Alexander Polynomials Multiply under Connected Sums

A Dynamics Proof

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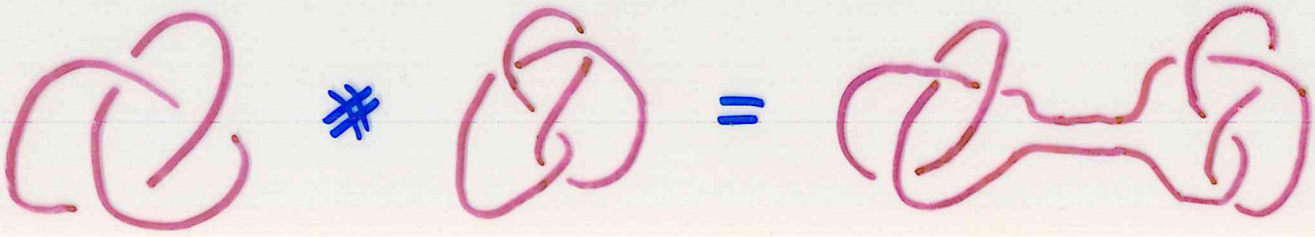
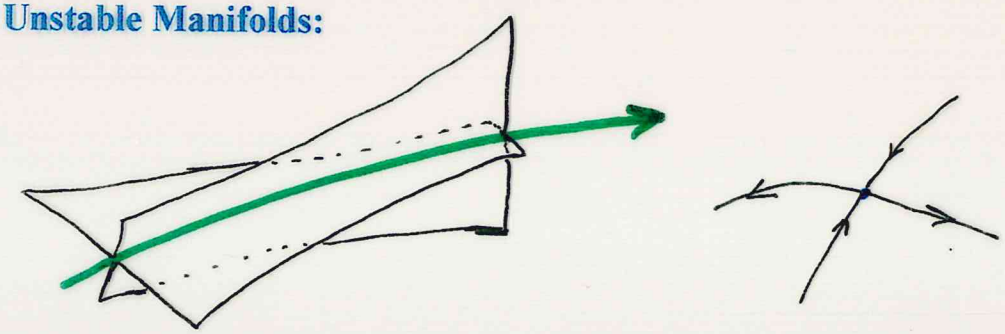


Flow Facts

Consider a smooth flow on a manifold.

- **Chain Recurrent Set:** **Points that flow back near to themselves:**

- **Hyperbolic Structure:** **No modulus one eigenvalues.**
- **=>Stable and Unstable Manifolds:**



- **Transversality Condition:** **stable and unstable manifolds meet transversely.**
- **Structural Stability:** **small pertibations do not break up orbits.**
- **H.S. on C.R.S. and T.C. => S.S.** **[Robinson,1975]**

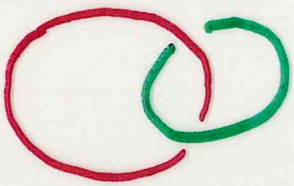
Morse-Smale Flows

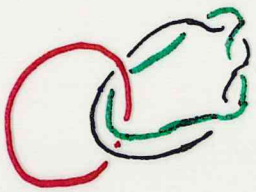
Definition: A smooth flow on a manifold is called Morse-Smale provided,

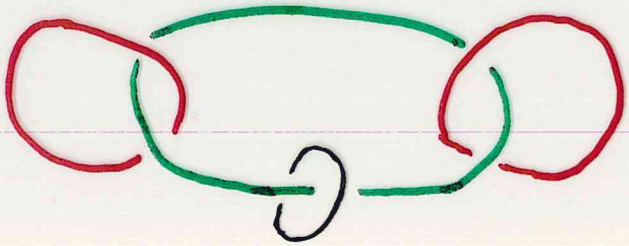
1. **The chain recurrent set consists of a finite collection of closed orbits and rest points,**
2. **There is a hyperbolic structure on the chain recurrent set, and**
3. **the transversality condition holds.**

Thus, Morse-Smale flows are structurally stable. It is also known that they have zero entropy.

