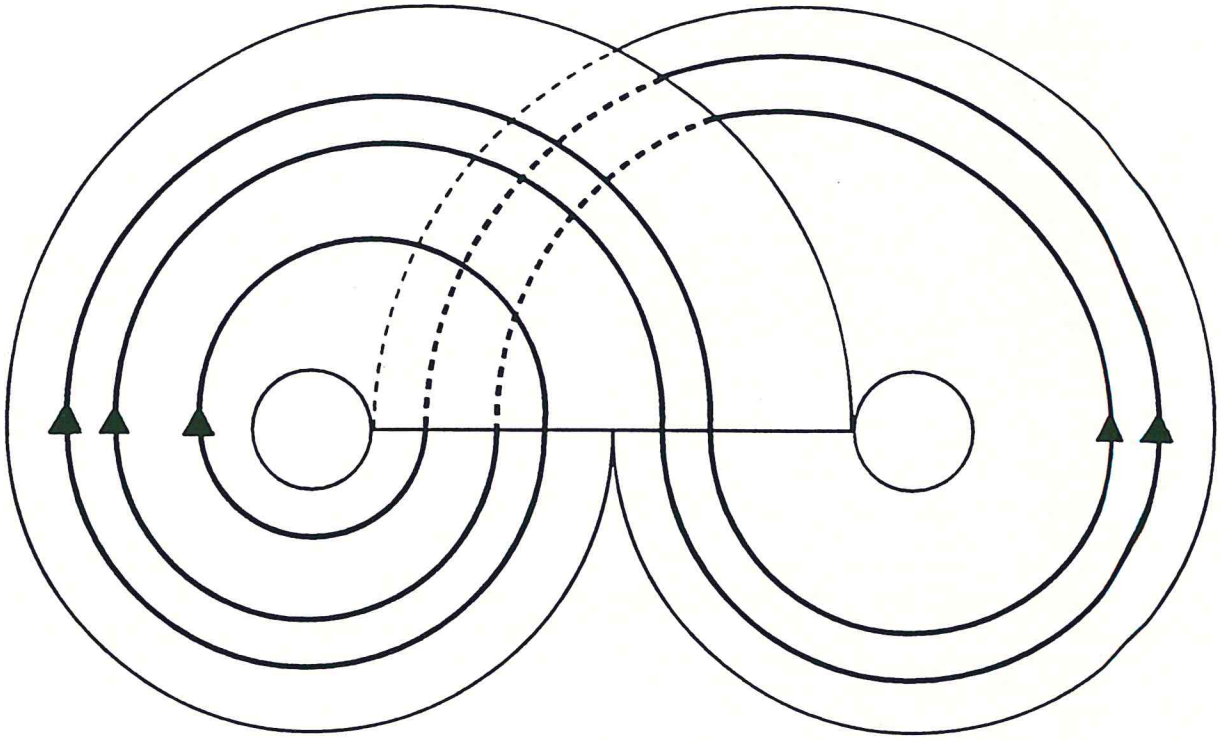
Template B



Template A+

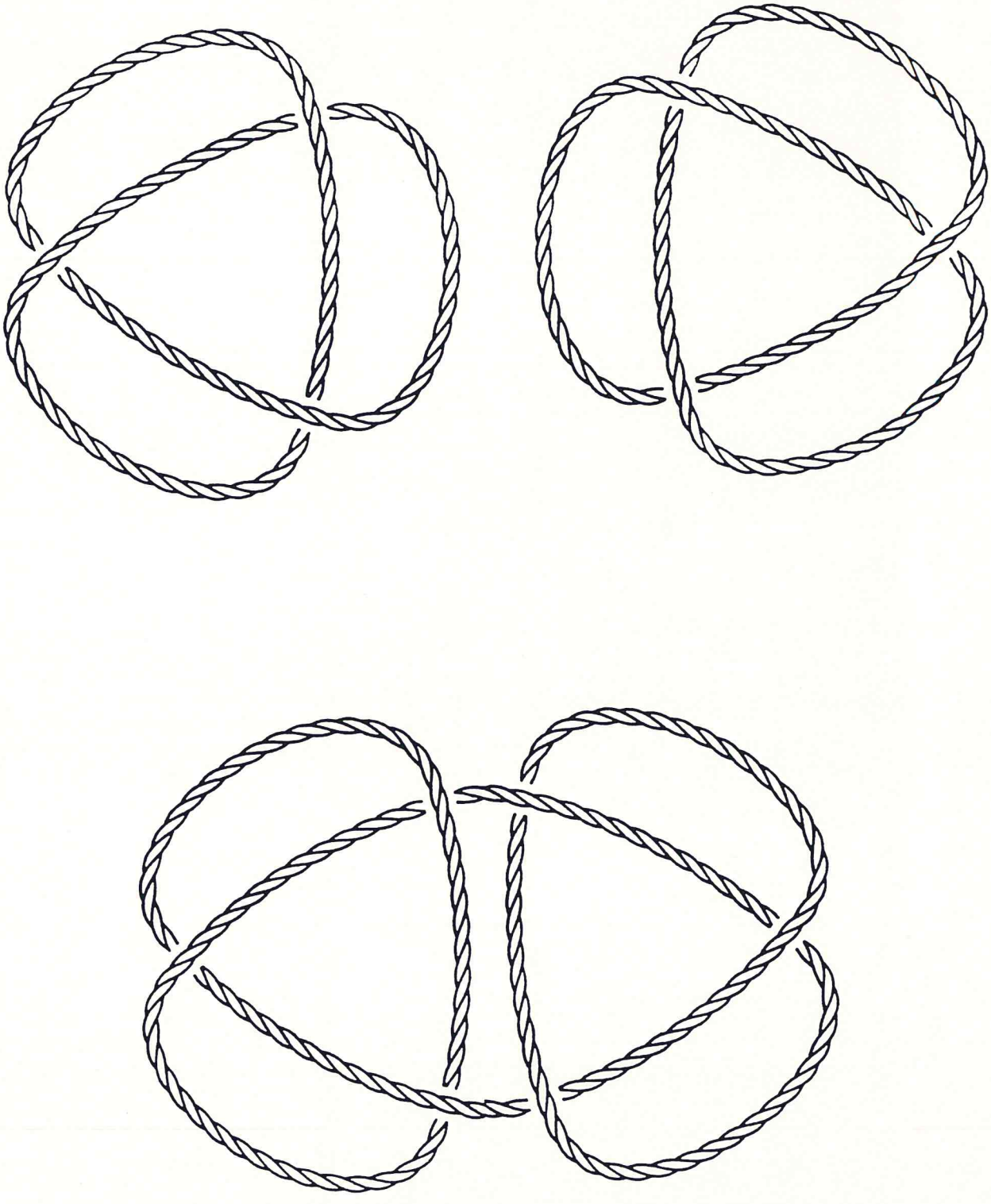


Template B+



Lorenz Template with a Trefoil Knot Shown

Lorenz knots are known to be prime [Williams, 83].



Definition: A knot $k \in S^3$ is *composite* if there exists a sphere S^2 such that $S^2 \cap k$ is just two points, p and q, and if γ is any arc on S^2 joining p to q, then the knots

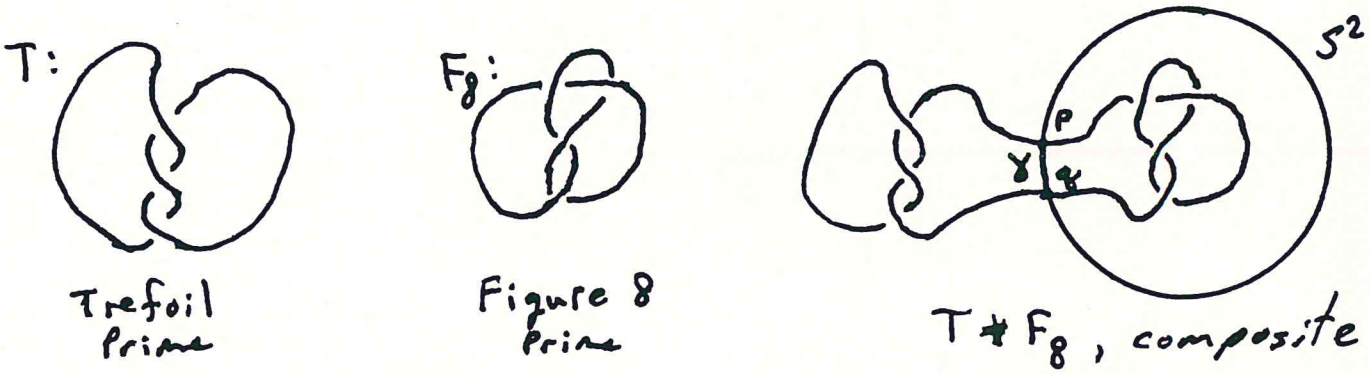
