Knots and Templates

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Lorenz Equations

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$$\dot{y} = \rho x - y - xz,$$
$$\dot{z} = -\beta z + xy,$$

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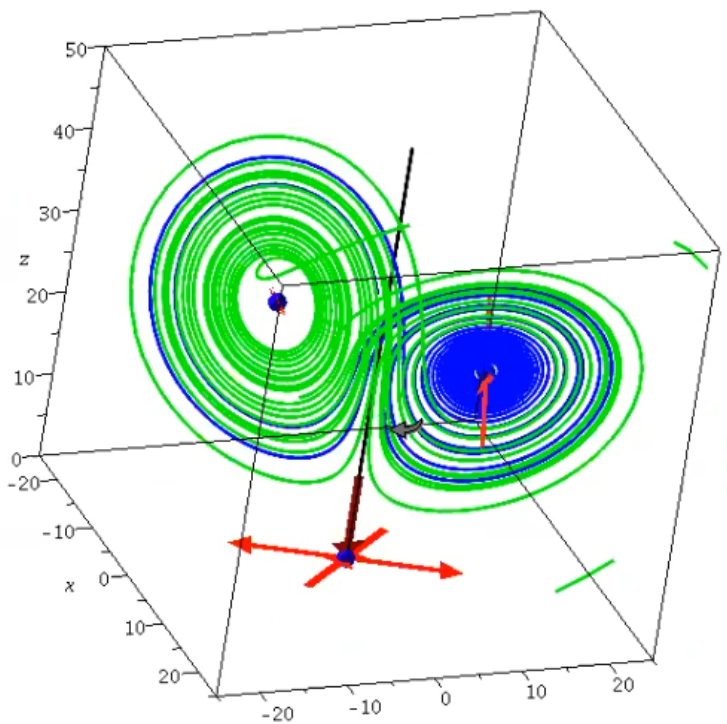
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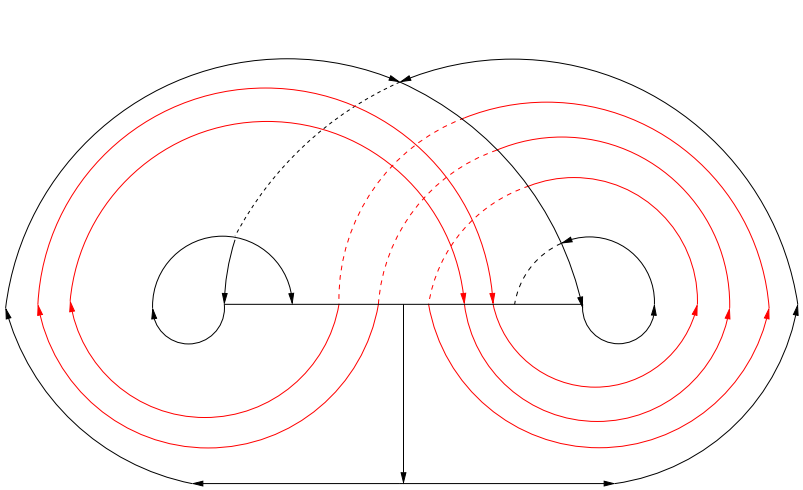


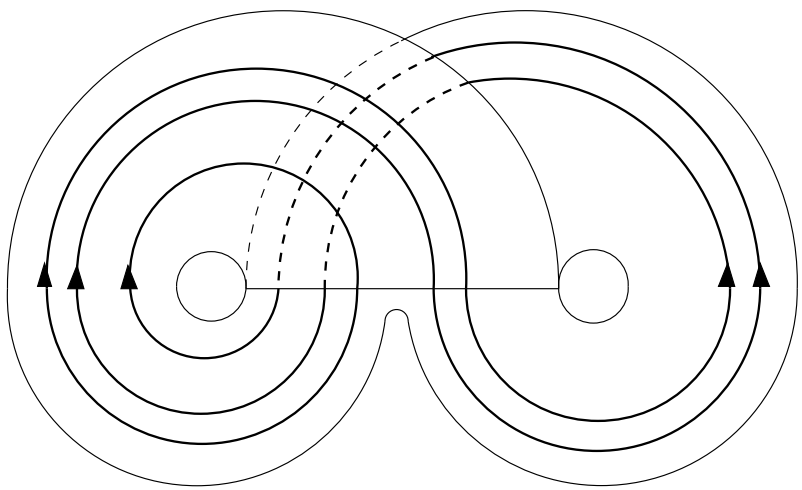
Lorenz Template

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Relation between Lorenz Equations and Lorenz Template

Warwick Tucker, while a Ph.D. student at Cornell University, was able to prove that the Lorenz equations do indeed have an attractor modeled by the Geometric Lorenz Attractor. His work was published in 2002.

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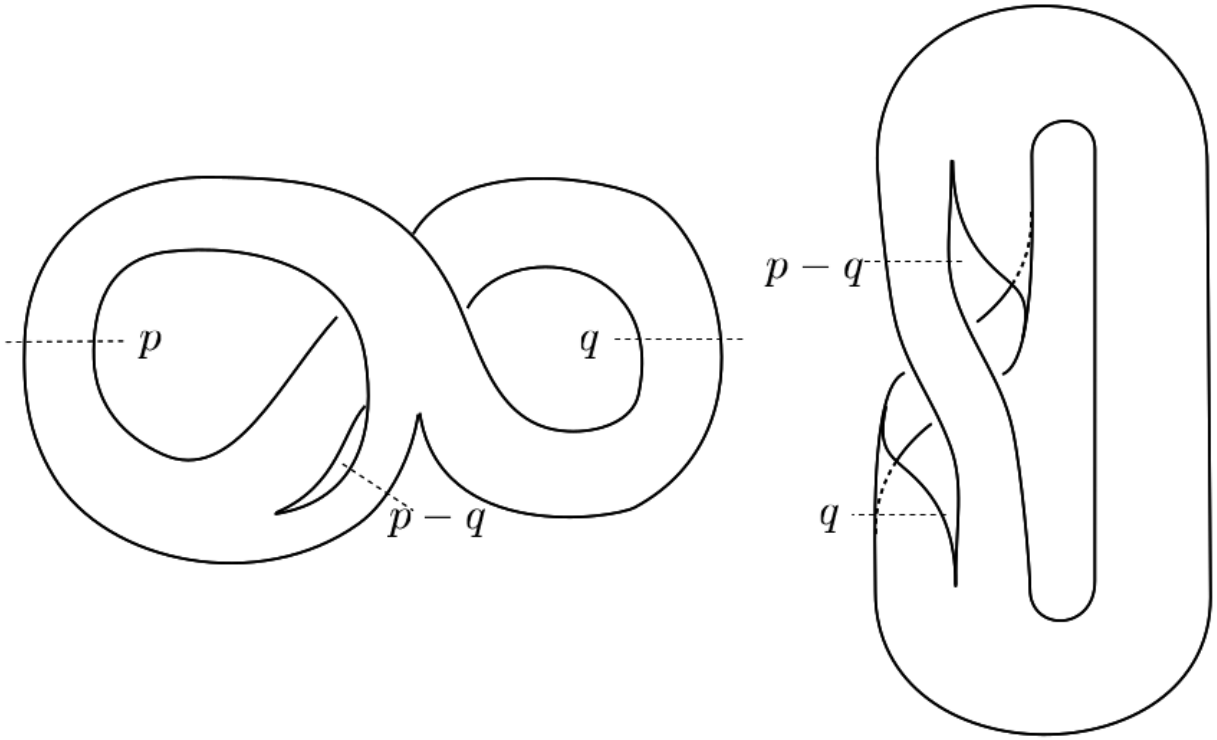
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- All Lorenz knots are positive braids with full twist. It follows that they have positive signature, are fibered and prime.

