Periodic orbits in a chaotic attractor introduced by Clark Robinson

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Lorenz Attractor & Templates

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\dot{x}(t) &= -10x + 10y \\
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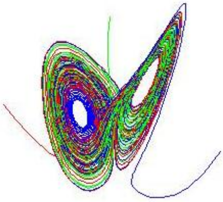

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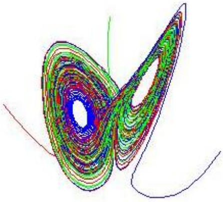

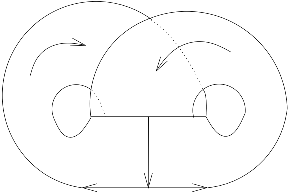

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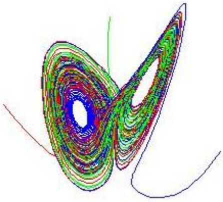

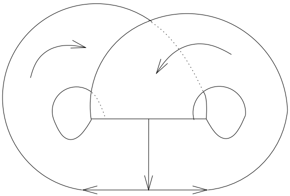

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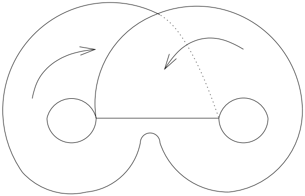

Lorenz Template, $L(0,0)$

Robinson's Attractor and Templates

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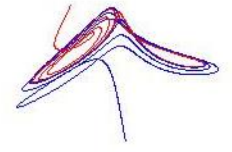

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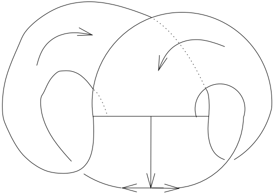

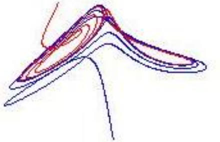

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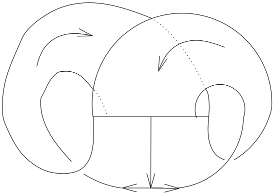

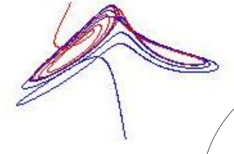

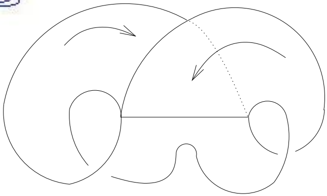

$L(-1,-1)$

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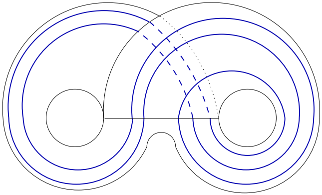

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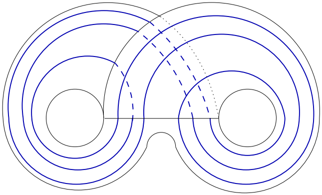

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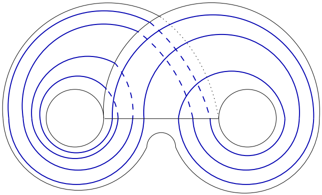

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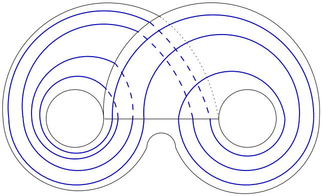

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The proof is harder to draw.

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On $L(-1,-1)$ the situation is a bit different. For orbits in $L(-1,-1)$ we have the following.

a. The orbit for xy is unlinked with all other closed orbits.
b. The orbit for x is unlinked to orbits of the form xy^n and the orbit for y is unlinked to orbits of the form $x^n y$,
c. Any pair of closed orbits not covered by (a) or (b) are linked.

For the proof see the next frame.

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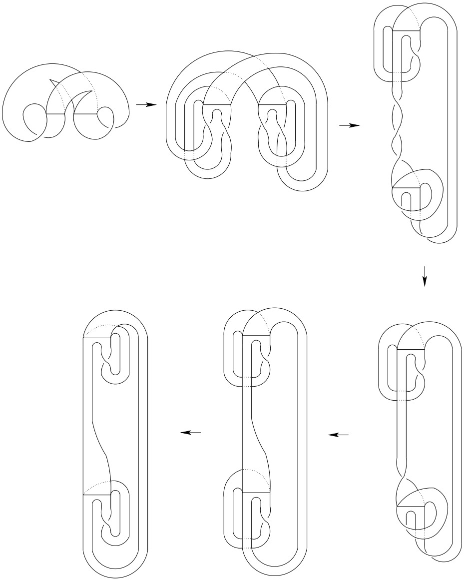

