Knotted Periodic Orbits in Flows

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Knots and Links

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NOTATION.

- $\mathbb{R}^n$ is the Euclidean space of dimension n.
- $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
- $S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1\} \approx \mathbb{R}^3 \cup \{\infty\}$.

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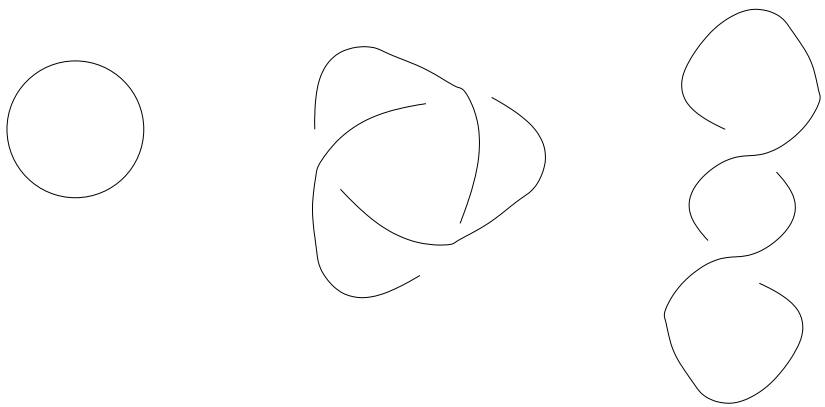

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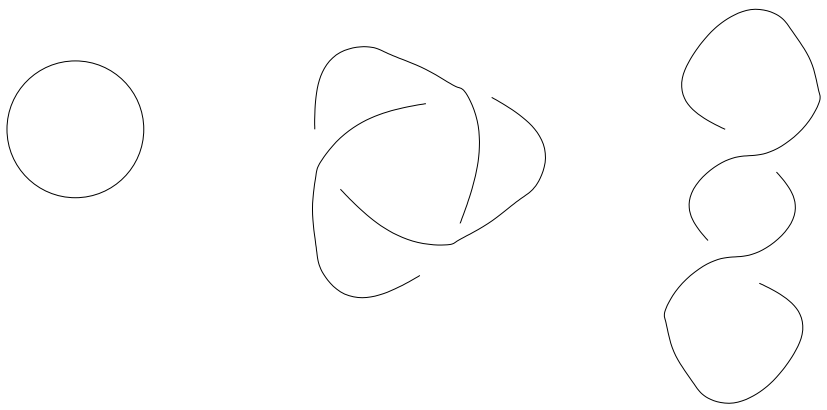

Two knots are **equivalent** (have the same **knot-type**) if one can be smoothly deformed into the other.

Knots can be Given an Orientation

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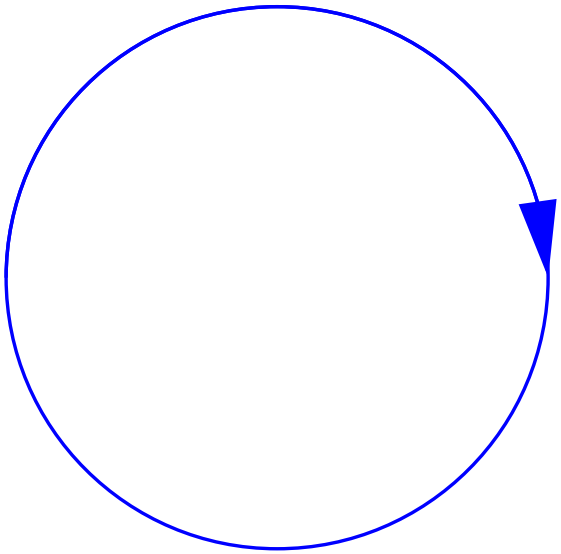

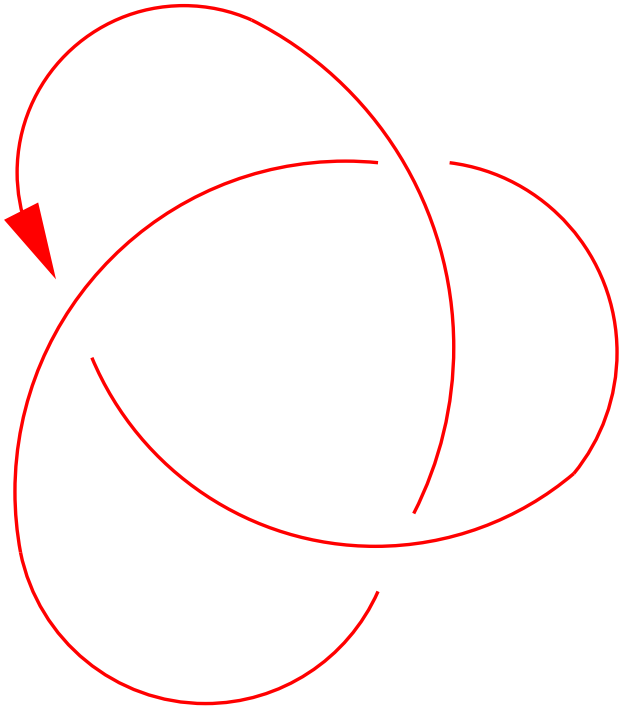

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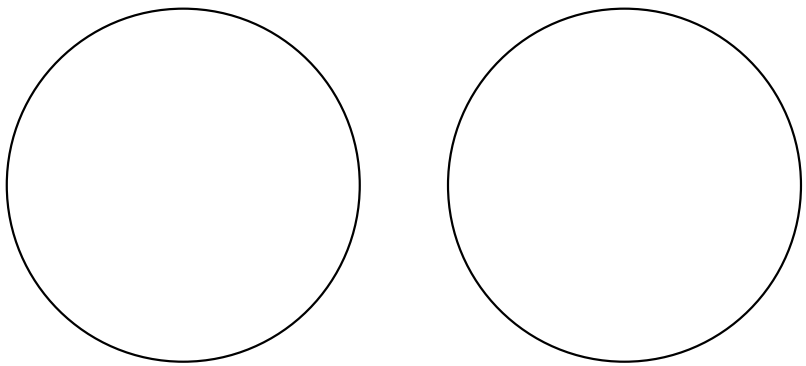
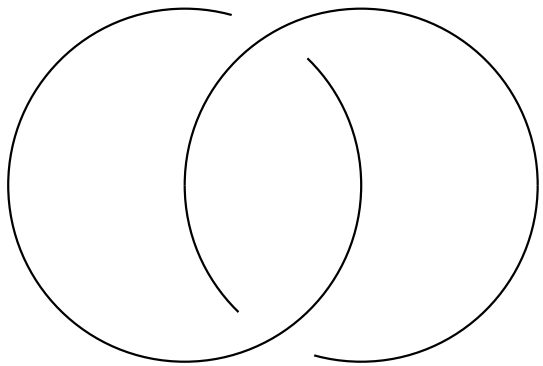
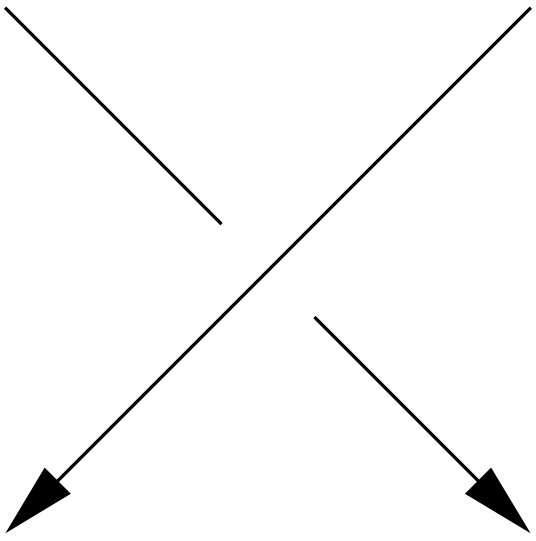