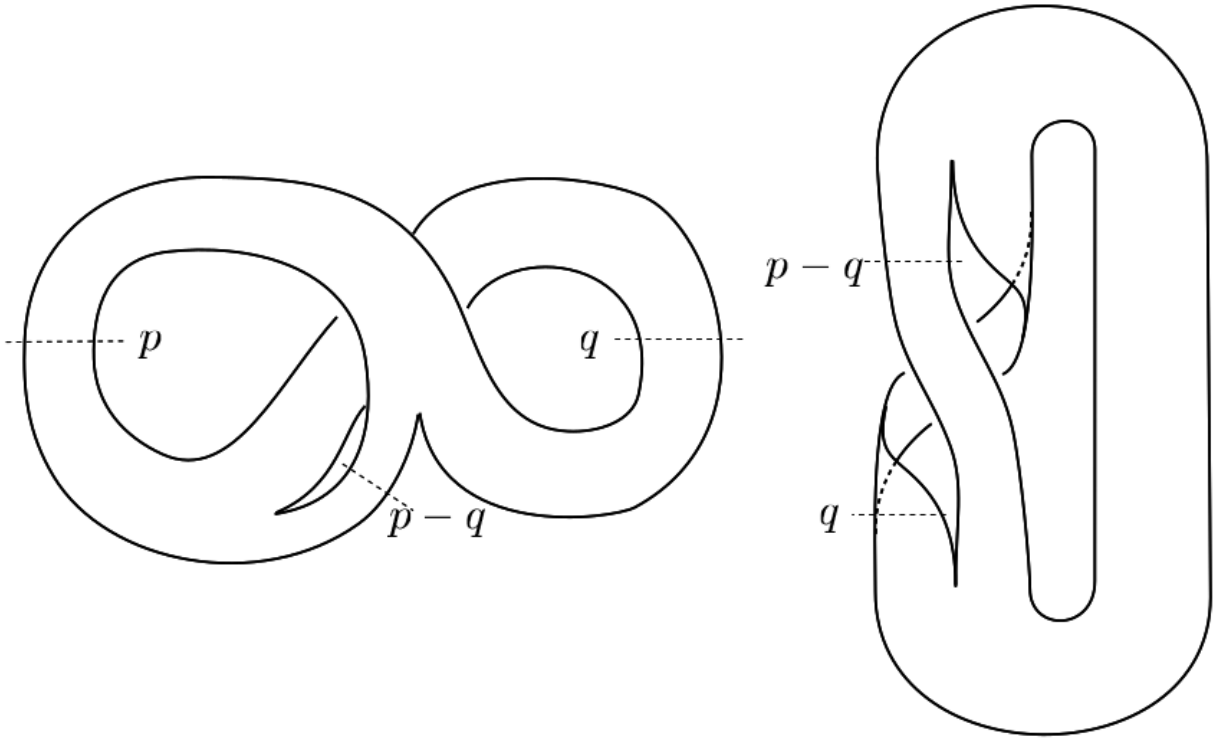
“Proof” Torus Knots are Lorenz Knots

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[Figure from GHS]

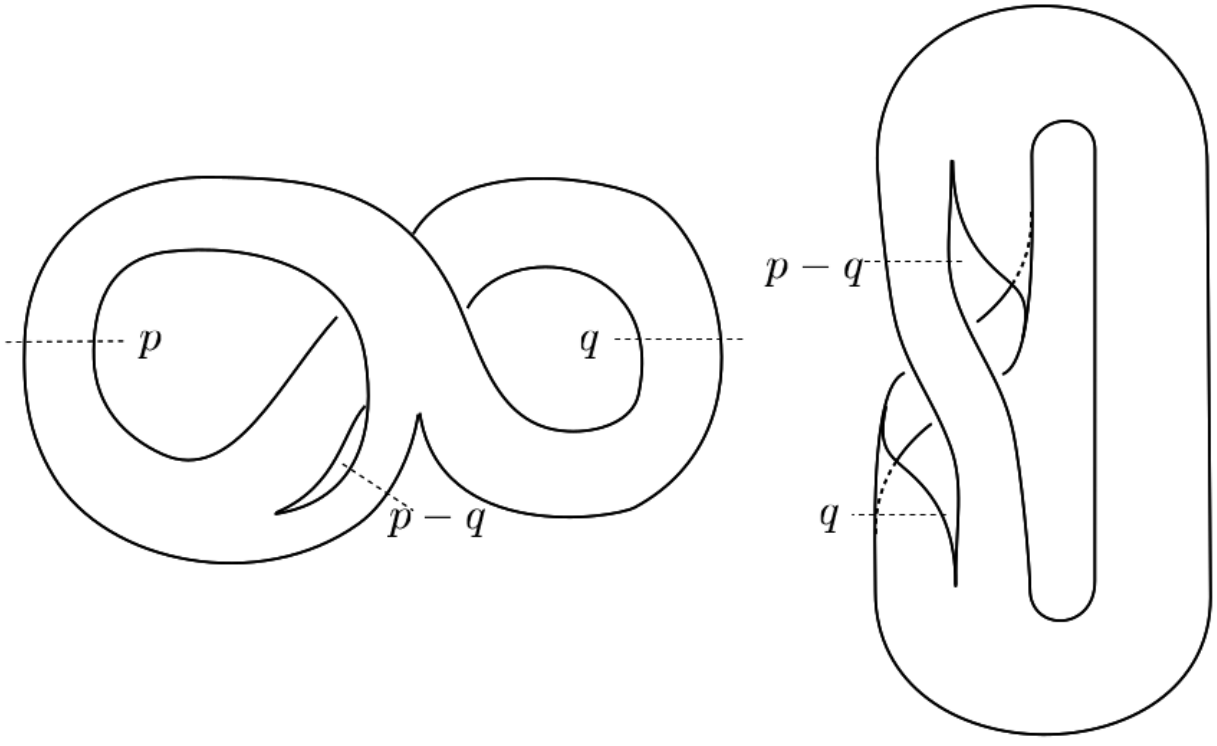
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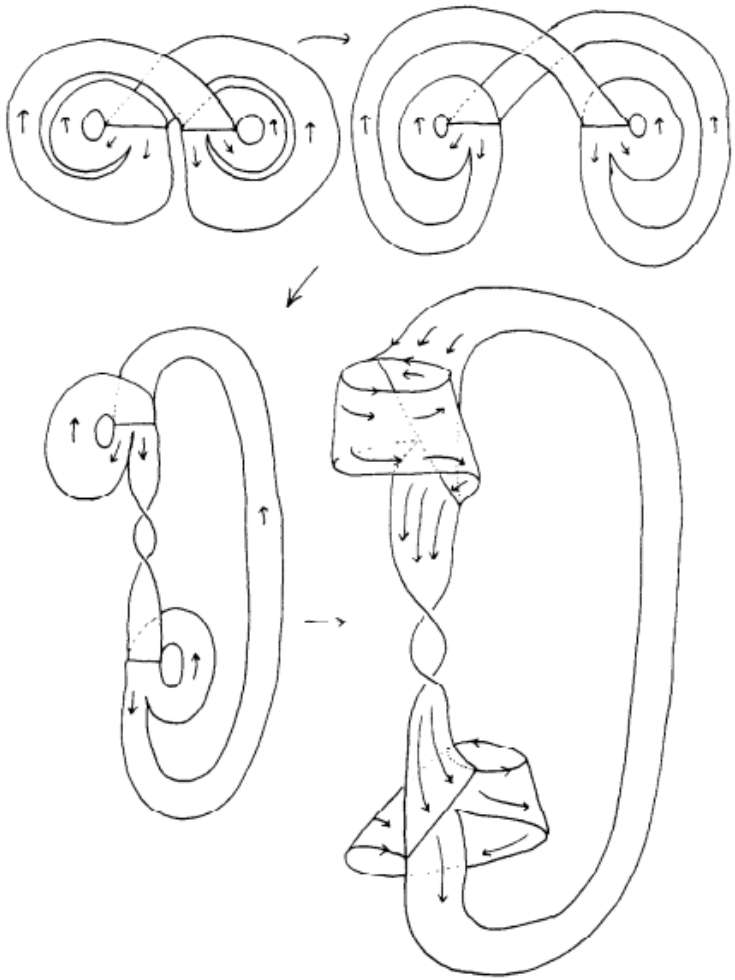