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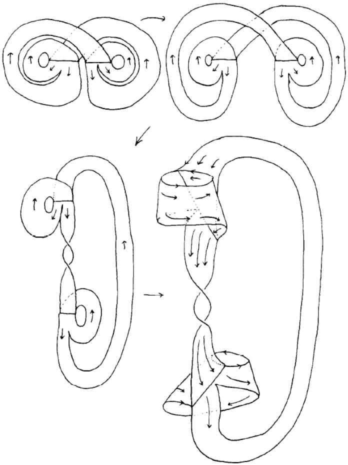

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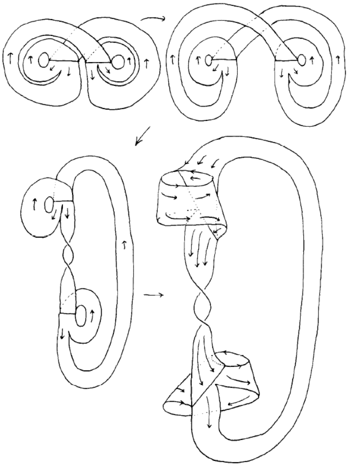

Figure is from Birman and Williams' 1983 paper.

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Fact: Postive braids with a full twist are prime. [Cromwell]

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Fact: Positive braids are fibered. [Stallings]

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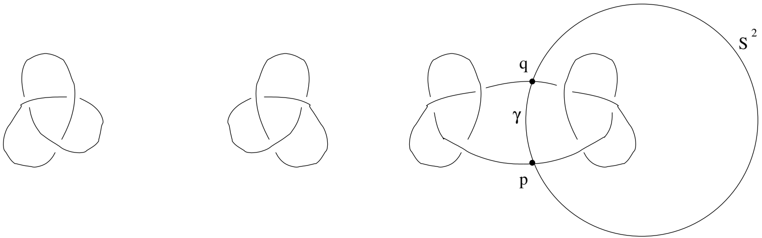

Left–hand Trefoil
Prime

Right–hand Trefoil
Prime

Square Knot
Composite

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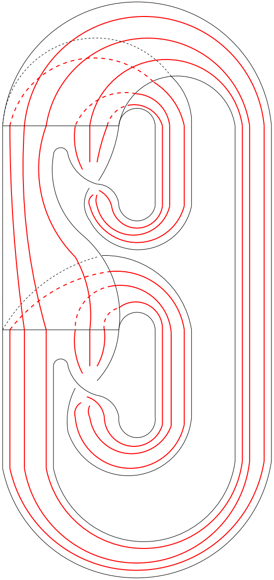

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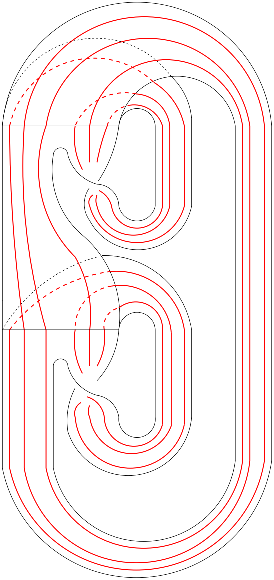

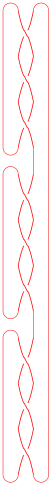

$L(-1,-1)$ has compisite knots

Composite knots on L(-1,-1) have either two or three prime factors. If there are three prime factors, they are all torus knots. If there are two prime factors at least one is a torus knot.

Reference: Factoring Families of Positive Knots on Lorenz-like Templates. *Journal of Knot Theory and Its Ramifications*, Vol. 17, No. 10, October 2008.

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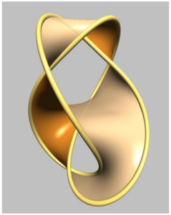

Figure from http://www.win.tue.nl/

∼vanwijk/seifertview/knot_gallery.htm

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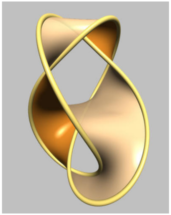

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If $S^3 - N(K)$ can be fibered by a Seifert surface of K, then K is a fibered knot.