There are Two Crossing Types



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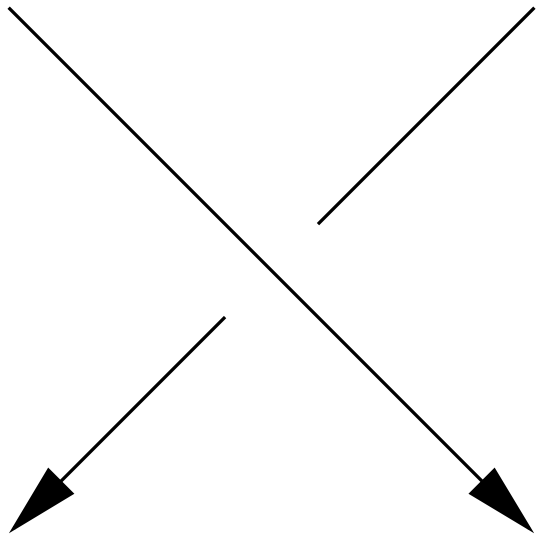

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Given an oriented link of two components the **linking number** is one half the sum of the crossing signs for each crossing of the two component knots.

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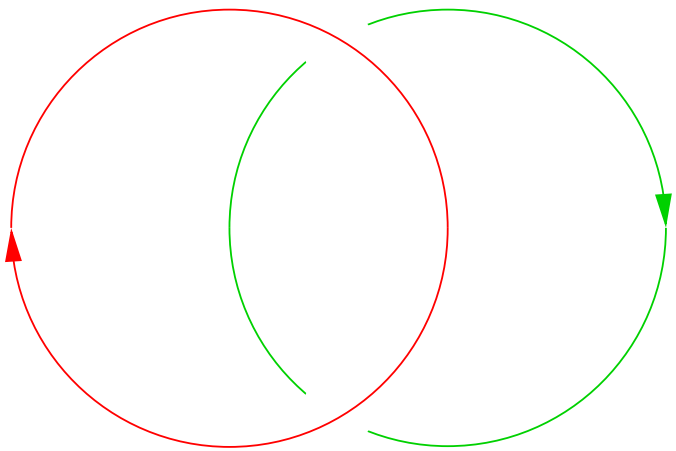
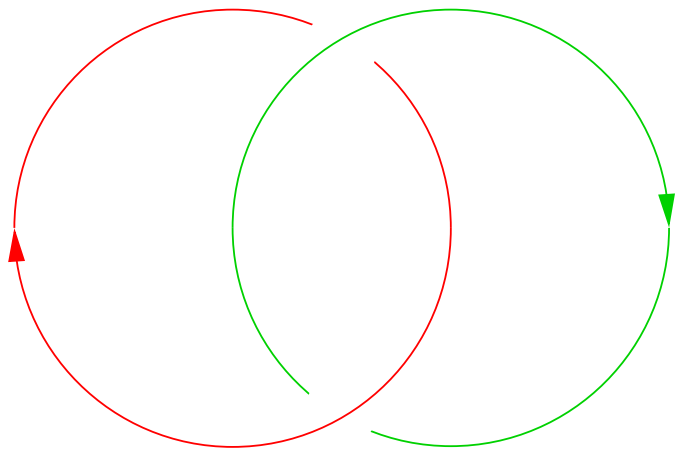


Linking Number = 0.



Linking Number = −1.

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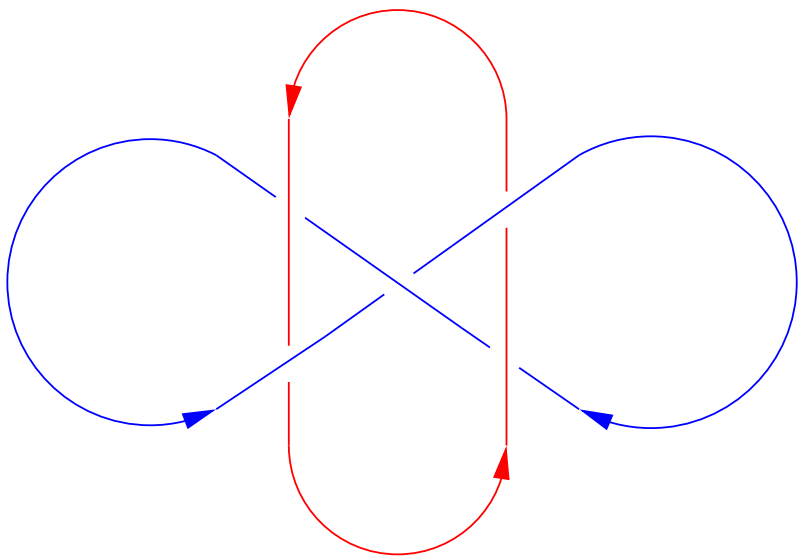

Whitehead link.

A Knot in a Flow

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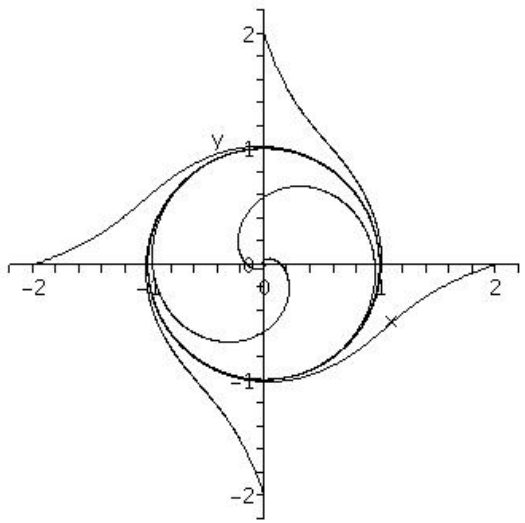

Source: Boyce & DiPrima, 10$^{\text{th}}$ edition, page 566.

A Flow with Lots of Knots

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Lorenz Equations.