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Proof all Lorenz Knots are Positive Braids with a Full Twist

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[Figure from BW-I]

Alexander Polynomials of Lorenz Knots

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Theorem (Dehernoy, 2015). Let k be a nontrivial Lorenz knot. Let g denote its genus and b its braid index. Then the zeros of the Alexander polynomial of k lie in the annulus

$$\left\{ z \in \mathbb{C} \,\middle|\, (2g)^{-4/(b-1)} \leq |z| \leq (2g)^{4/(b-1)} \right\}.$$

T-links and Lorenz Links

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In 2009 Birman & Kofman showed that Lorenz links and a new class of links they called *T-links* are almost the same. T-links are a generalization of torus links. They expand the definition of Lorenz links to include parallel push-offs on the Lorenz template of any sublinks. We will call these *expanded Lorenz links*. T-links which are knots coincide exactly with the traditional Lorenz knots.

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Theorem. [Birman & Kofman] Expanded Lorenz links and T-links are the same.

Definition. Let r_i and s_i for $i = 1, \ldots, k$ be integers such that $2 \leq r_1 \leq \cdots \leq r_k$, and $0 < s_i$. These determine a *T-link*, denoted $T((r_1, s_1), \ldots, (r_k, s_k))$, as the closure of the braid

$$(\sigma_1\sigma_2 \cdots \sigma_{r_1-1})^{s_1}(\sigma_1\sigma_2 \cdots \sigma_{r_2-1})^{s_2} \cdots (\sigma_1\sigma_2 \cdots \sigma_{r_k-1})^{s_k}.$$

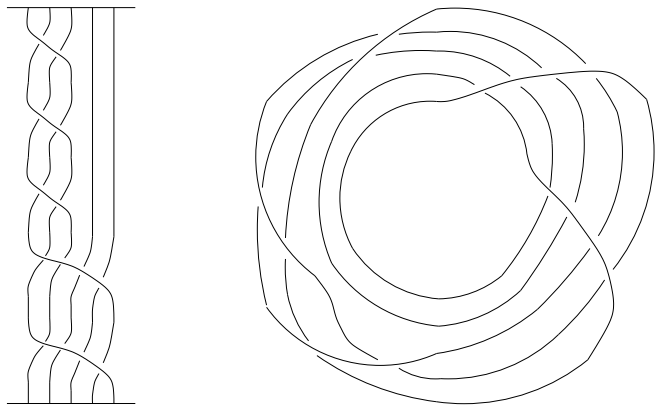
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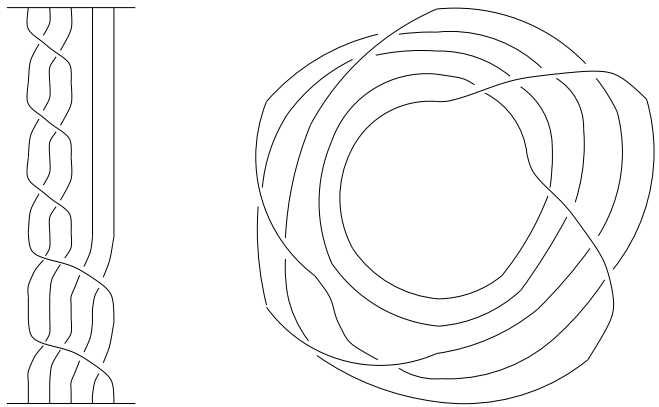
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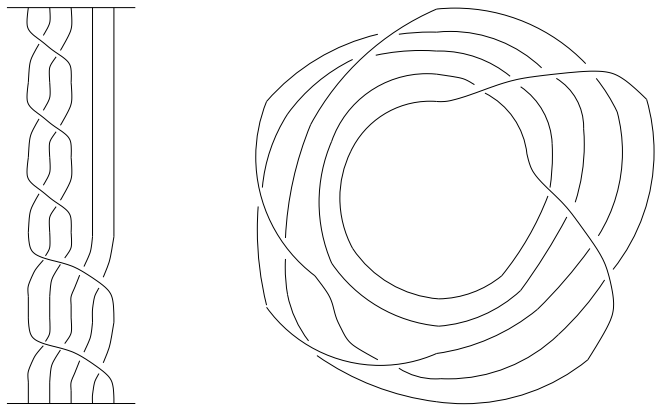
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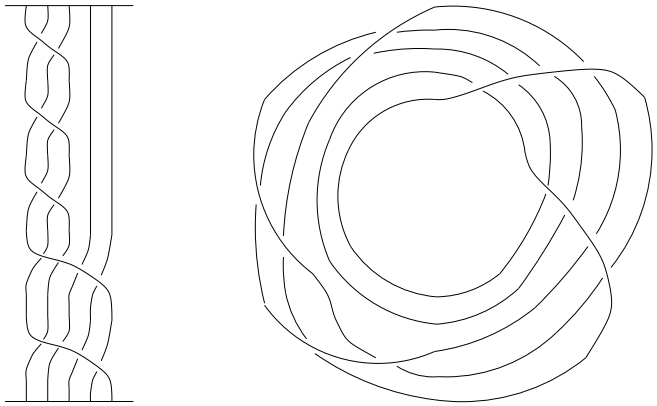


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They "show that both hyperbolic volume and the Mahler measure of Jones polynomials are bounded for infinite collections of hyperbolic Lorenz links."

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Theorem. Let L be a link in S^3 and let $\mathrm{Vol}(L)$ denote the sum of the volumes of the hyperbolic pieces in the JSJ decomposition of $S^3 - L$. Then

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This builds on an earlier preprint [28 Jul 2026], *Generalized t-links represent all links in the 3-sphere*, by the same authors.

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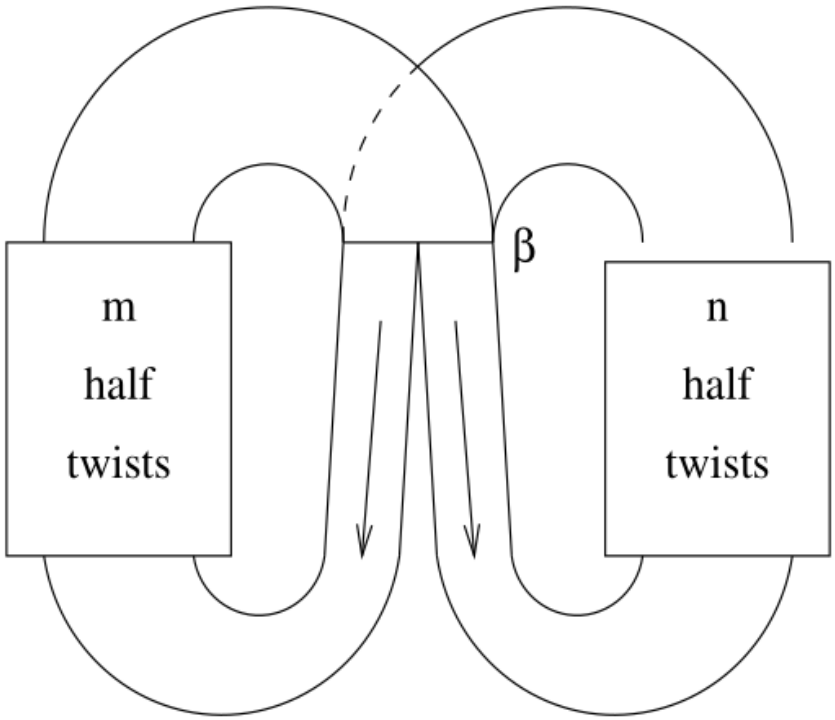
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Can other templates arise in similar processes?