Wada's Theorem [1989] tells us which (indexed) links can be realized as Chain Recurrent sets of nonsingular Morse-Smale flows on S^3.



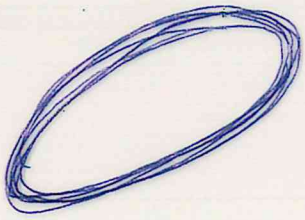




Smale Flows

Definition: A smooth flow on a manifold is called Smale provided,

1. **the chain recurrent set is no more than one dimensional,**



2. **there is a hyperbolic structure on the chain recurrent set, and**
3. **the transversality condition holds.**

Thus, Smale flows are structurally stable. It is also known that they have positive entropy, if they are not Morse-Smale.

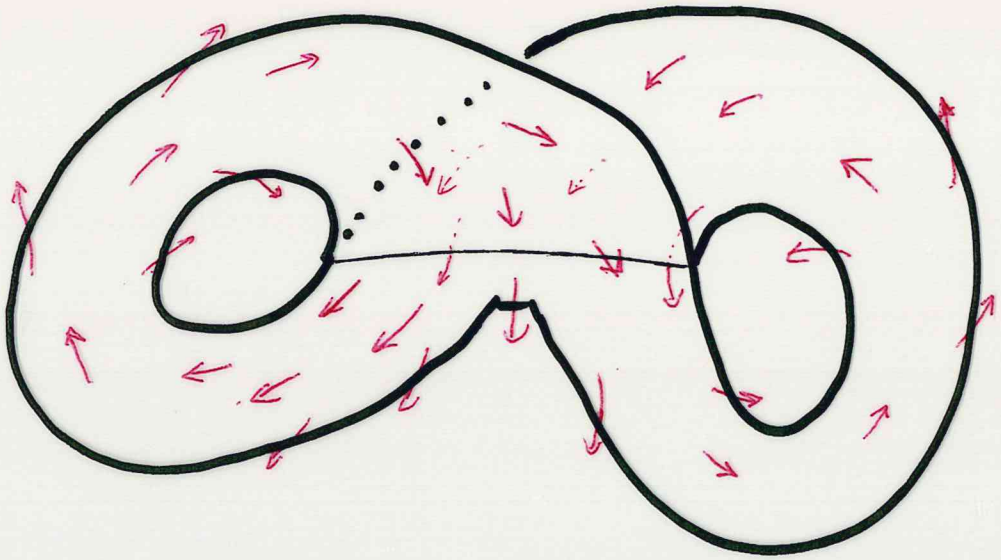
On S^3, Smale flows are the structurally stable flows.
[Hayashi,1997]

But, what do they look like?