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned}$$

$\sigma = 10$ $\qquad$ $r = 28$ $\qquad$ $b = 8/3$

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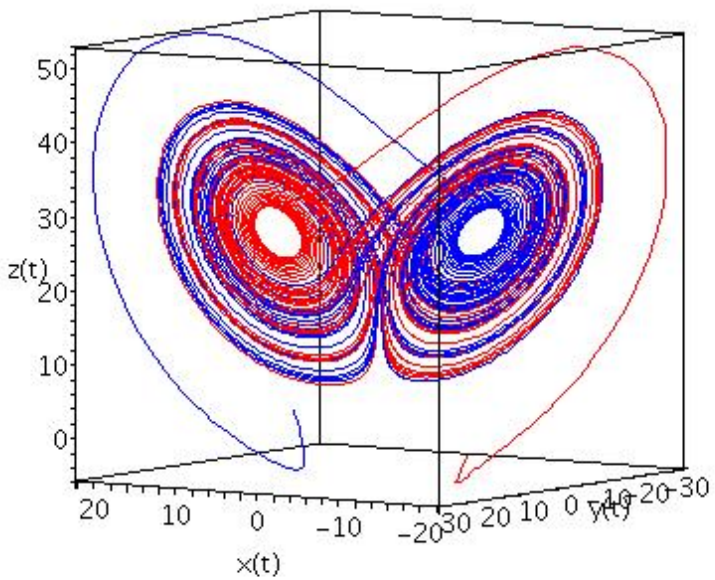
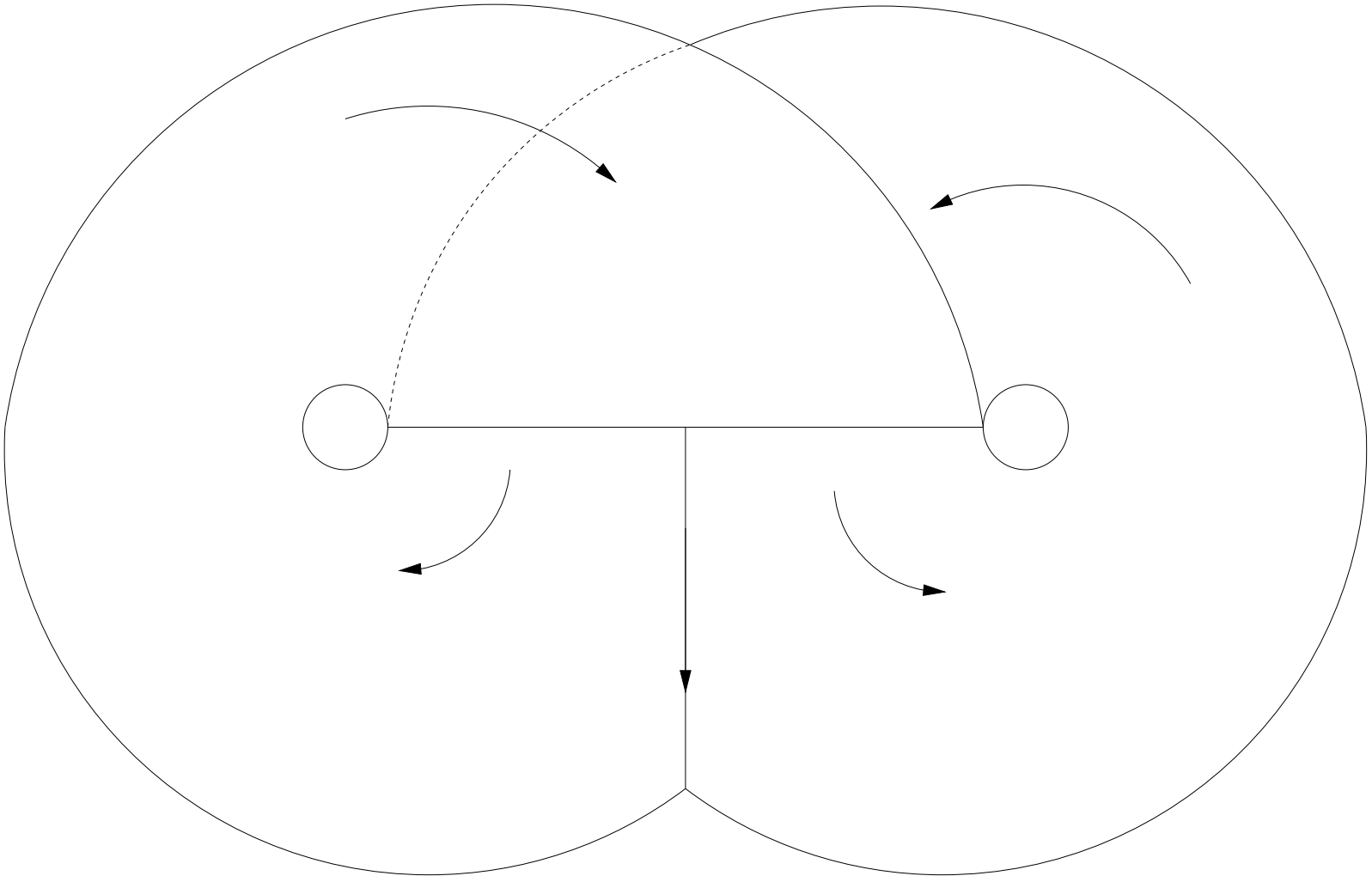


A Template for Lorenz Orbits



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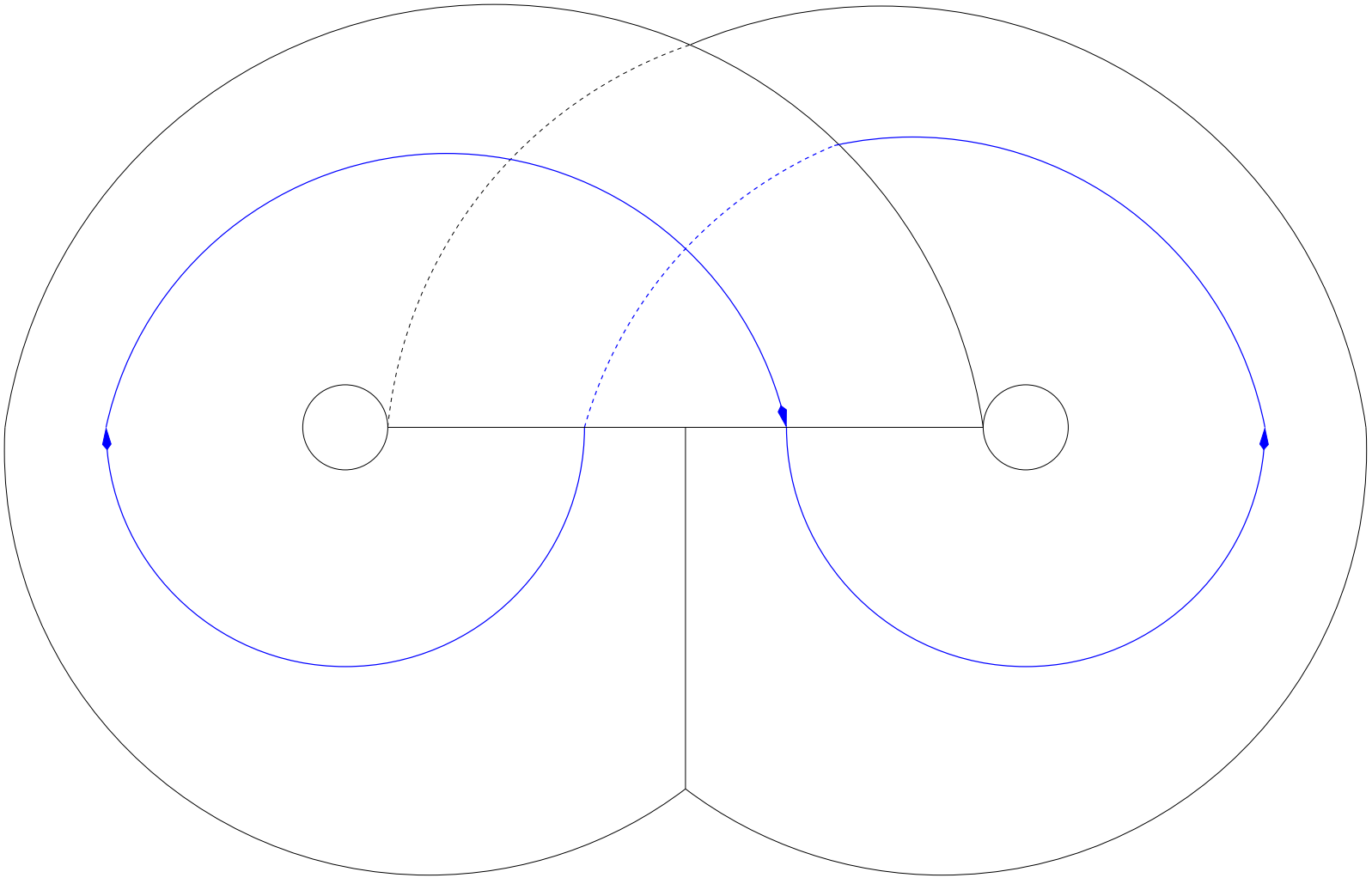