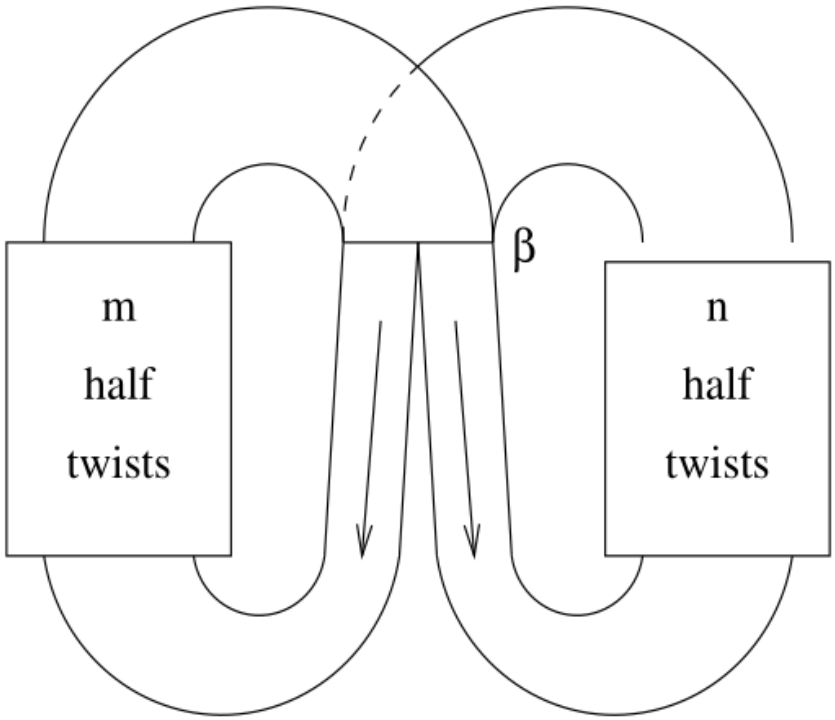
Lorenz-like Templates

Definition of $L(m, n)$:



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$L(m, n) = L(n, m)$ via a 180^o rotation.

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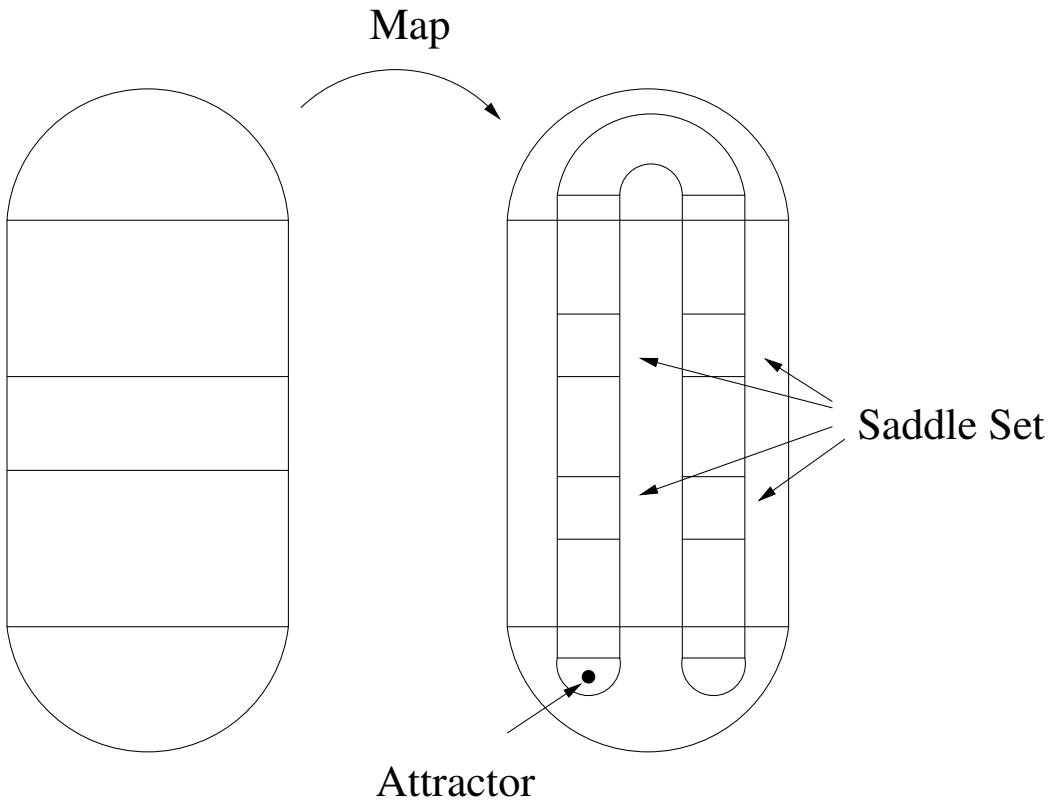
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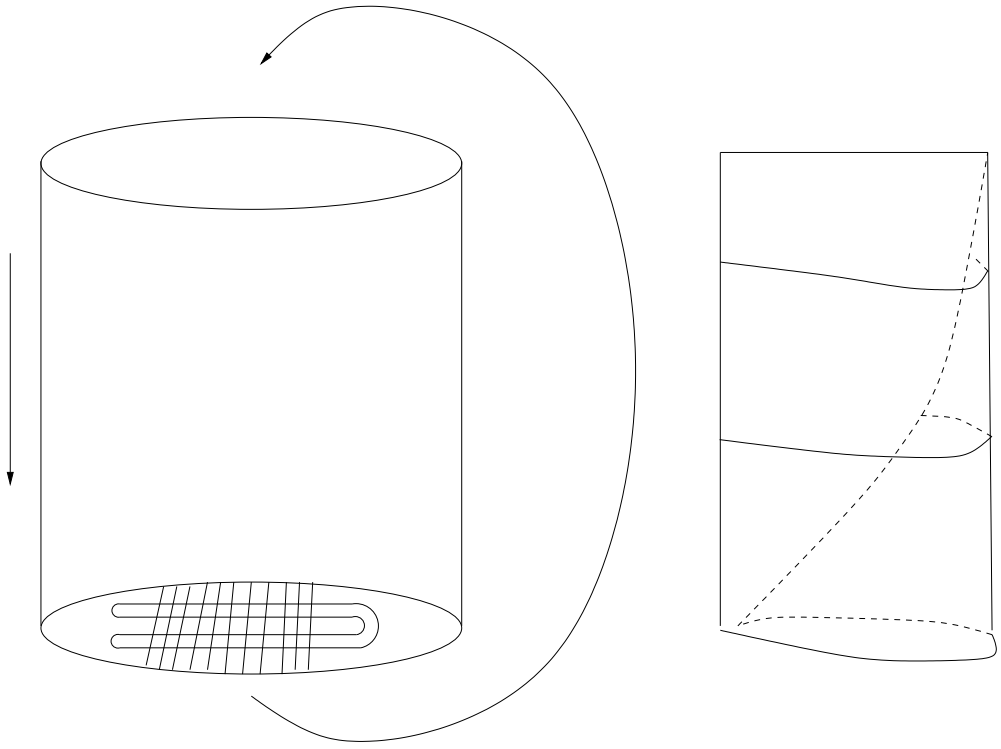
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$L(0,1)$ and Smale's horseshoe map