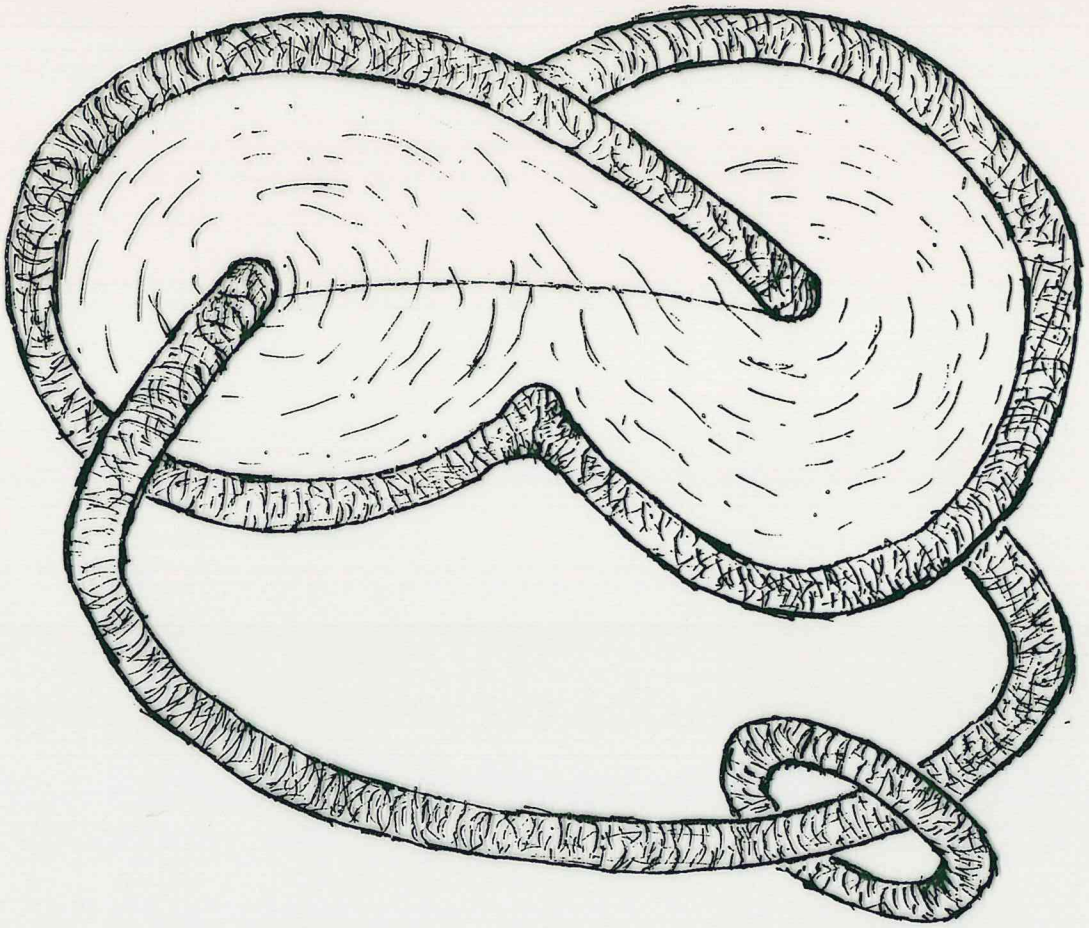
Templates for Chaotic Saddle Sets

Theorem: [Birman and Williams, 1983]