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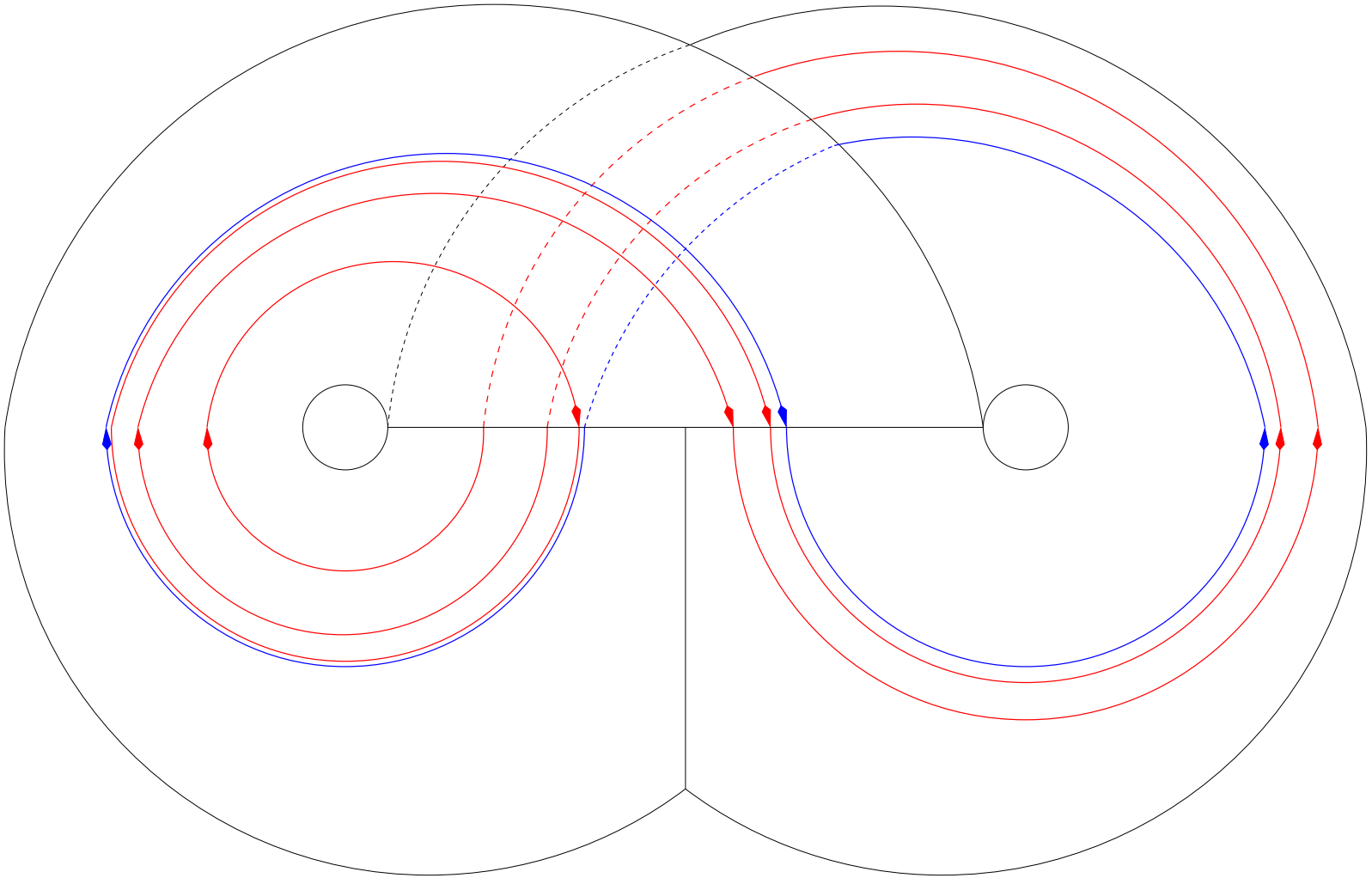

How many different Lorenz knots are there?

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Theorem. There are infinitely many distinct Lorenz knots. [Franks & Williams, 1983]

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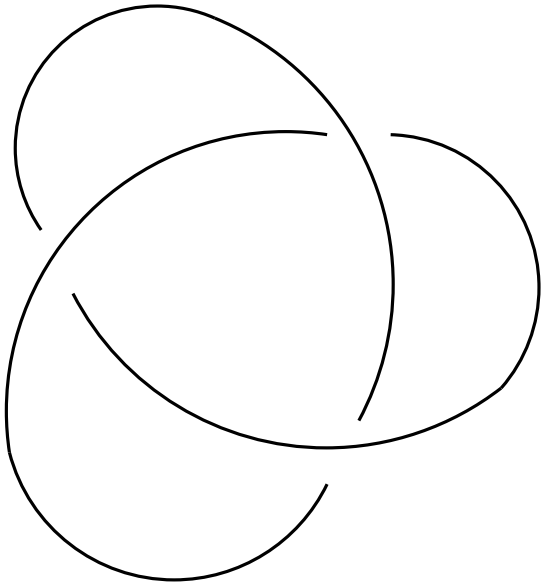
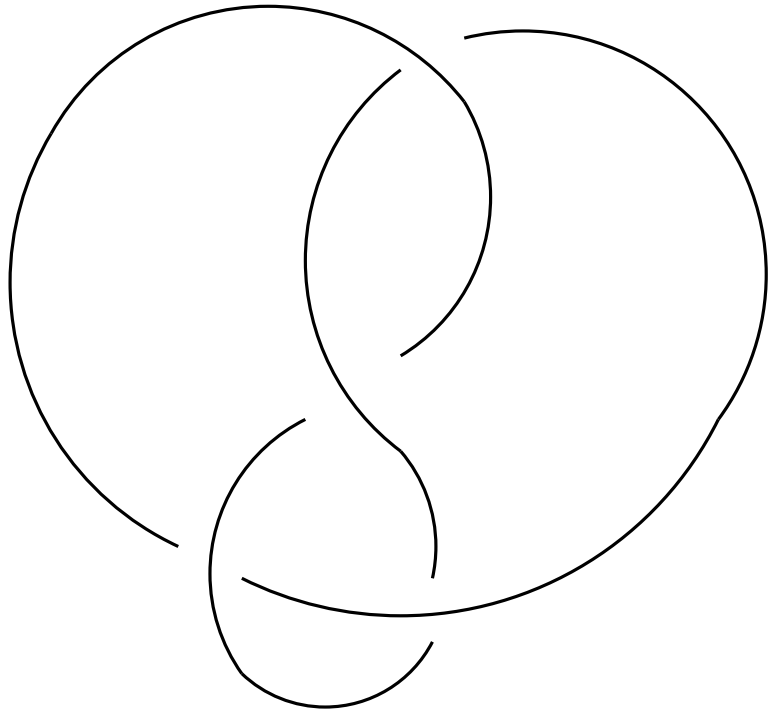