$$k_1 = \gamma \cup (k \cap \text{outside of } S^2) \quad \textit{and}$$
$$k_2 = \gamma \cup (k \cap \text{inside of } S^2),$$

are nontrivial. We call k_1 and k_2 *factors* of k and write

$$k = k_1 \# k_2.$$

If a knot isn't composite then it is *prime*.

Examples:

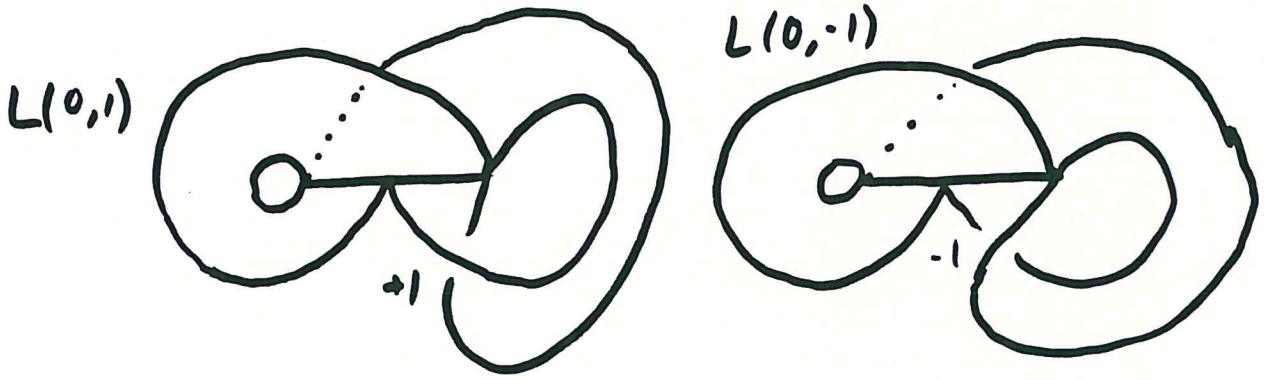


Theorem of Schubert [1953]: Knots can be factored uniquely into primes. See Rolfsen: Knots and Links.