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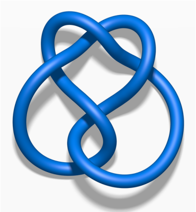

Figure from http://katlas.org/wiki/File:Blue_Three-Twist_Knot.png

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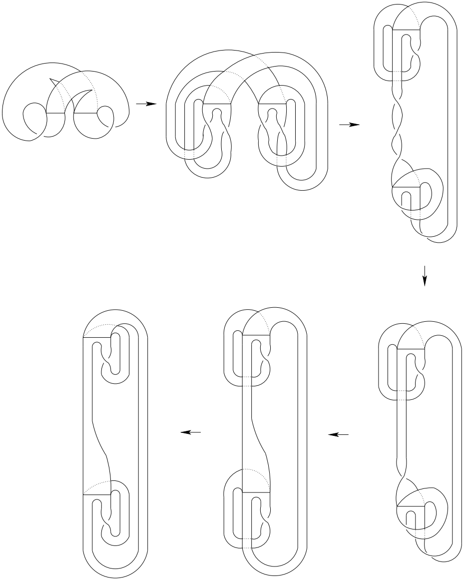

Not all are positive briads

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Example: The orbit with word $xy^4x^2yx^4y^2$ in $L(-1,-1)$ can be presented as the following braid on five strands, $(32233232221\bar{4})^2$. A calculation shows that its Conway polynomial has leading coefficient 3. [S., 2005] Hence it is not a positive braid. [James M. van Buskirk, 1983]

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The proof is a bit complicated to present here, but it closely follows Stallings' proof that postive briads are fibered. We give an brief outline.

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If μ_* is an isomorphism, then K is fibered. [Stallings]

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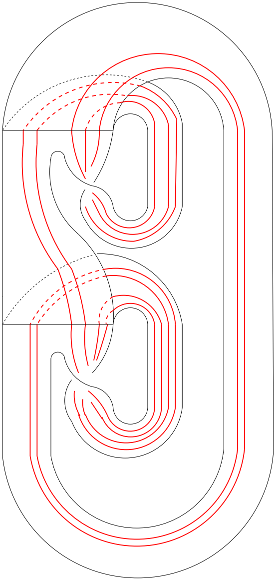
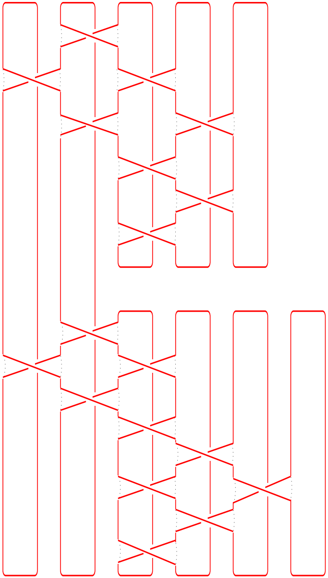
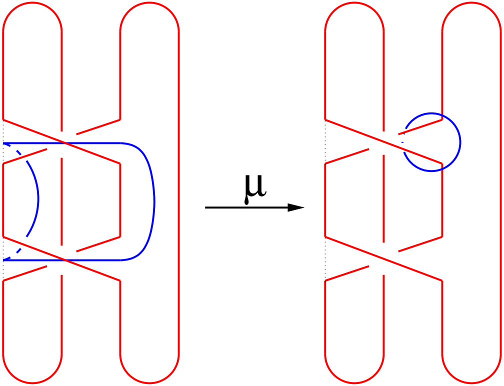
Then we show that having all the twists of the same type forces μ_* to be surjective.

Seifert Surface Example





Push Off: Generator to Generator



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Thus, while it is known that any template supports infinitely many distinct knot types the collection of prime knots in $L(-1,-1)$ seems rather narrow.

Zeta Functions

For diffeomorphisms, $f : M \to M$, we can count the number of fixed points of each iterate; let $N_p = \#$ fixed points of f^p. Assume this is alway finite. Then define

$$\zeta(t) = \exp\left(\sum_{p=1}^{\infty} \frac{N_p t^p}{p}\right).$$

If f has a hyperbolic chain recurrent set then $\zeta(t)$ is rational and if A is an incidence matrix for f, and

$$\zeta(t) = \frac{1}{\det(I - tA)}.$$

Example

Let $f : D^2 \to \mathbb{R}^2$ be Smale's horseshoe map. Then, for the usual Markov partition on the invariant set, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Thus,

$$\zeta(t) = \frac{1}{1-2t}.$$

Different Markov partitions of the invariant set do not change $\zeta(t)$.

Flows

For flows this breaks down as periodicity is not necessarly well defined. Different cross sections can give different zeta functions. To get a zeta function for flows I used the twisting in the local stable manifold instead of the period. Sometimes this works! But not always.