Trefoil

Prime



Figure–8

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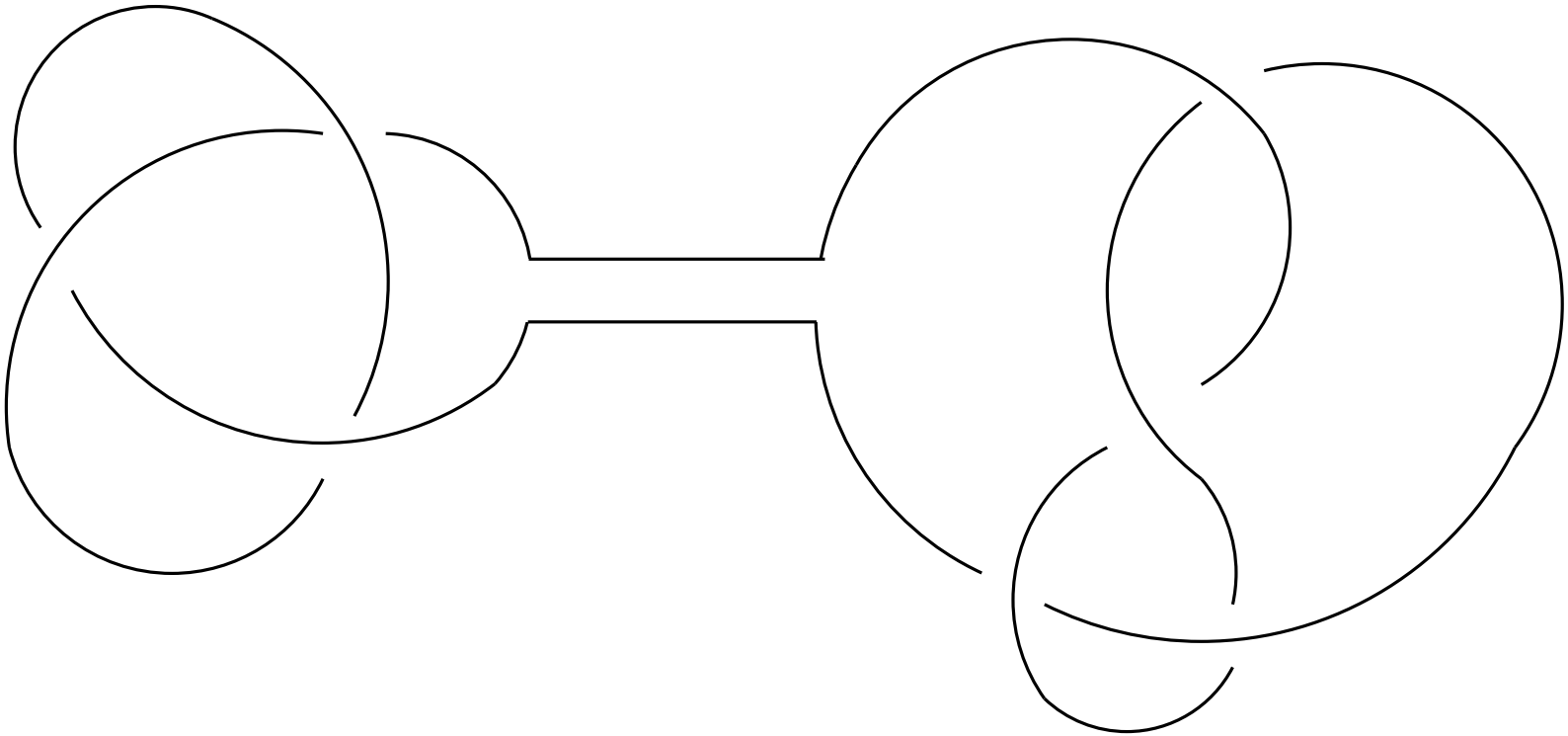

Trefoil # Figure–8

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Proof: Any pair of knots in the interior of the Lorenz template must cross and all the crossings in the Lorenz template are of the same sign. Thus, the linking number will be nonzero.

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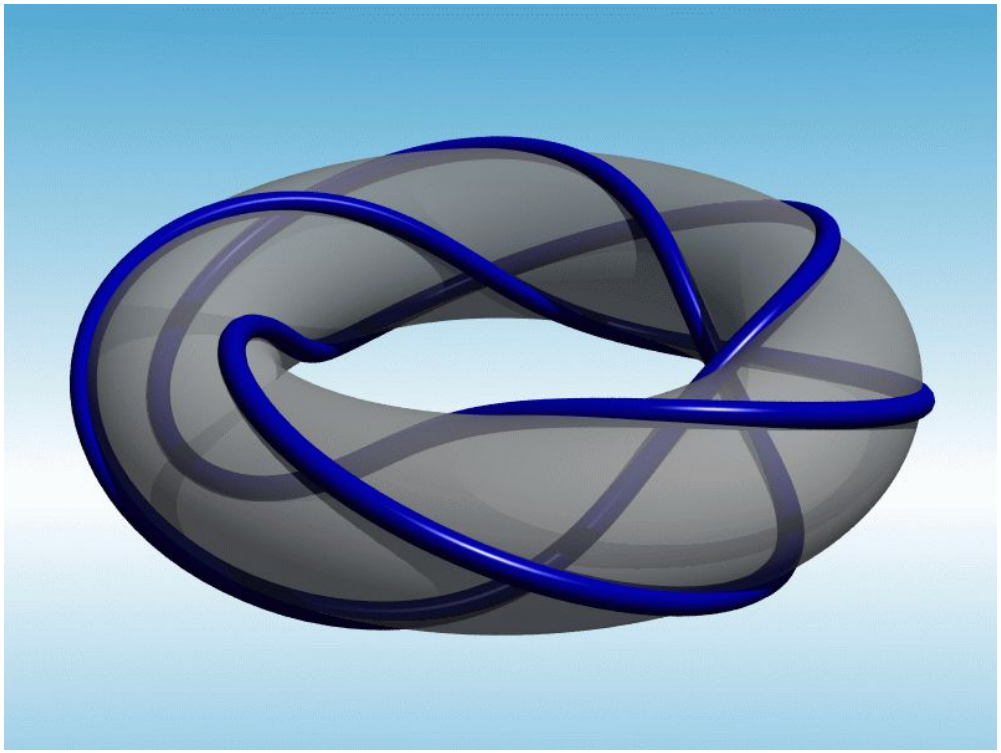