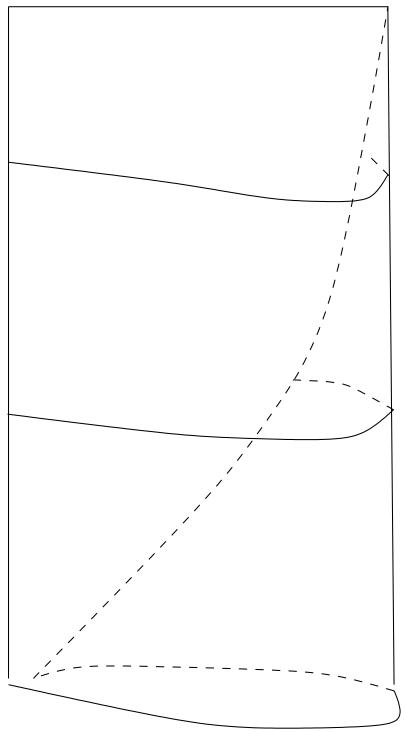
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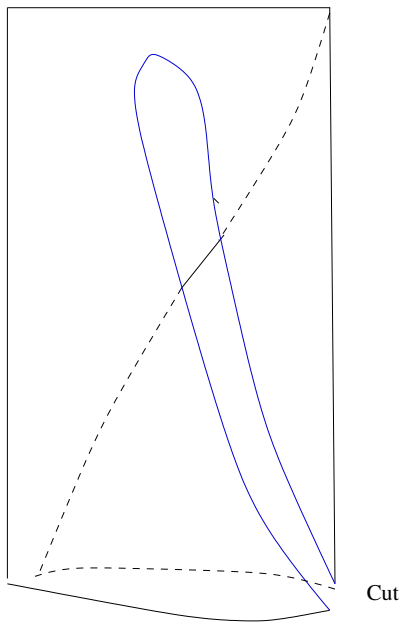
$L(0,1)$ template for suspension of saddle set



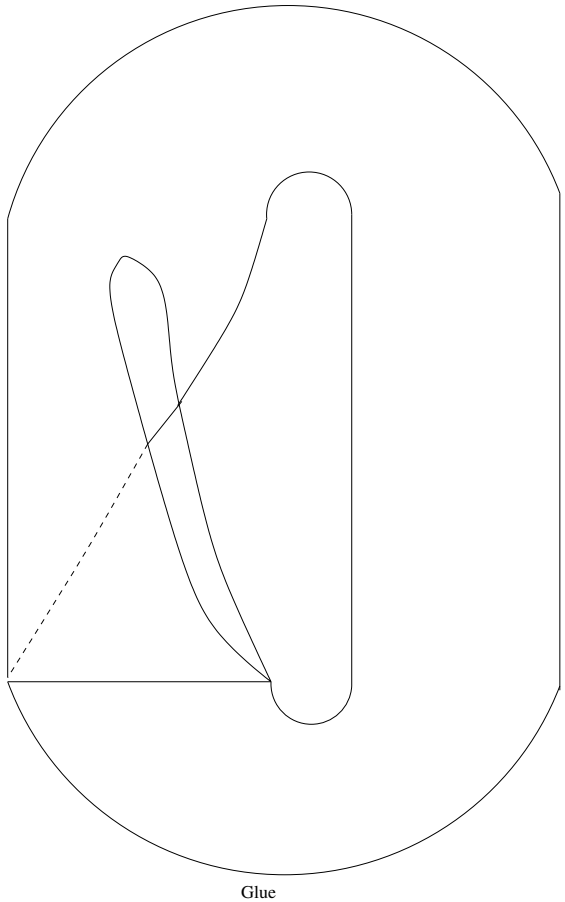
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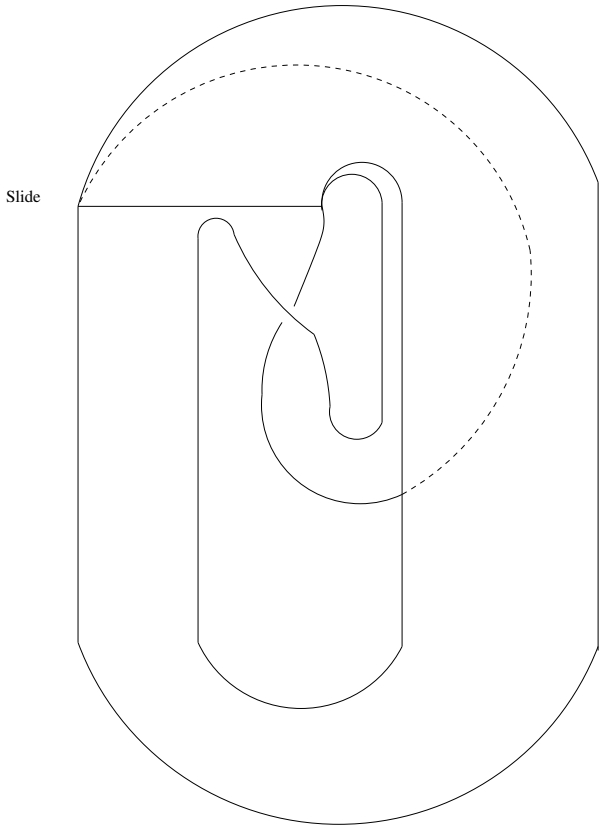
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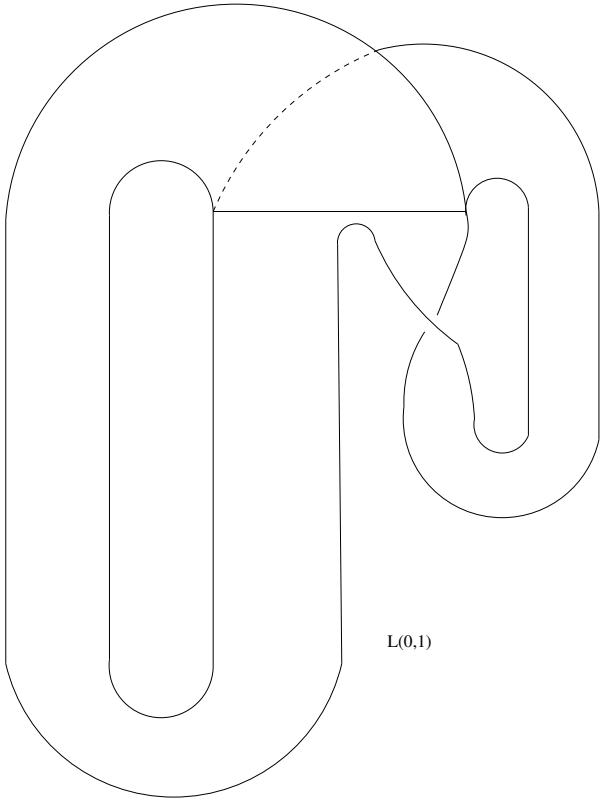
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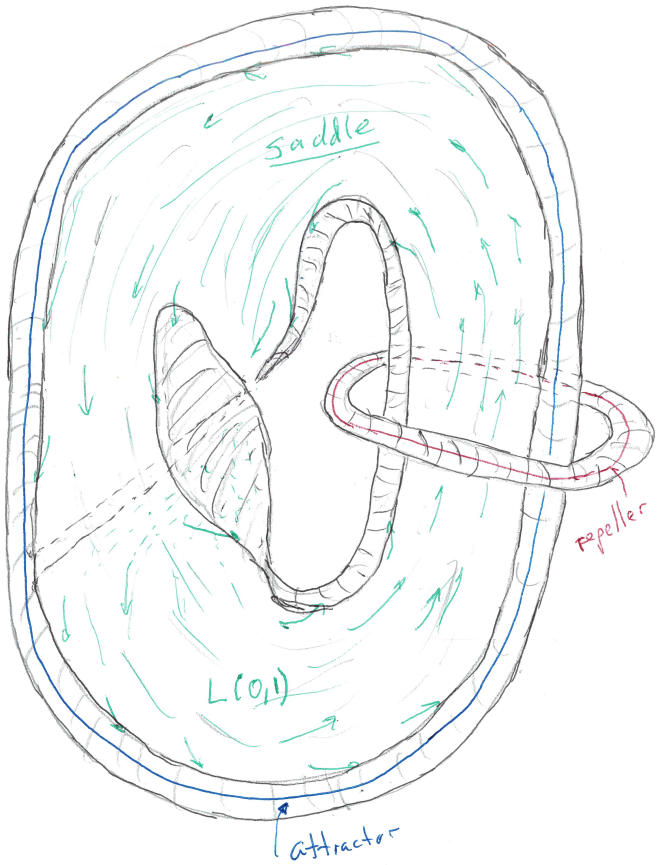
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A Smale flow on S^3



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Can these be related to T-links?