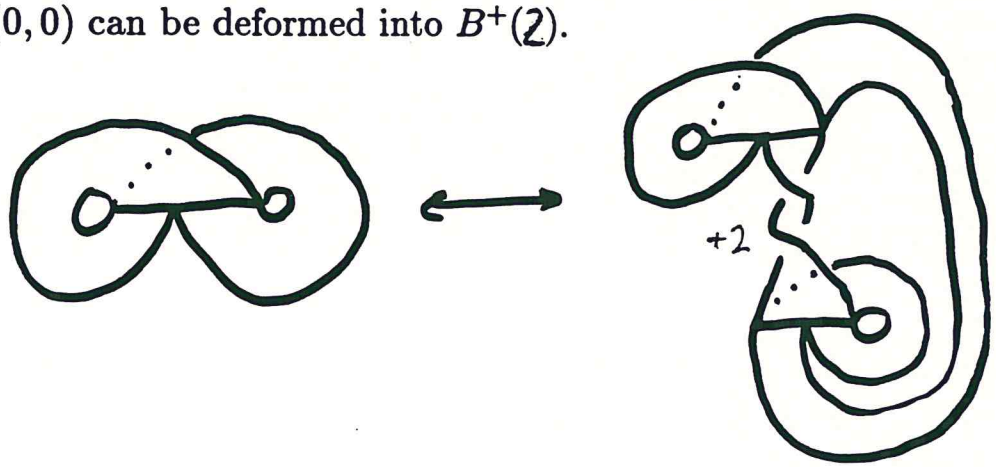
When are two templates the "same" ?
Are some templates more "complex" than others ?

EXAMPLES:

1) $L(0,1)$: All knots prime [Williams 83]. $L(0,-1)$: Has composite knots [W 83] with any number of prime factors [S 92]. Also, $L(0,1)$ is a "subtemplate" of $L(0,-1)$. Is $L(0,-1)$ more "complex".

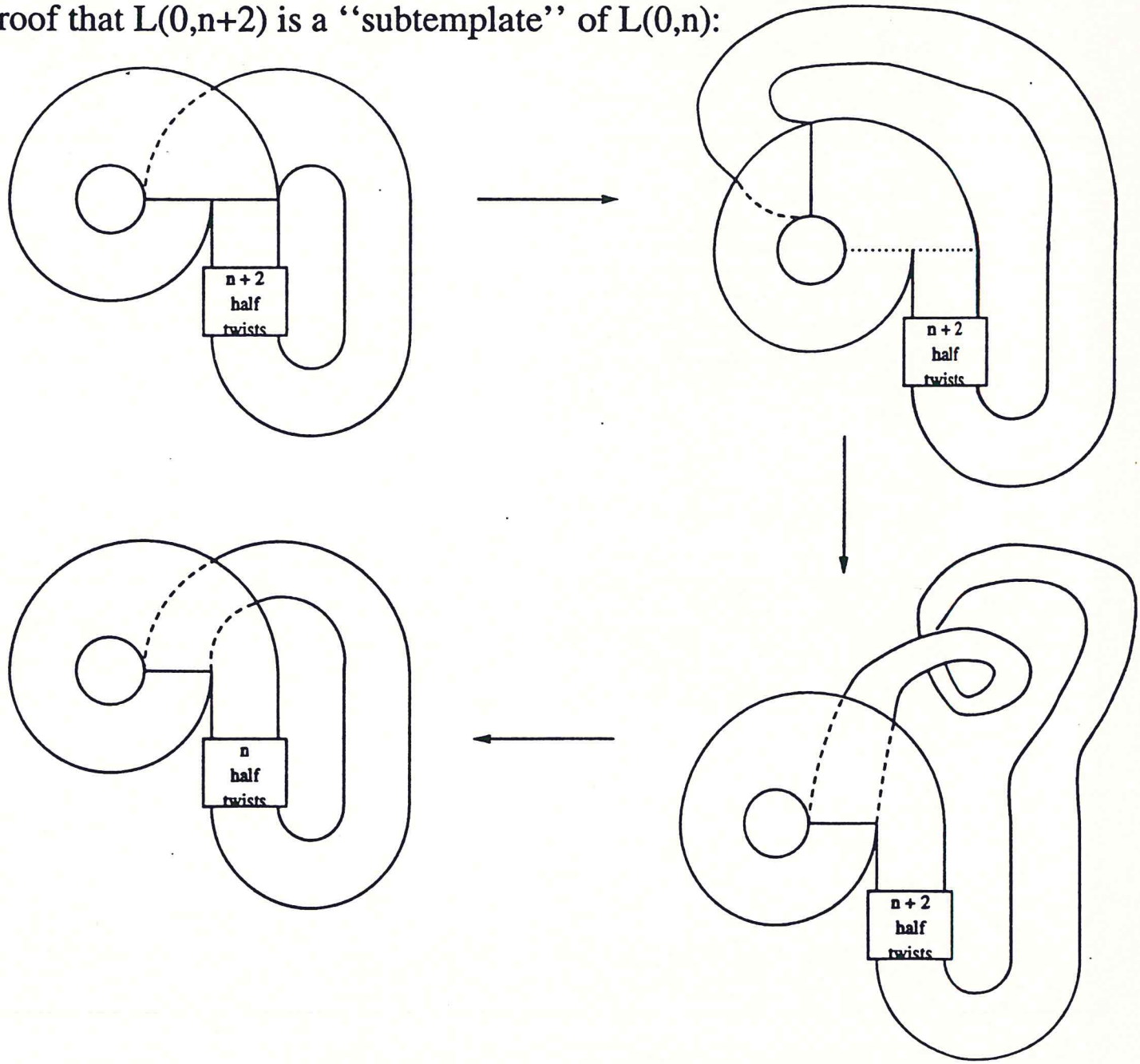


2) $L(0,0)$ can be deformed into $B^+(2)$.