Image source: http://makerhome.blogspot.com/2014/01/day-150-trefoil-torus-knots.html

Torus Knots are Lorenz Knots

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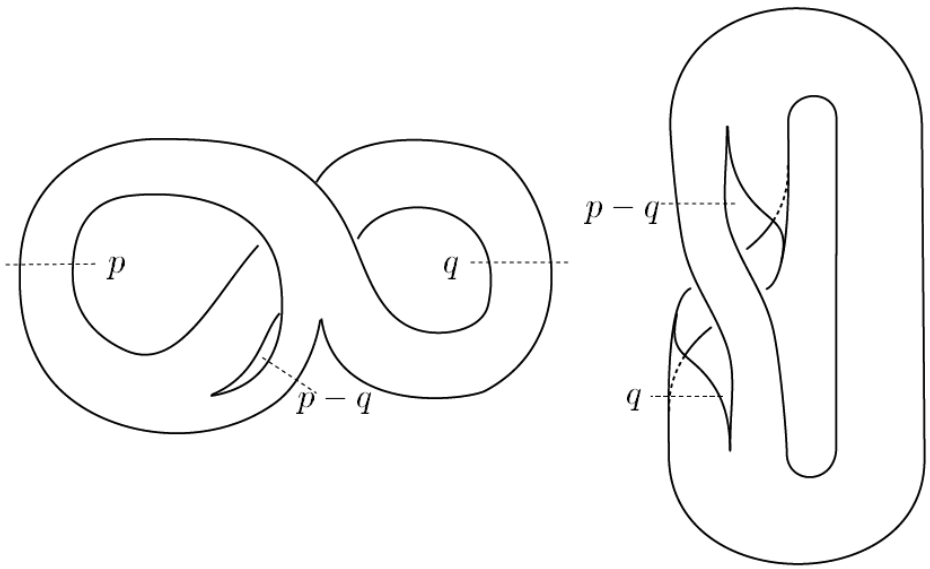

Image source: Knots and Links in Three-Dimensional Flows, by Ghrist, Holmes, Sullivan.

Lorenz-like Templates

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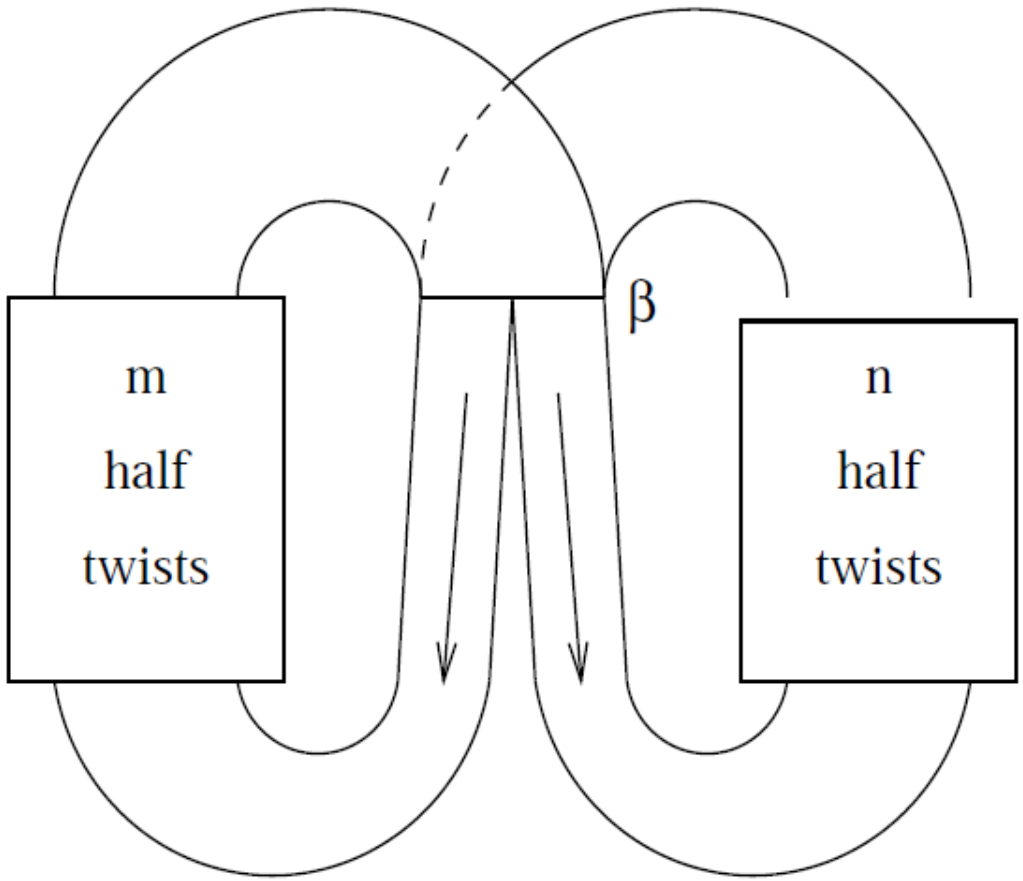