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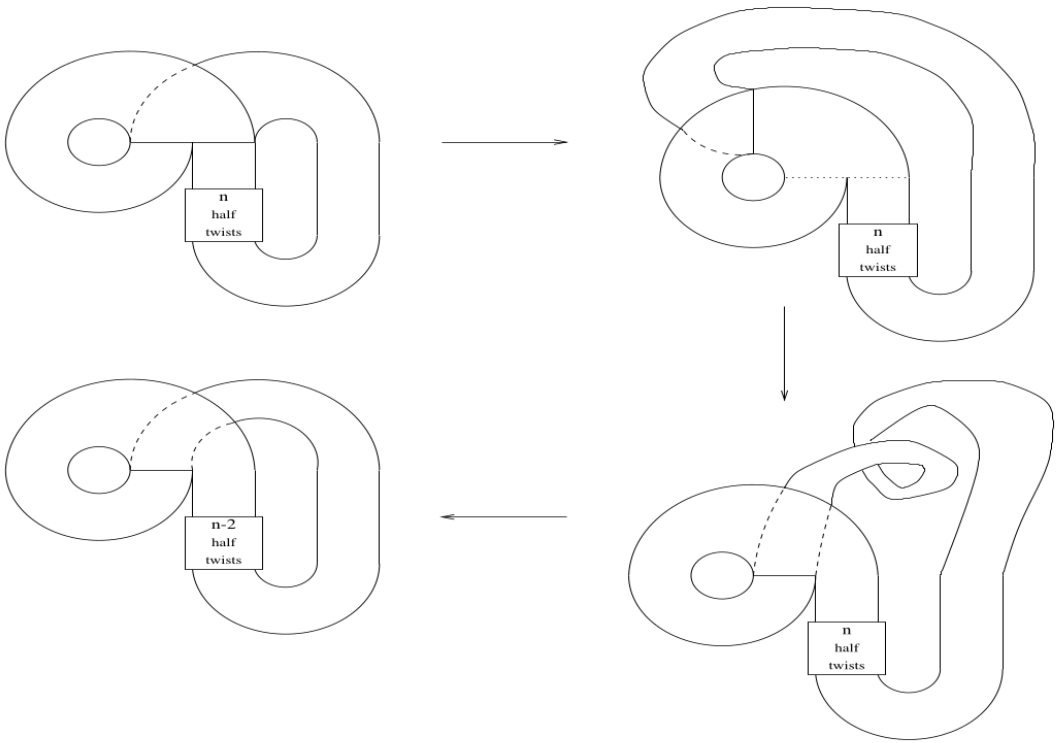
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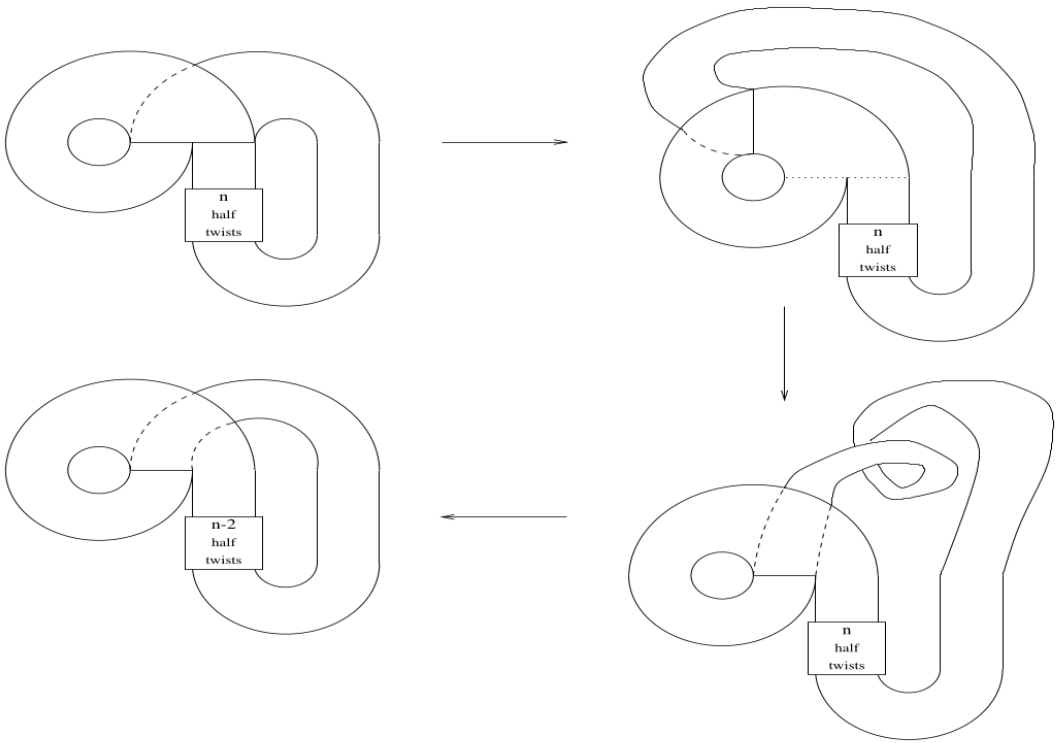
Theorem C. Any knot formed by the connected sum of a knot in $L(0, 2)$ and a knot in the mirror image of $L(0, 2)$ is in $L(0, -2)$.

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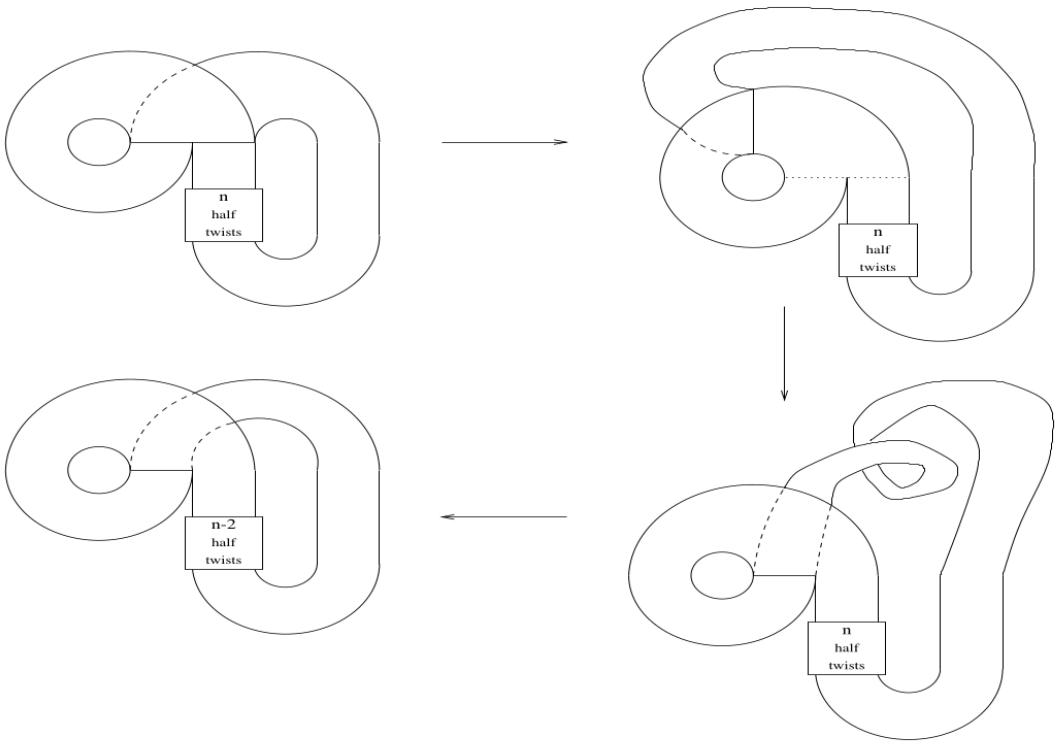


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Robinson's Attractor

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In 1989 Clark Robinson studied the 3×3 system of ODEs below.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y + yz, \\ \dot{z} &= -\gamma z + \delta x^2, \end{aligned}$$

where $\alpha = -0.71$, $\beta = 1.8690262$, $\delta = 0.1$, and $\gamma = 0.6$.