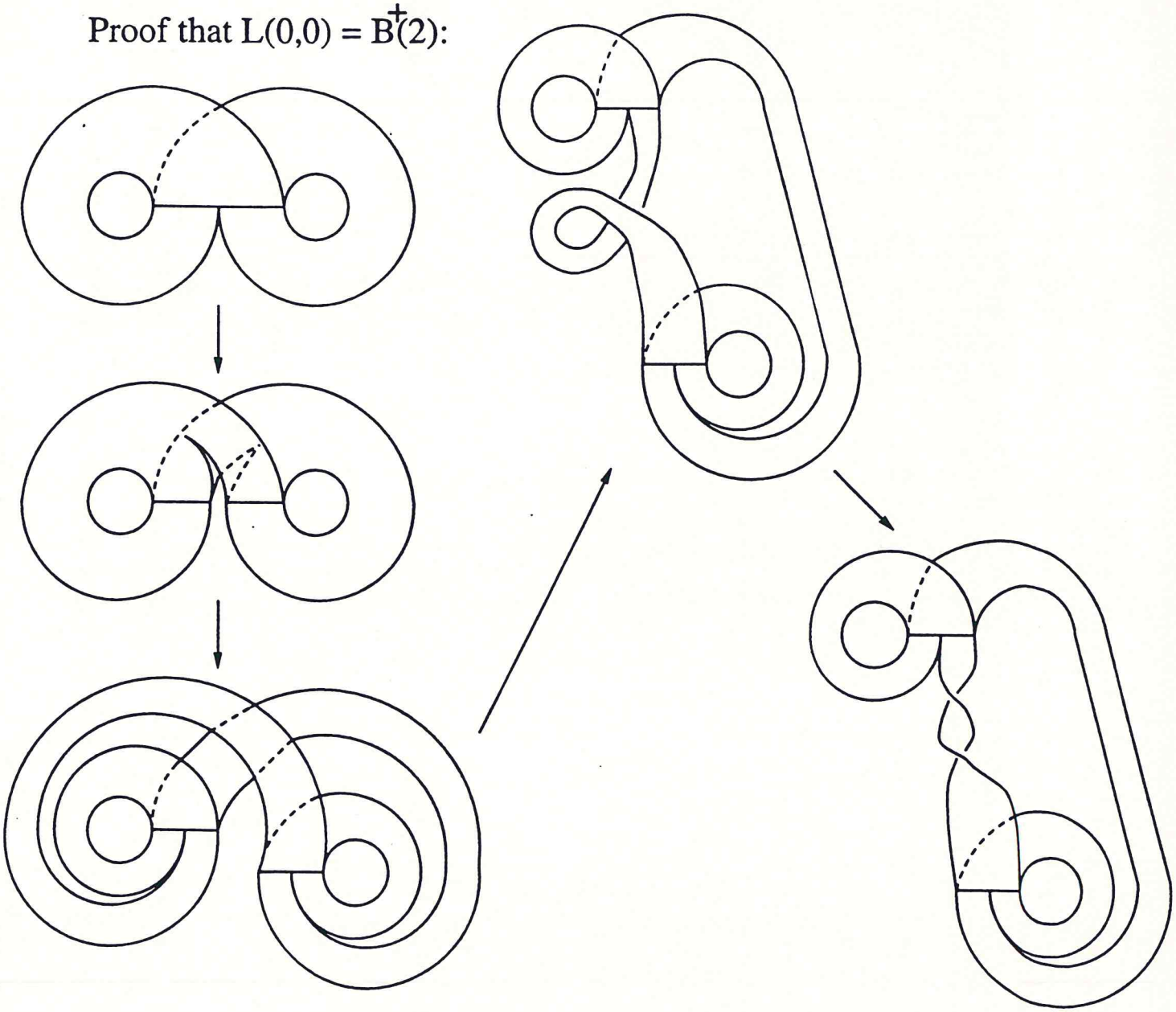


3) $A^+(0)$: All knots prime. $B^+(0)$: Composite knots but with only two prime factors. But $A^+(1) = B^+(1)$.

