Smale Flows

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given at

Saint Louis University
April 15, 2025

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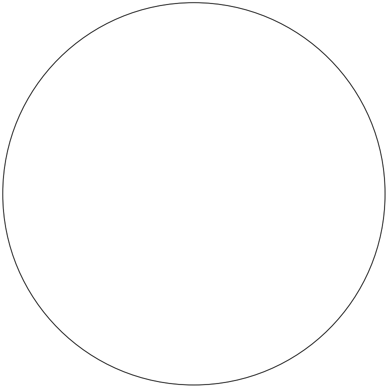
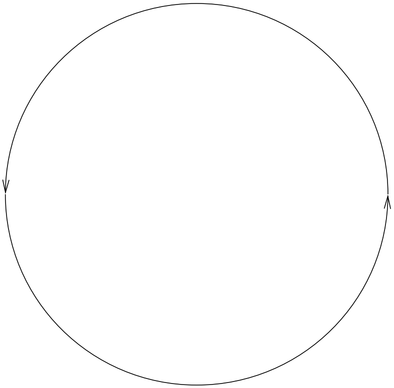
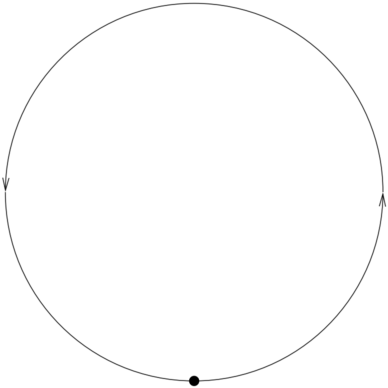
A 1-dimension manifold without bountary



A 1-dimension manifold without bountary with a nonsignular flow



A 1-dimension manifold without bountary with a flow with a rest point



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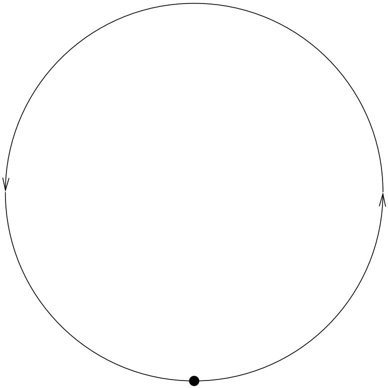
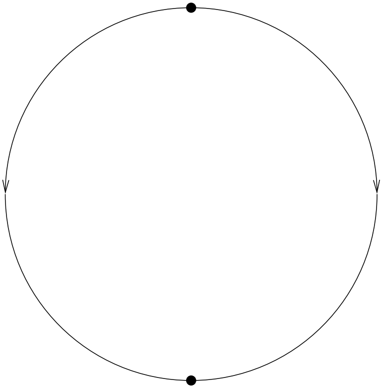
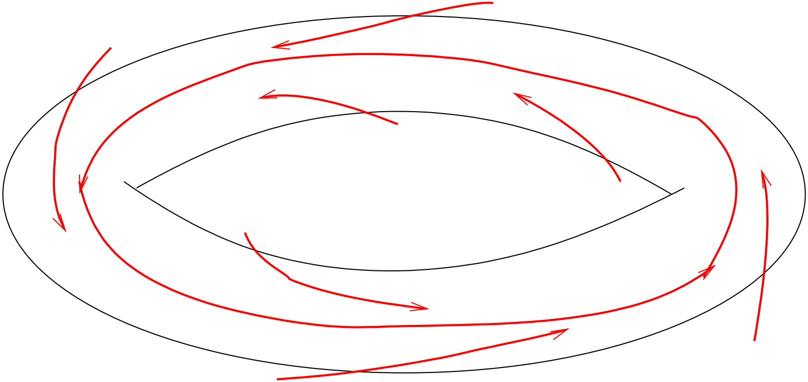


... that is not structurally stable.

A 1-dimension manifold without bountary with a flow with that is structurally stable



Let's jump to 3-D



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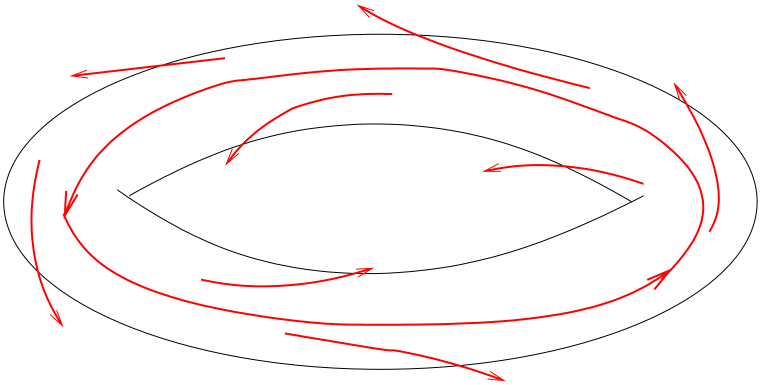