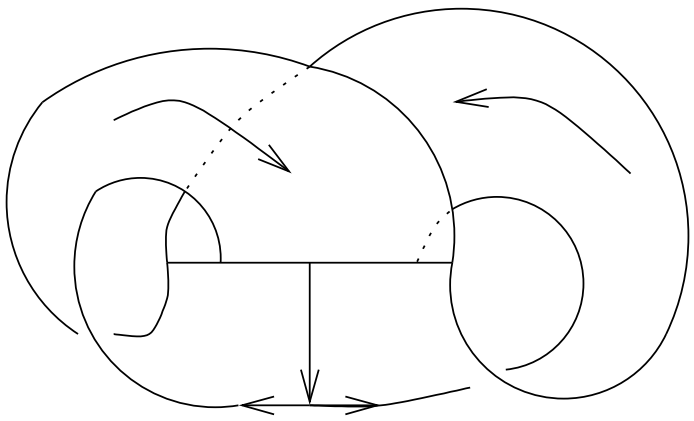
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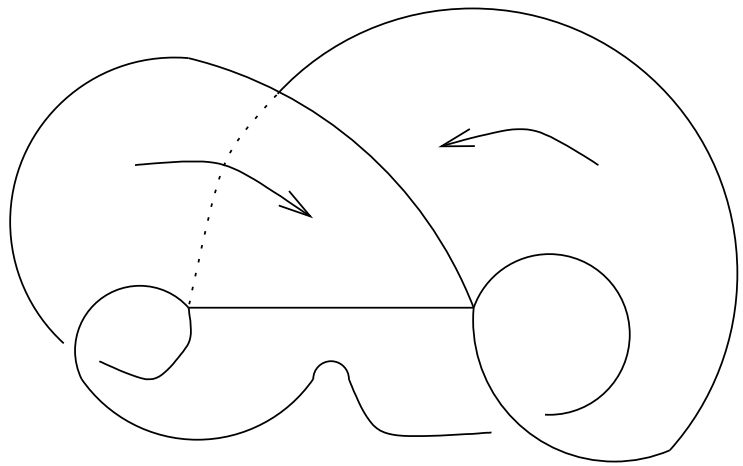
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$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y + yz, \\ \dot{z} &= -\gamma z + \delta x^2, \end{aligned}$$

where $\alpha = -0.71$, $\beta = 1.8690262$, $\delta = 0.1$, and $\gamma = 0.6$.
He showed that the system had a strange attractor similar to the Lorenz attractor except that the bands had half twists. Then (S.) determined that $L(-1, -1)$ was a template model for this system.

$L(-1,-1)$





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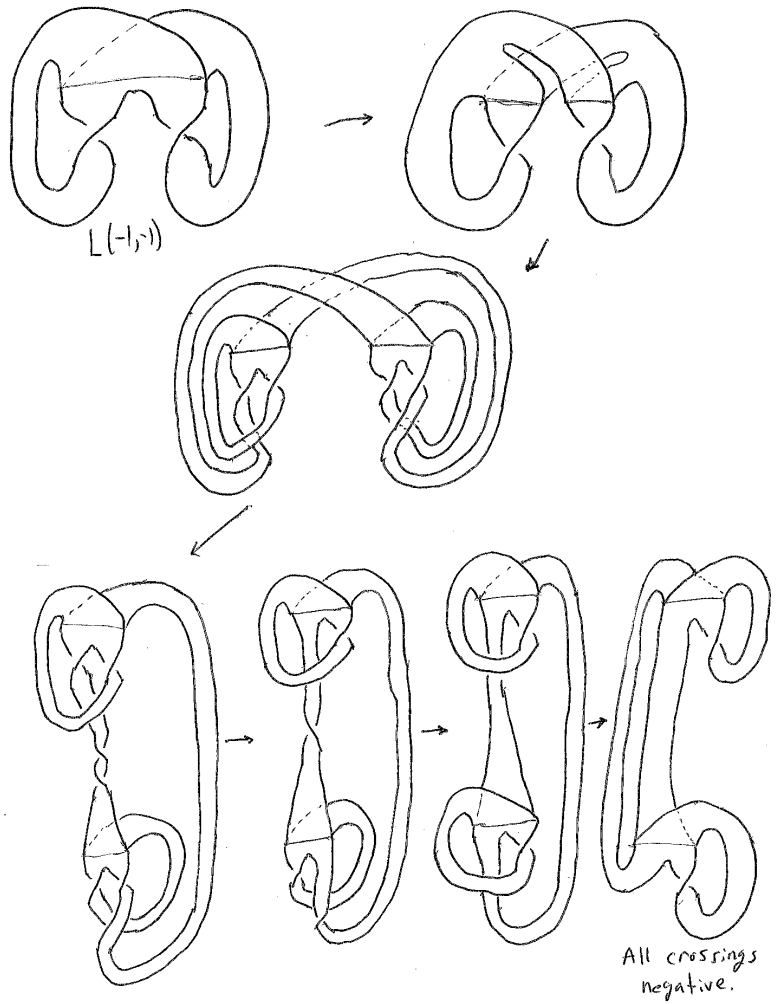
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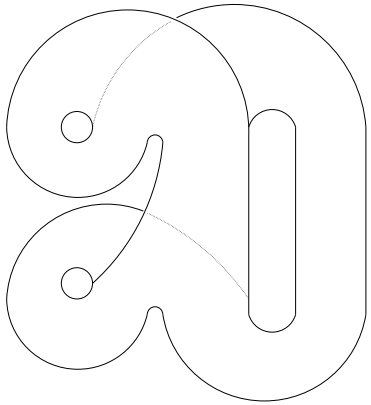
Is there a way to characterize which templates support only fibered knots and links?

$L(-1,-1)$ Links are Negative: Proof

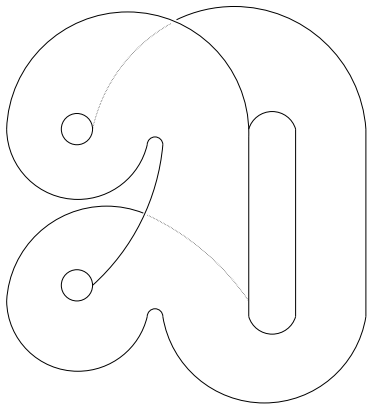


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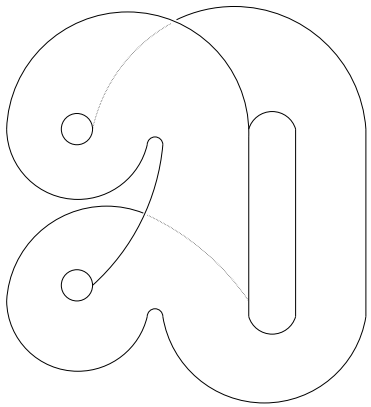