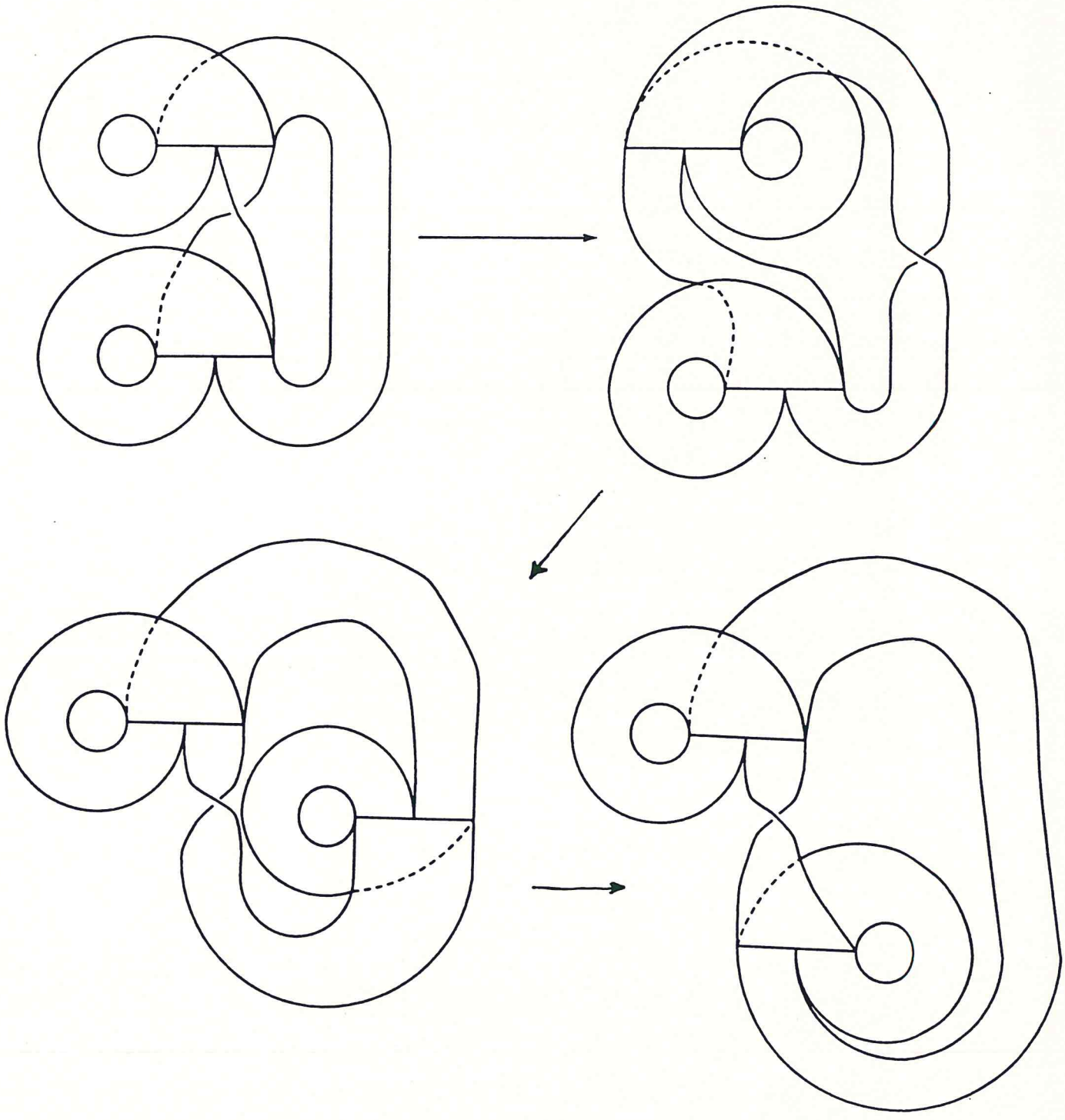
Proof that L(0,n+2) is a ''subtemplate'' of L(0,n):



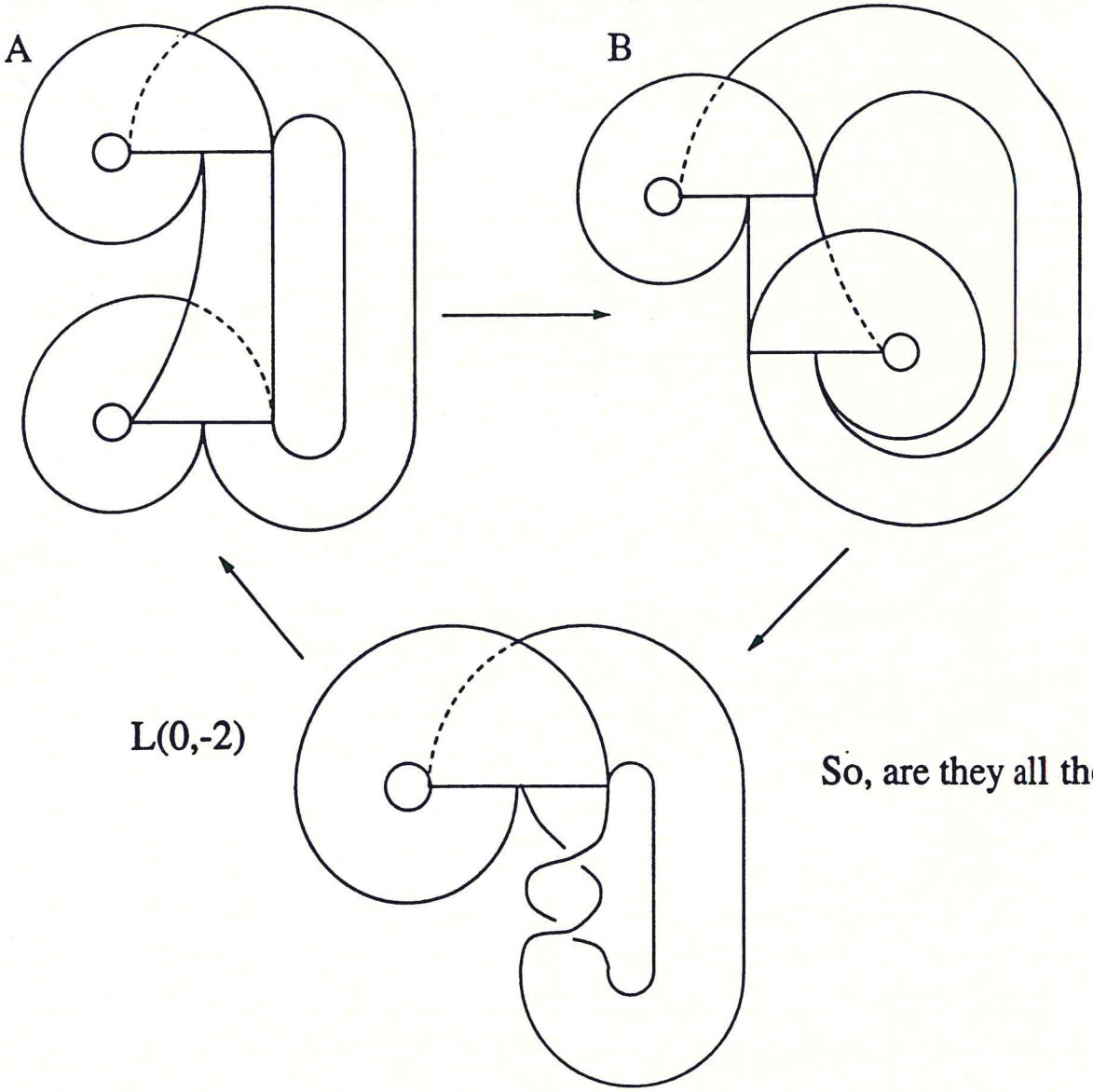
Proof that $L(0,0) = B^{+}(2)$:



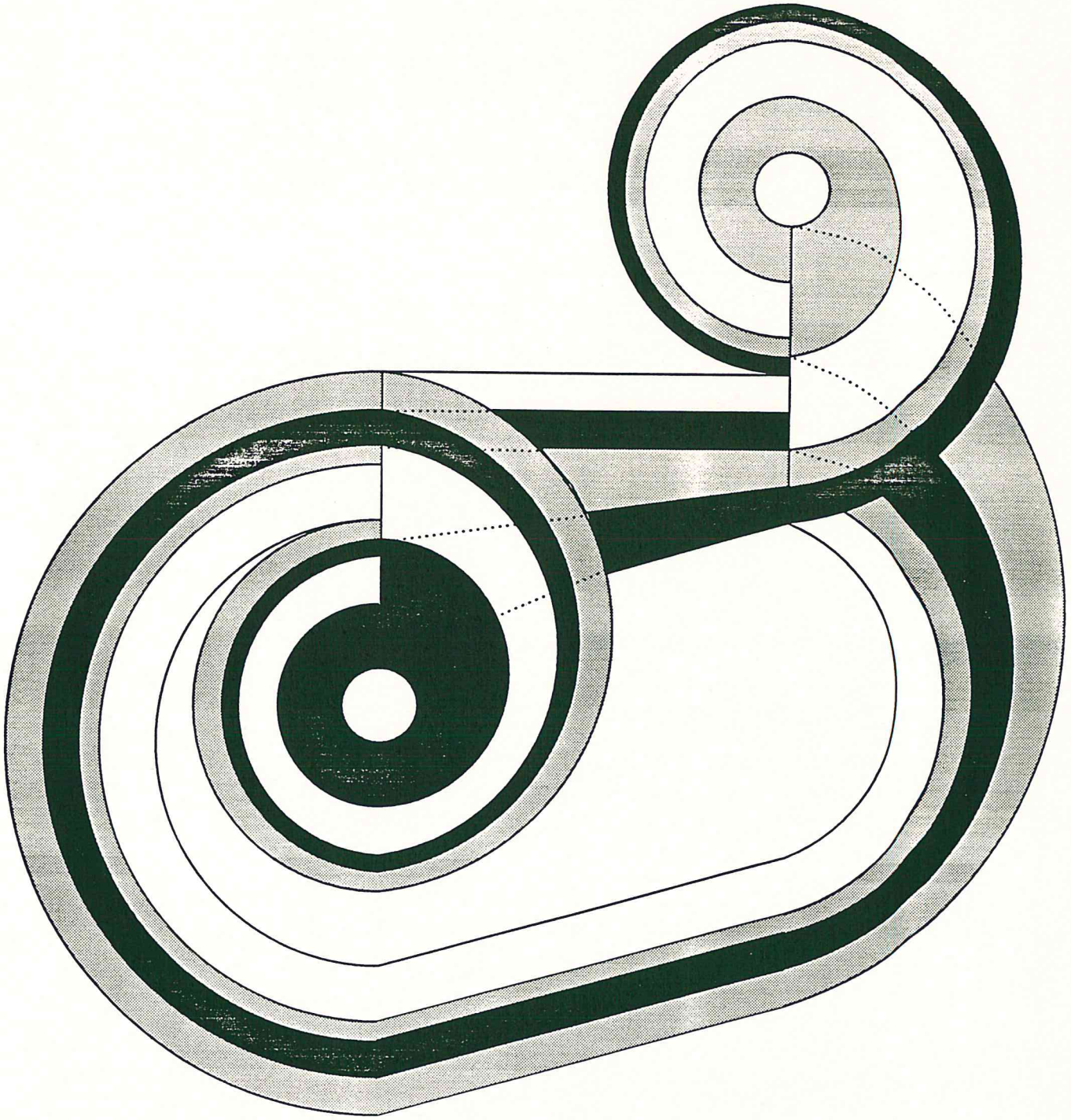
Proof that $A^{+}(1) = B^{+}(1)$:



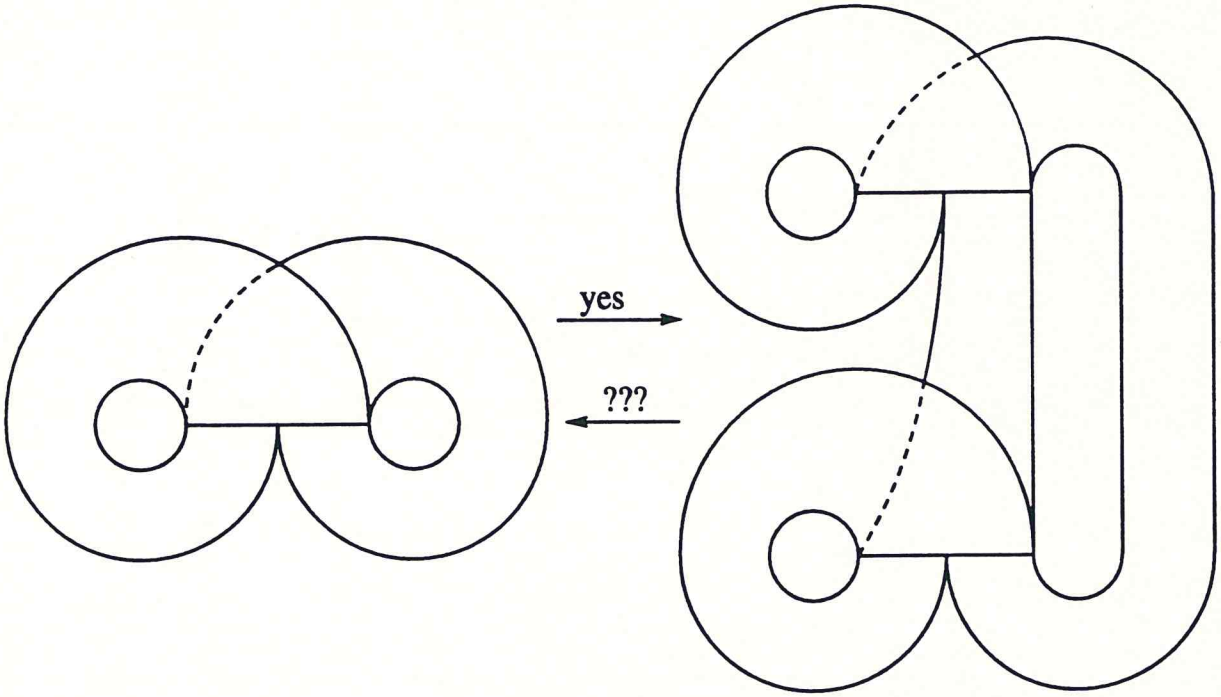
A weird example: each template is a subtemplate of the others.



So, are they all the ''same'



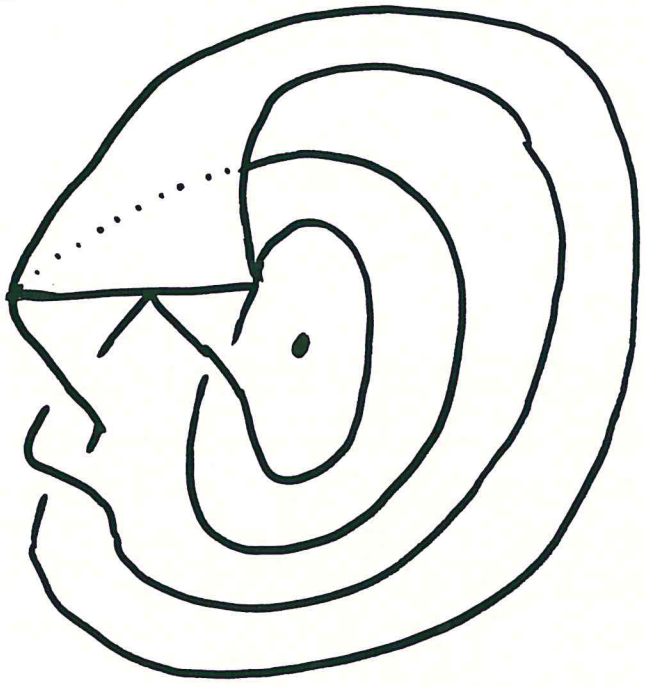
A Problem: L(0,0) is in $A^+(0)$. Is $A^+(0)$ in L(0,0) ?



The answer is no. Every closed orbit can be assigned a TWIST INDEX by gluing a ribbon normal to the template along the orbit and counting the number of twists in the ribbon. $A^+(0)$ has three twist zero orbits while L(0,0) has only two. Why didn't this work with A,B, and L(0,-2) ? They had positive and negative crossings and infinitely many twist zero orbits.