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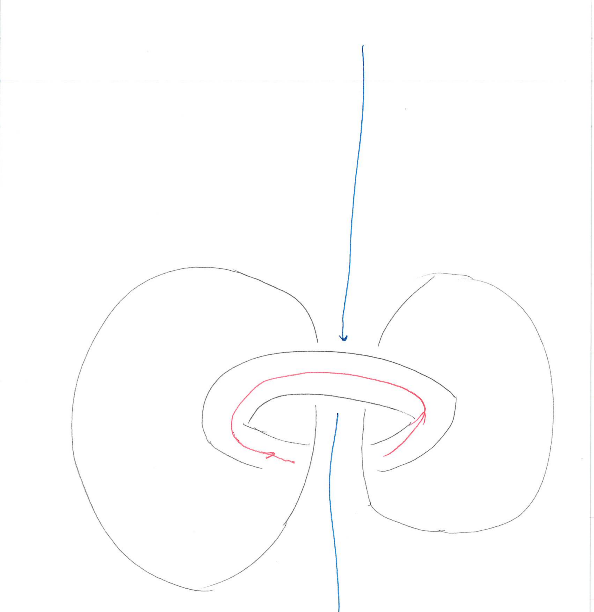
If the map just swaps the longitudes and meridians, we get good old S^3
With a nonsignular structurally stable flow. It is an example of a Morse-Smale flow.

Our Flow



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$$\phi : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$$

C^∞, such that $\phi(x, 0) = x$ and $\phi(x, s + t) = \phi(\phi(x, s), t)$.

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We will mostly work with the 3-sphere.

Chain Recurrent Set of a Flow

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A point $x \in \mathcal{M}$ is *chain recurrent* for a flow ϕ_t if for every $\epsilon > 0$ and $T > 0$ there exist a chain of points $x = x_0, \dots, x_n = x$ in M, and real numbers $t_0, \dots, t_{n-1}$ all bigger than T such that $d(\phi_{t-i}(x_i), x_{i+1}) < \epsilon$.

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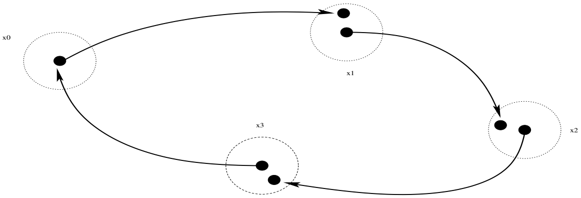

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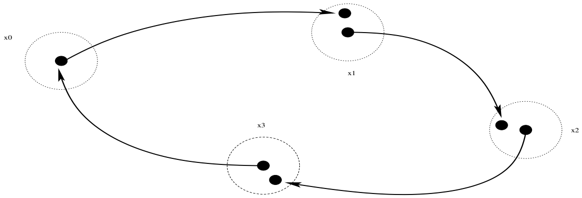

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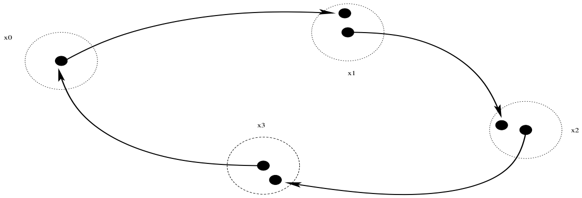

The set of all such points is called the *chain recurrent set*, $\mathcal{R}$. It is compact and invariant under the flow.
Any rest points and period orbits will be in the chain recurrent set.
But, stranger objects may be lurking.

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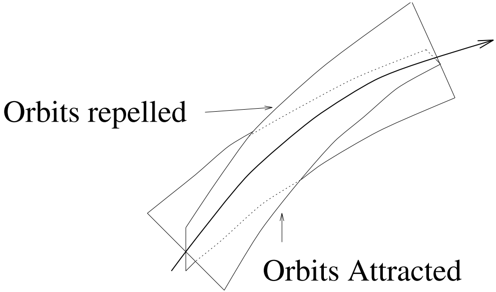

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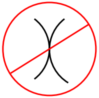

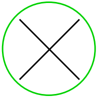

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The term was introduced by John Franks in the 1970's.

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No far no one has been able to provide such a classification on other 3-manifolds.

Back to Smale Flows

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Attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a nontrivial shift of finite type - we call these chaotic saddle sets.