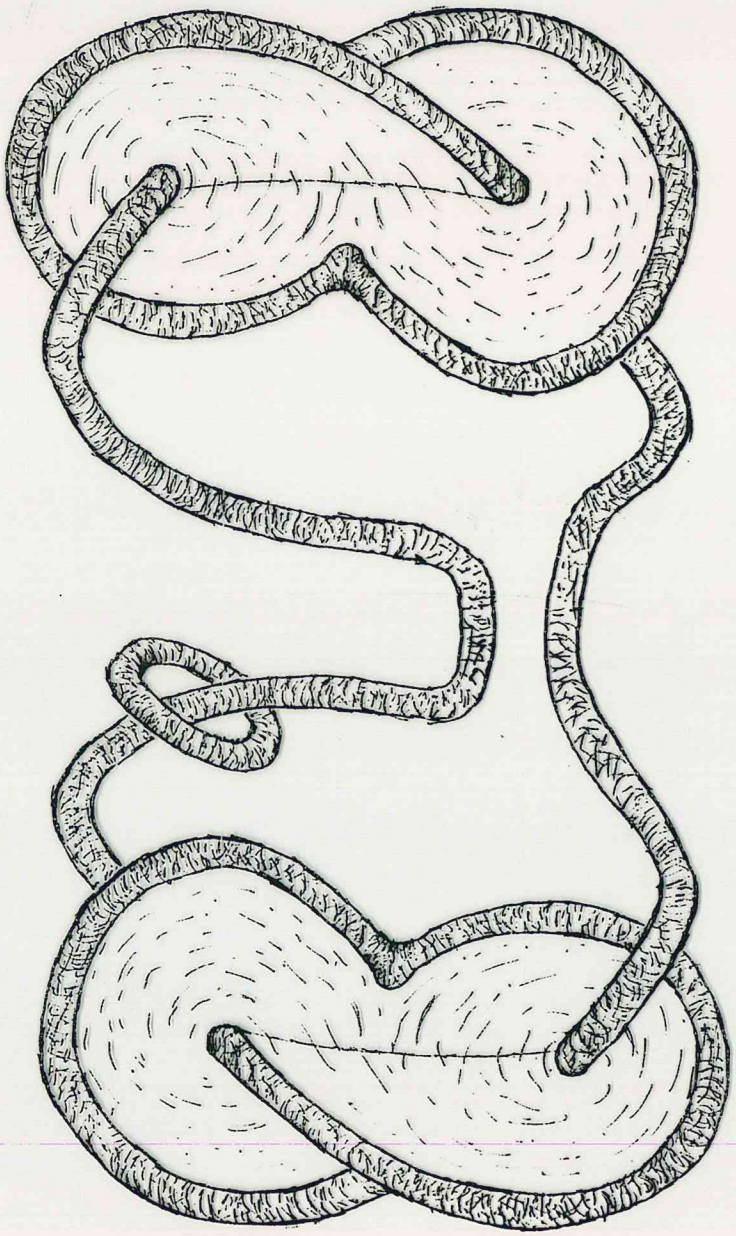
If a minimal invariant saddle set of a Smale flow on a 3-manifold is not a single closed orbit (so, it contains infinitely many closed orbits) then there is an isolating neighborhood, N, foliated by stable manifolds, whose quotient by leaves of the foliation is a branched 2-manifold, called a template. Any link of closed orbits in the saddle set is isotopic to a link of orbits in a naturally induced semi-flow on the template.