Template Relationships

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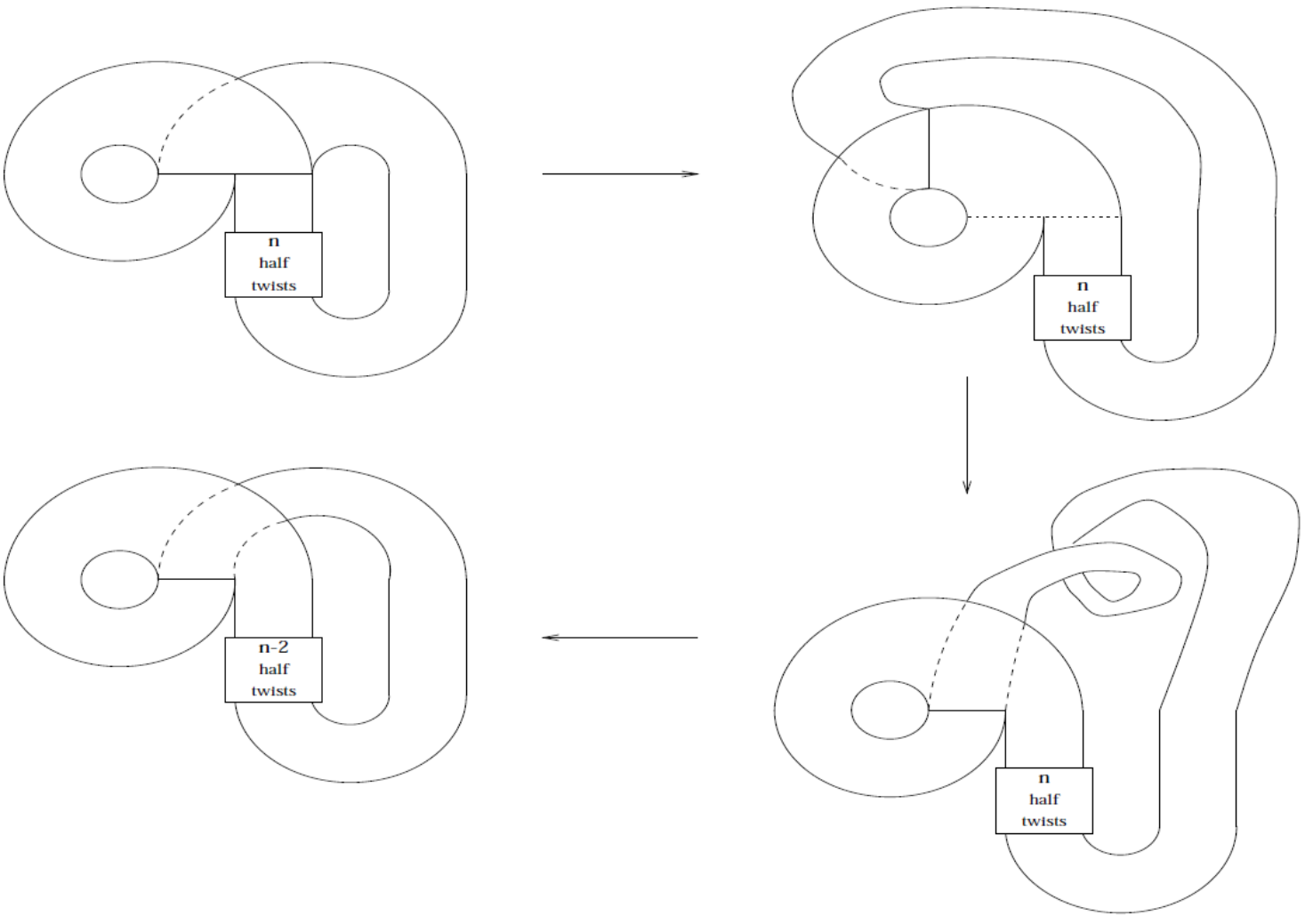
Theorem. Regarding $L(m,n)$ as the set of all knots in $L(m,n)$ we have

$$\cdots \supset L(0,-4) \supset L(0,-2) \supset L(0,0) \supset L(0,2) \supset \cdots$$

and

$$\cdots \supset L(0,-3) \supset L(0,-1) \supset L(0,1) \supset L(0,3) \supset \cdots$$

Proof!



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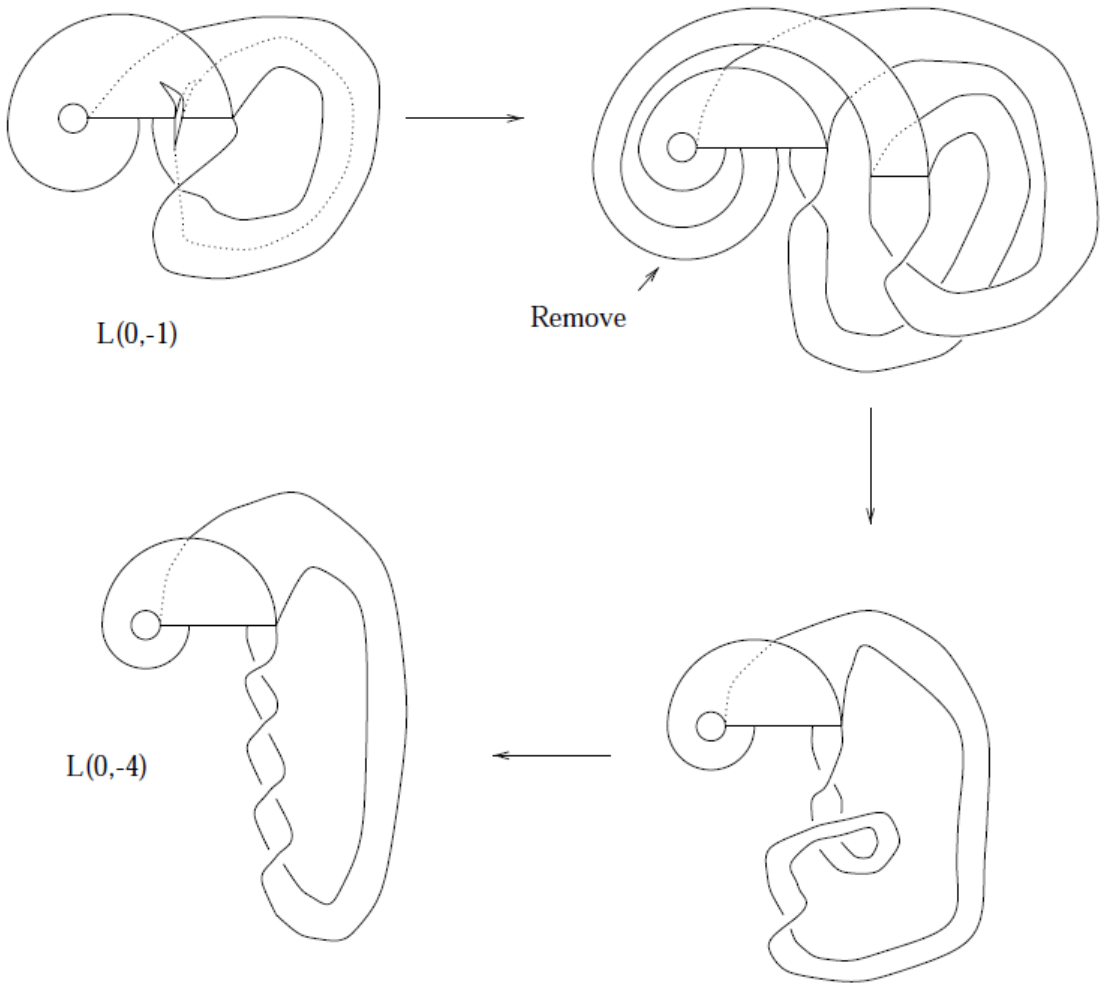

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The proof is hard.