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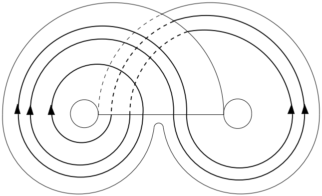

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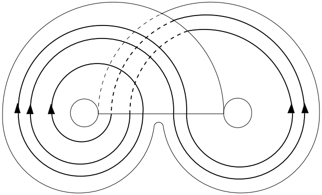
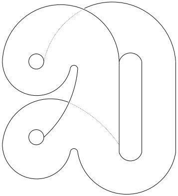


Here are some others.



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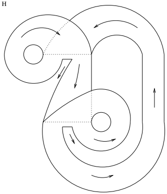

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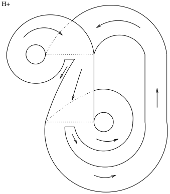

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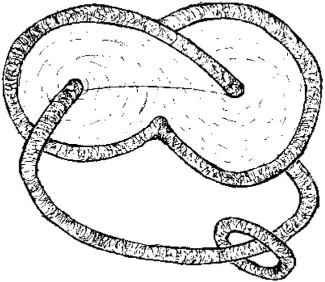
Theorem (S. 2000)

For a Lorenz-Smale flow on S^3, the link $a \cup r$ is either a Hopf link or a trefoil and meridian.

This is a Lorenz-Smale Flow



Lorenz-like Smale flows

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Definition

We use $L(0,0)$ to denote the Lorenz template, $L(0,1)$ to denote a variation where one band contains a Möbius band, and $L(1,1)$ to denote a variation where each band contains a Möbuis band.

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Bin Yu goes on to describe all 3-manifolds that can support these cases.

Four-band Templates

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Theorem (Haynes & S., 2014)

A simple Smale flow with saddle set modeled by U will have $a \cup r$ a Hopf link or a figure-8 knot and meridian; if the saddle set is modeled by U^+ then $a \cup r$ a Hopf link or a trefoil and meridian.

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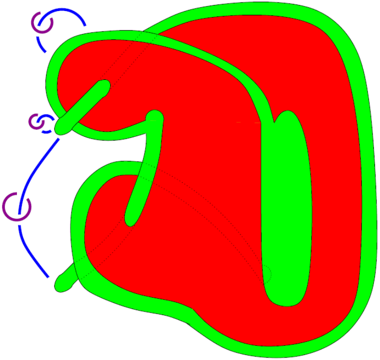

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Theorem (Adhikari & S., 2015)

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`https://www.youtube.com/watch?v=sJB09zqRqPg`

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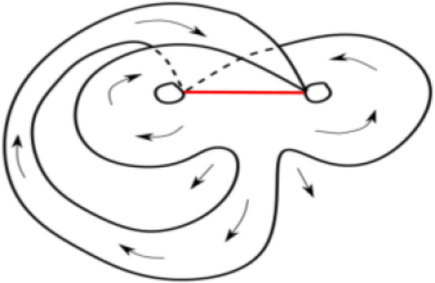

Y

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Theorem (M. Alshamrai, 2025)

For a simple smale flow in S^3 with saddle set modeled by the given template Y, the only realizable configuration is the attractor-repeller link $a \cup r$ forms a closed braid with braid word σ_1 and an unknot serving as its braid axis. Thus, a and r have linking number 2.

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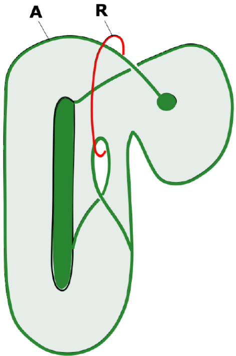

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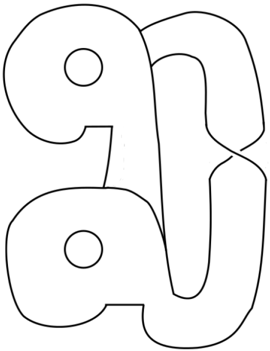

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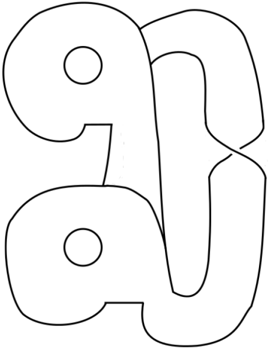

Theorem (A. Sloan, 2024)

For a simple Smale flow on S^3 with saddle set modeled by $U^+(0,0,0,1)$, the link $a \cup r$ is either a Hopf link or a figure eight knot and meridian.

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Draw lots of pictures.

References

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