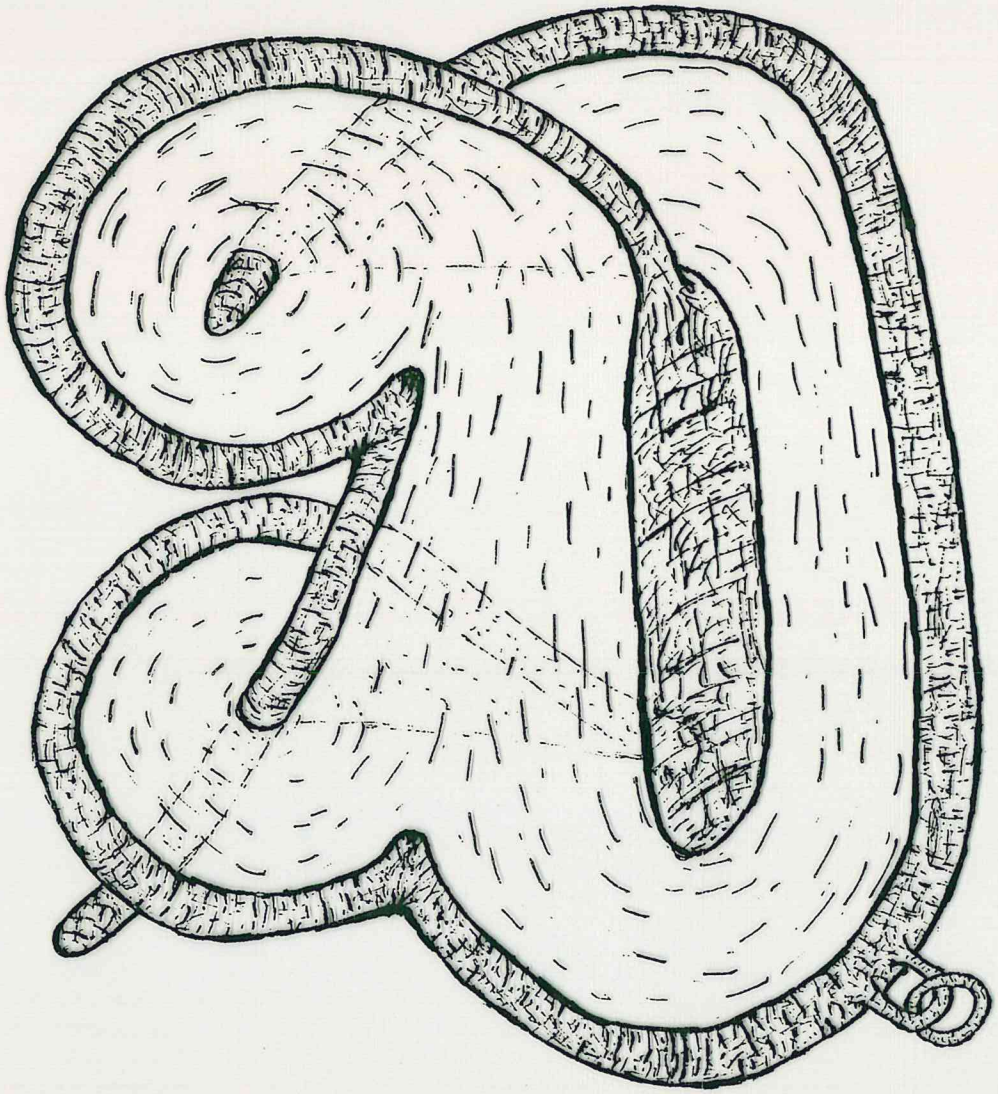


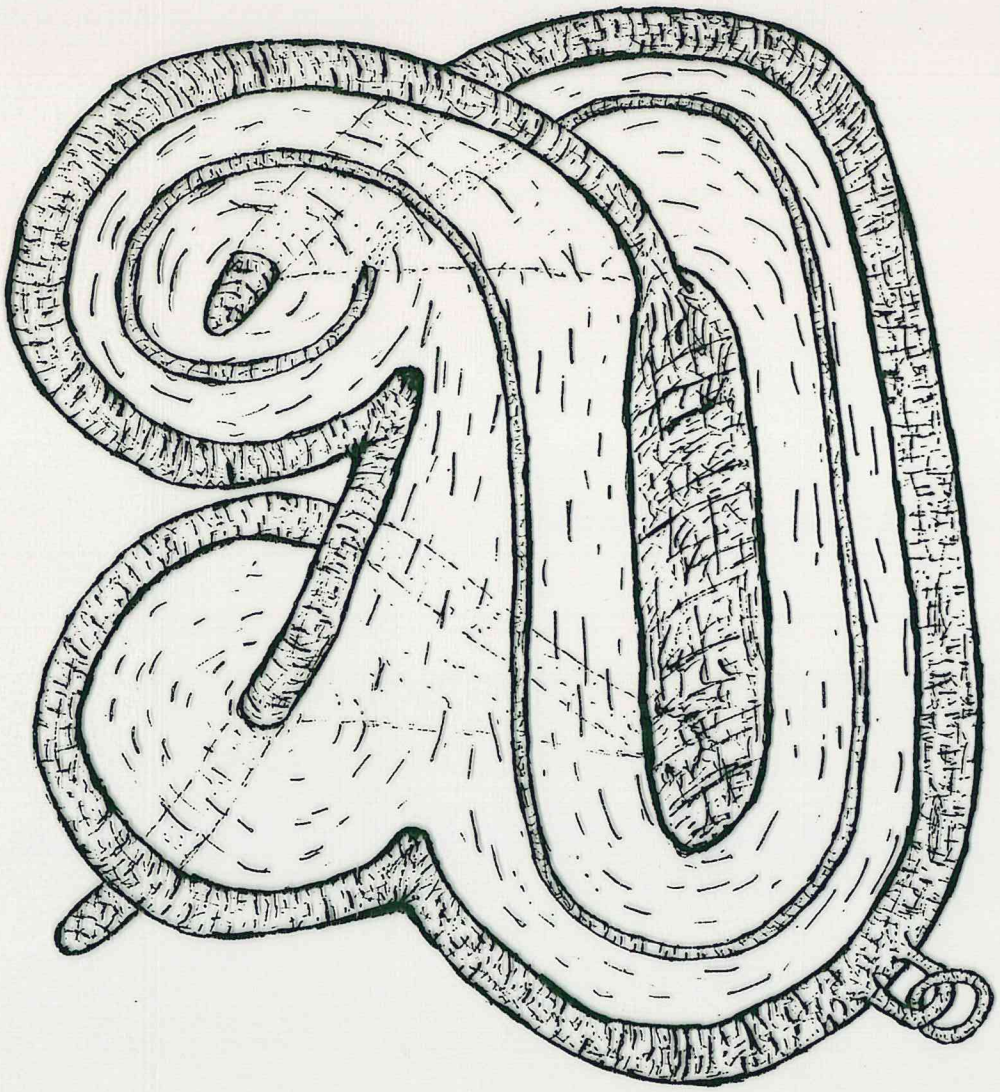
Example of a Smale flow



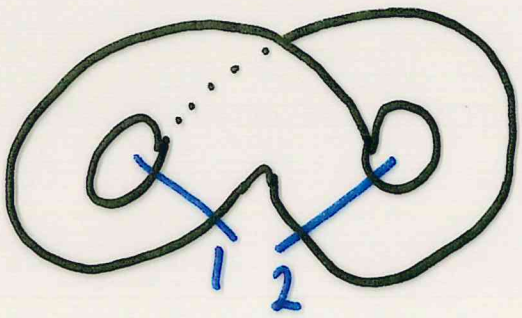
"Connected Sum" of two Smale flows







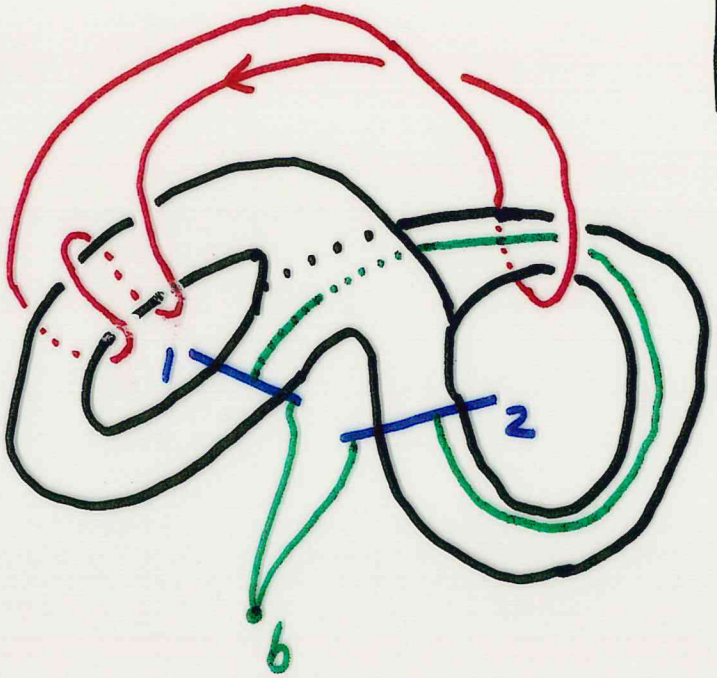
Incidence Matrix:



$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Linking Matrix:

$$\begin{bmatrix} t^2 & t^2 \\ t & t \end{bmatrix} = L$$



Alexander Polynomials

Theorem: [Franks, 1981]

Let k be a knotted closed orbit in a nonsingular Smale flow on S^3.