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Consider $L(0,0)$ and $L(0,1)$. These have respective **twist-matrices**

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The choice of the cross section or the Markox partition can change the twist-matrices, but not the zeta functions.

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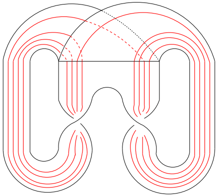

On the template $L(1, -1)$ all orbits given by words of the form $x^n y^n$ will have exaclty one full twist and hence a twist Zeta function cannot be defined.

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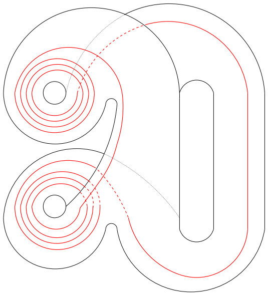

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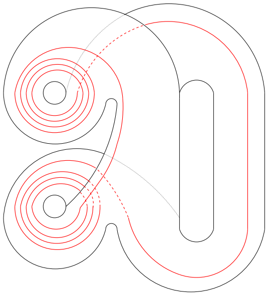

On the template above there are infinitely many orbits of zero twist. It was shown by Rob Ghrist to contain all knots and links. It is known that this template can be found inside $L(0, -n)$ for $n = 1, 2, 3 \ldots$

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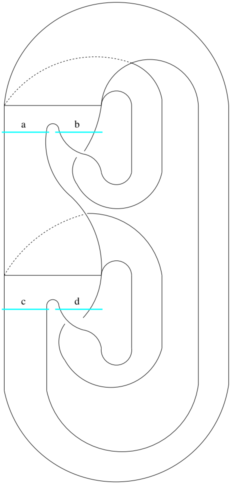

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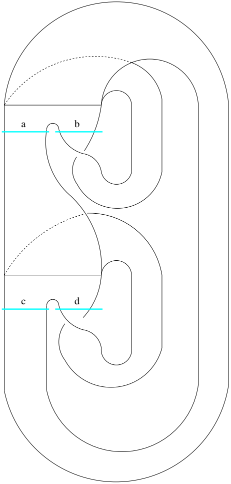
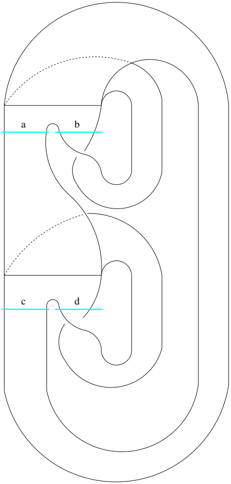


$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ t^3 & t^3 & 0 & 0 \\ t^2 & t^2 & 0 & 0 \\ 0 & 0 & t^3 & t^3 \end{bmatrix}$$

But the twist-zeta function works on $L(-1,-1)$



$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ t^3 & t^3 & 0 & 0 \\ t^2 & t^2 & 0 & 0 \\ 0 & 0 & t^3 & t^3 \end{bmatrix}$$

$$\zeta(t) = \frac{1}{t^6 - 2t^3 - t^2 + 1}$$

Comparing Series

$\log(1/\det(I - A(t))) =$

$$t^2 + 2t^3 + \frac{t^4}{2} + 2t^5 + \frac{4t^6}{3} + 2t^7 + \frac{13t^8}{4} + \frac{8t^9}{3} + \frac{6t^{10}}{5} + \cdots$$

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$$\sum_{p=1}^{\infty} \frac{A^p(t)}{p} =$$

$$2t^3 + \frac{2t^2 + 2t^6}{2} + \frac{6t^5 + 2t^9}{3} + \frac{2t^4 + 12t^8 + 2t^{12}}{4} + \frac{10t^7 + 20t^{11} + 2t^{15}}{5} + \cdots$$

Tracking Some Orbits

b | d — $t^3 + t^3$

bb | dd | ac ca — $t^6 + t^6 + 2t^2$

bbb | ddd | acb bac cba | adc cad dca — $t^9 + t^9 + 3t^5 + 3t^5$

bbbb | dddd | acac caca | addc cadd dcad ddca | cbba acbb bacb bbac | adcb badc cbad dcba

$$t^{12} + t^{12} + 2t^4 + 4t^8 + 4t^8 + 4t^8$$

Under what conditions will a template have a well defined twist-zeta function?

Under what conditions will a template have a well defined twist-zeta function?

I do not know!

Some References

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