An ODE with all knots

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Theorem. There exists an open set of parameters $\beta \in [6.5, 10.5]$ for which periodic solutions to the differential equation

$$\begin{aligned} \dot{x} &= 7|y - \phi(x)| \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{aligned}$$

where

$$\phi(x) = 2x/7 - 3\left[|x+1| - |x-1|\right]/14,$$

are modeled by $L(0, -1)$.
[Ghrist & Holmes, 1996]

Robinson's Attractor and Templates

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\begin{aligned}
\dot{x}(t) &= y \\
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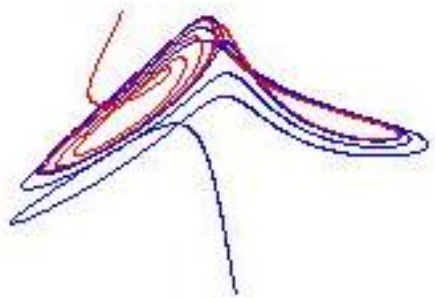

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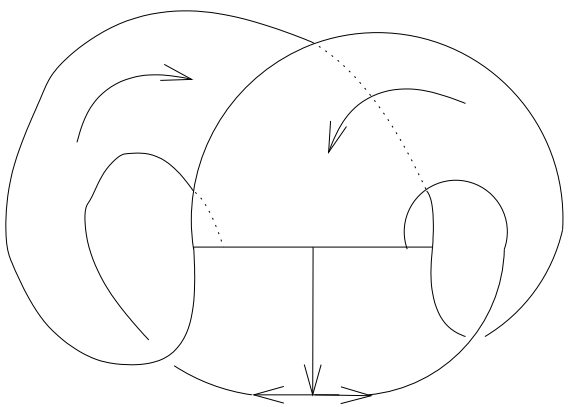

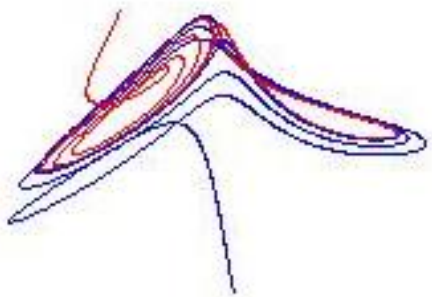

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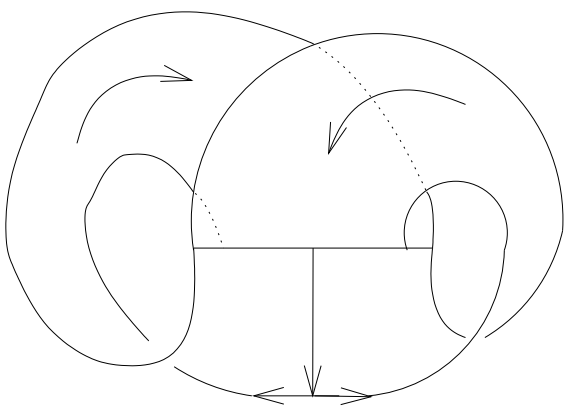

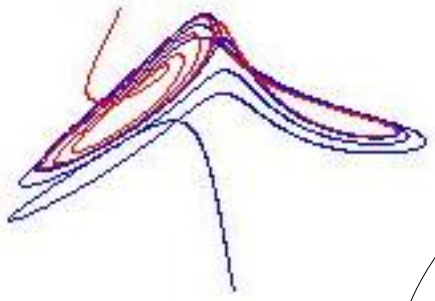

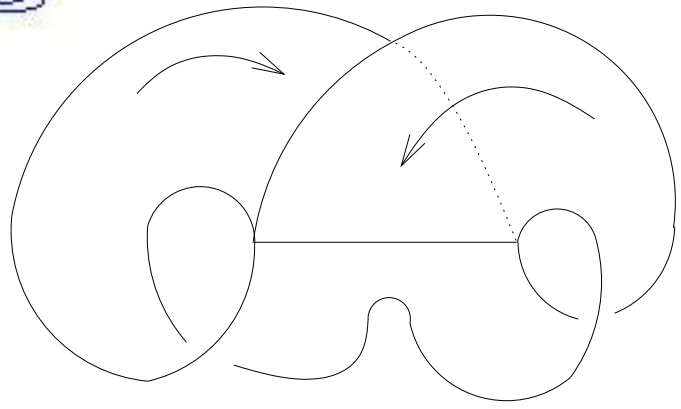

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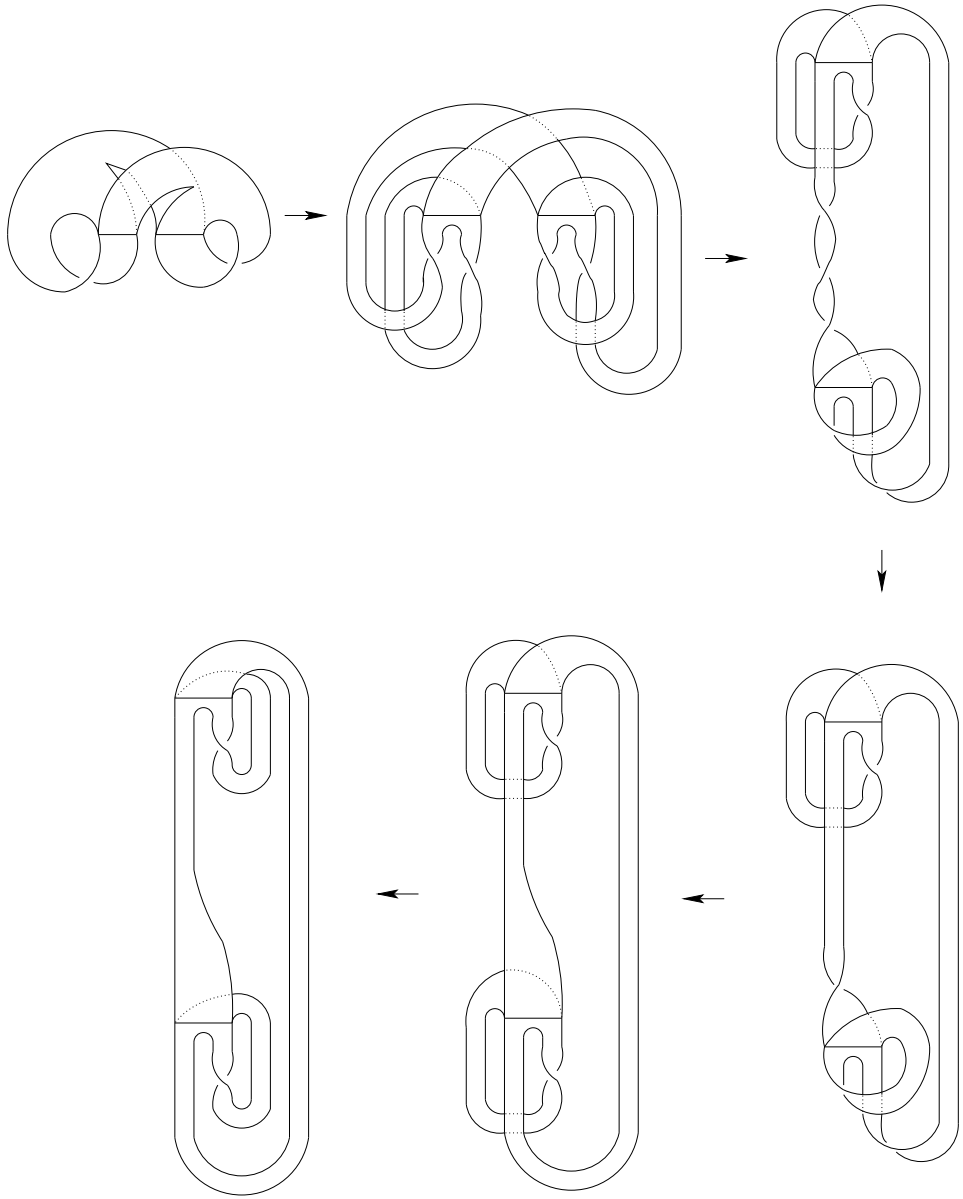

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$L(-1,-1)$ contains composite knots. It is now known what the maximum number of prime factors is three.