We can generalize this. Let Ti = # orbits with twist index i, an integer. For positive templates Ti is always finite. Rough proof: Arrange the template as a braid [Franks-Williams]. Every time we go around the braid axis the twist index increase by at least one full twist. Since the semi-flow is expansive, the number of ''words'' is one-to-one with the number of orbits. There are only finitely many words of a given length. See Example:

L(2, 1)



Review of zeta functions after Franks [82] & Smale [67]

DEFINITION (via Franks): The *zeta function* of a map $f : M \longrightarrow M$ is the series in t

$$\zeta(f) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m\right)$$

where N_m is the cardinality of the fixed point set of f^m. When f has a hyperbolic chain recurrent set, $N_m < \infty, \forall m$ and $\zeta(f)$ is a rational function. This means that a finite set of numbers determine all the N_m.

If O_l is the number of periodic orbits of length l then

$$N_m = \sum_{l|m} l O_l.$$

According to Narasimhan (see Smale):

$$O_l = \frac{1}{l} \sum_{m|l} \mu(m) N_{l/m},$$

where

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } \exists \text{ a prime } p \ni p^2 | m \\ (-1)^r & \text{if } m = p_1, \ldots, p_r, \text{ for } r \text{ distinct primes.} \end{cases}$$

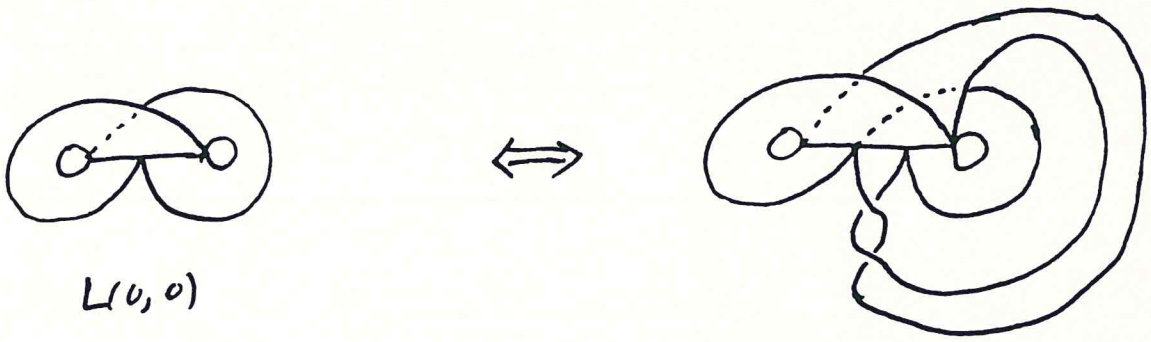
For many functions there exist a square matrix A such that $N_m = \operatorname{tr}(A^m)$. Then

$$\zeta(f) = 1/\det(I - tA).$$

EXAMPLE: The Lorenz template induces a map on the branch line. The matrix is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Which gives $\zeta = 1/(1-2t)$.



Notice that the zeta function is the same for all $L(m,n)$. We can deform $L(0,0)$ to the template above. Now

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Which gives a different $\zeta = 1/(1-3t+t^2)$.

We define a new zeta function, which will count the number of periodic orbits according to their twist index. For the templates above we get

$$A_1 = \begin{bmatrix} t^2 & t^2 \\ t^2 & t^2 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} t^2 & t^2 & t^2 \\ t^4 & t^4 & t^4 \\ 0 & t^2 & t^2 \end{bmatrix},$$

respectively.

These give:

$$\zeta_1 = 1/\det(I - A_1) = 1/(1-2t^2) = \zeta_2 = 1/\det(I - A_2).$$

For $L(m,n)$ we get

$$\zeta_{m,n} = 1/(1 - t^{m+2} - t^{n+2}).$$

Formally we define:

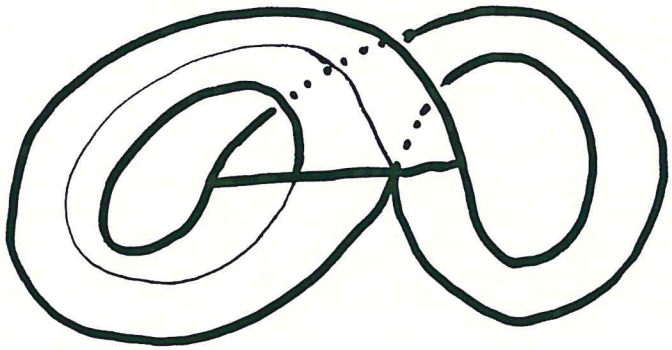
$$\zeta = \exp\left(\sum_{n=2}^{\infty}\left(\sum_{i|n} iT_{i-1}\right)\frac{t^n}{n}\right)$$

For $A^+(0)$ we get

$$\begin{bmatrix} t^2 & t^2 & t^2 \\ 0 & t^2 & t^2 \\ t^2 & t^2 & t^2 \end{bmatrix}.$$

Which gives $\zeta(A^+)(t) = 1/(1-3t^2+t^4)$.

An Attractive Example:



Here the matrix is:

$$\begin{bmatrix} 0 & t^2 & 0 \\ 0 & 0 & t^2 \\ t^2 & t^2 & 0 \end{bmatrix}.$$

This gives $\zeta = 1/(1-t^4-t^6)$.

T_i	$L(0,0)$	$L(0,1)$	A^+	sub-L
T_0	2	1	3	0
T_1	0	1	0	0
T_2	1	0	2	1
T_3	0	1	0	0
T_4	2	0	5	1
T_5	0	1	0	0
T_6	3	1	10	0
T_7	0	1	0	0
T_8	6	1	24	1
T_9	0	1	0	0
T_{10}	9	1	50	1
T_{11}	0	2	0	0