Let L_i, $i = 1, 2, \ldots, n$ be the linking matrices of k with each saddle set.

Let q_i, $i = 1, 2, \ldots, m$ be the linking numbers of k with the other attractors and repellers.

Then

$$\Delta_k(t) = \frac{(1-t)\prod_{i=1}^{n} \det(I - L_i)}{\prod_{i=1}^{m} (1 - t^{q_i})}$$

If there is only one other closed isolated orbit besides k and its linking number with k is one, then

$$\Delta_k(t) = \prod_{i=1}^{m} \det(I - L_i)$$

Corollary

Let k_1 and k_2 be knots.

Suppose there are two nonsingular Smale flows on 3-spheres such that k_1 is the only attractor of one and k_2 is the only attractor of the other, and that if both the only repeller is a meridian of the attractor.

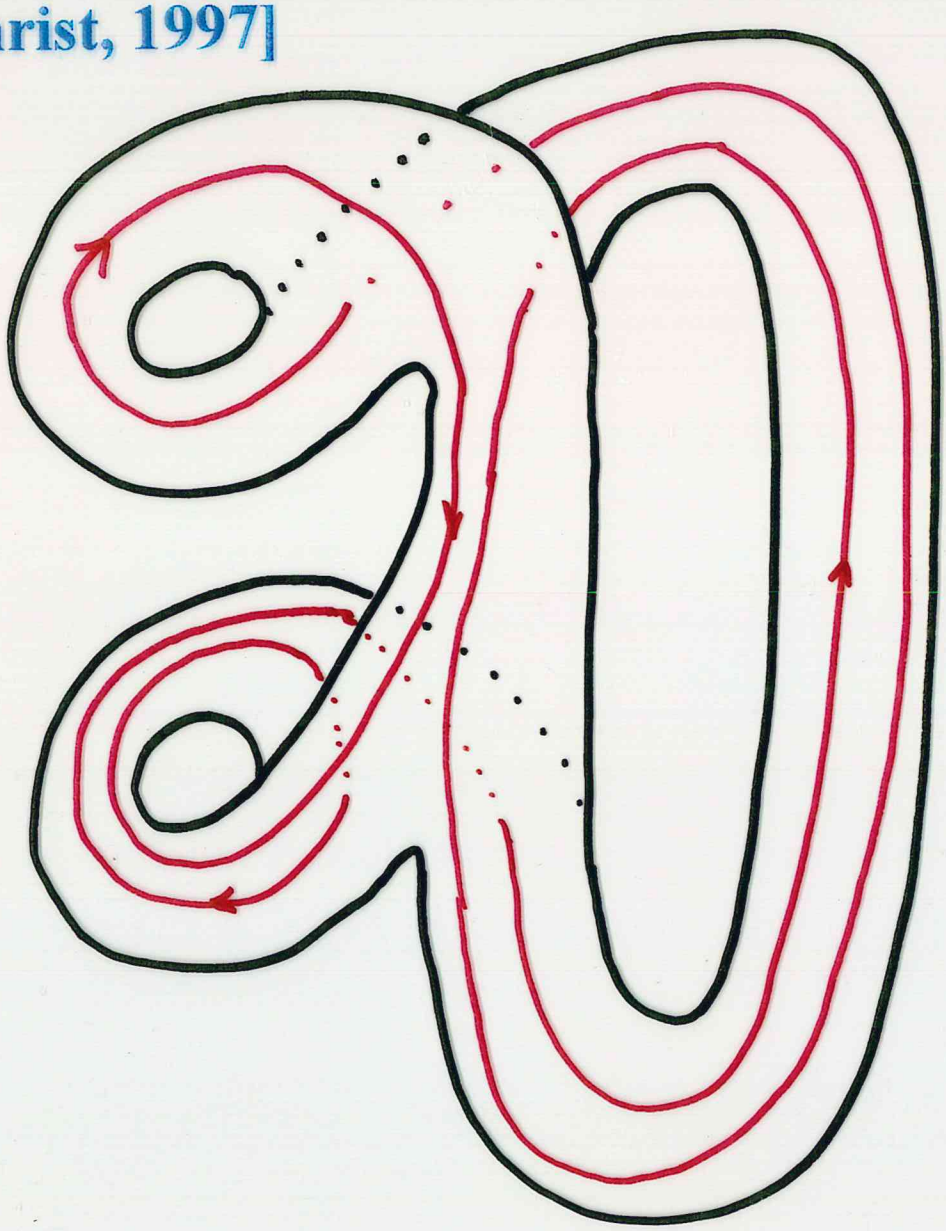
Then

$$\Delta_{k_1 \# k_2}(t) = \Delta_{k_1}(t) \cdot \Delta_{k_2}(t)$$

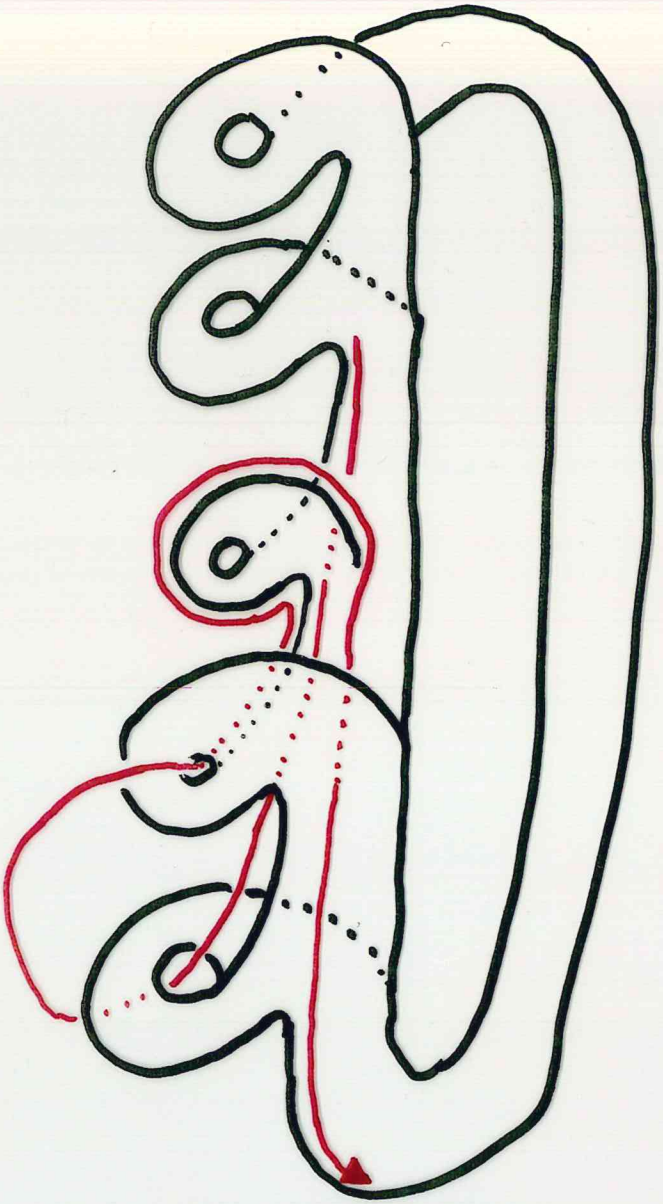
Universal Templates

Let k be any knot.

Then k lives on this universal template as an orbit. [Ghrist, 1997]



Weird Looking Object



The red curve is still the knot k.

Q.E.D.

In this figure k is realized as an attractor in a nonsingular Smale flow on S^3. The green torus contains a meridianal repeller.