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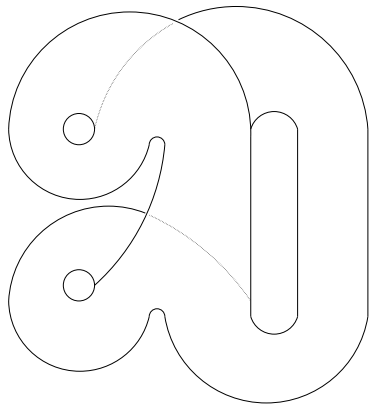
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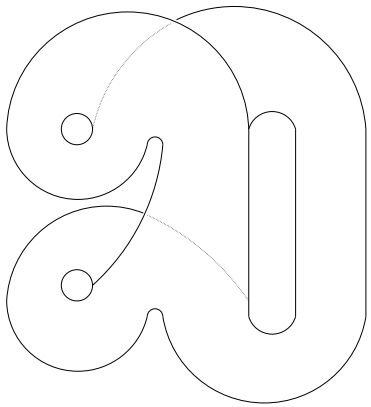
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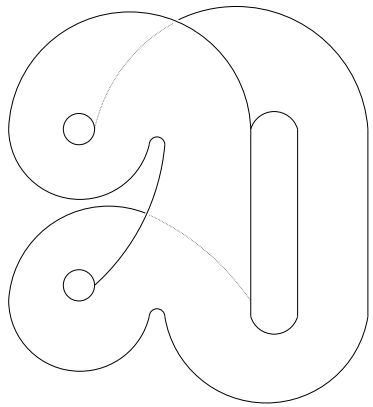
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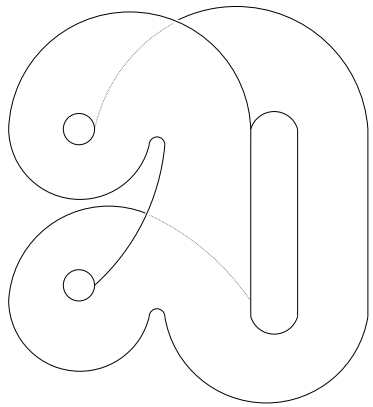


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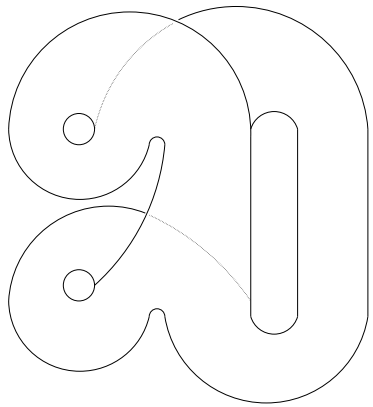


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Corollary. $L(0,-n)$ for $n > 0$ are universal.

Figure-8 knot in U

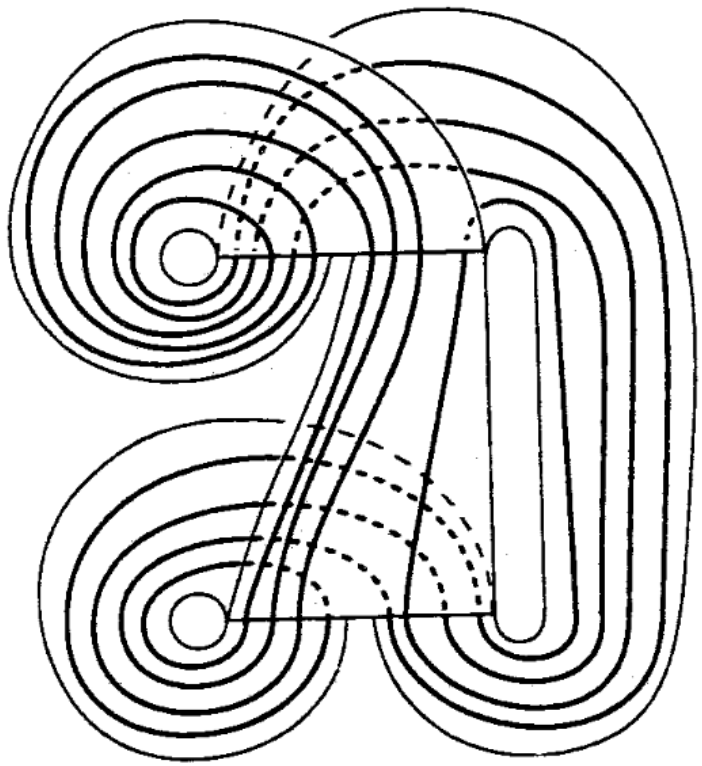
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[Figure from Meleshuk's dissertation.]

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But, finding more general answers proved to be very difficult.

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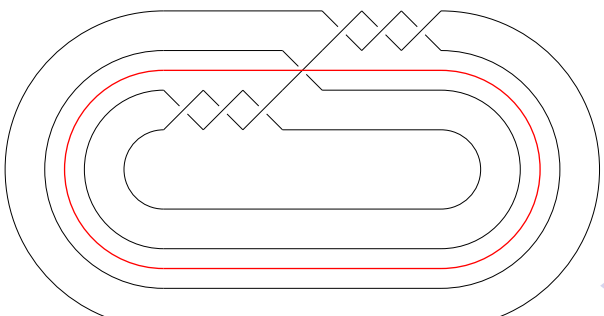
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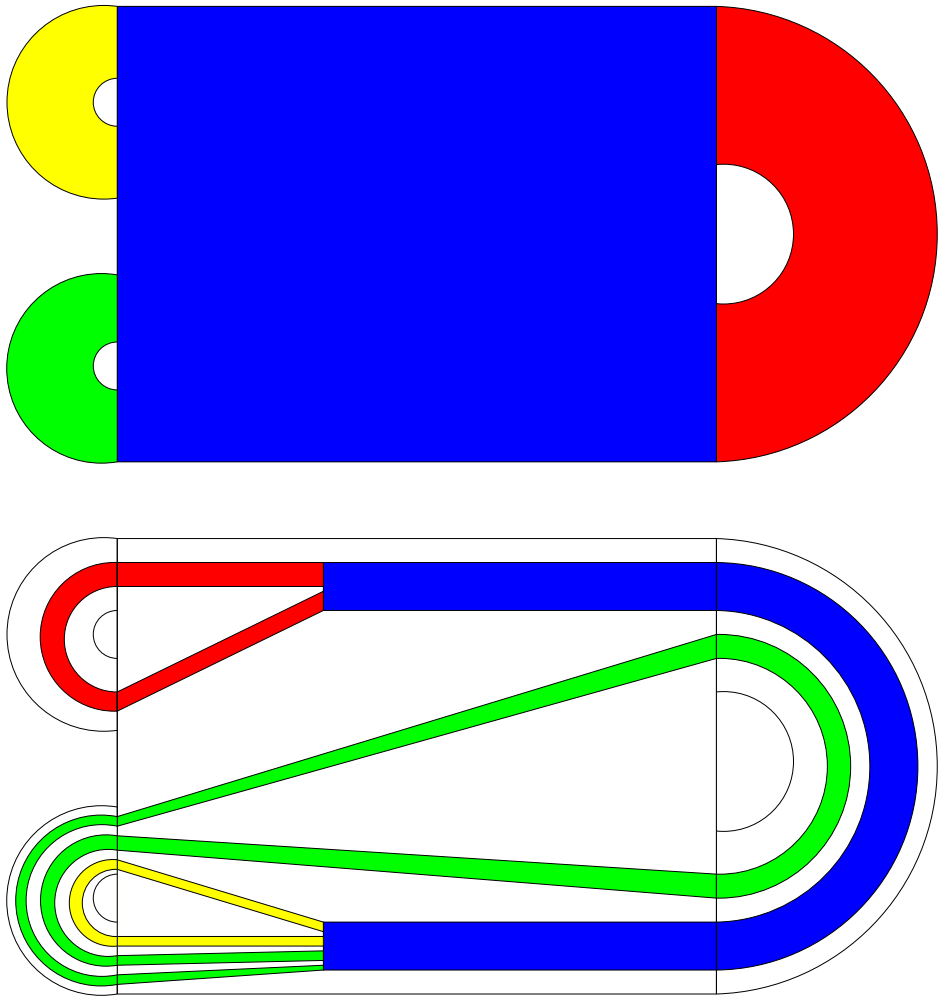
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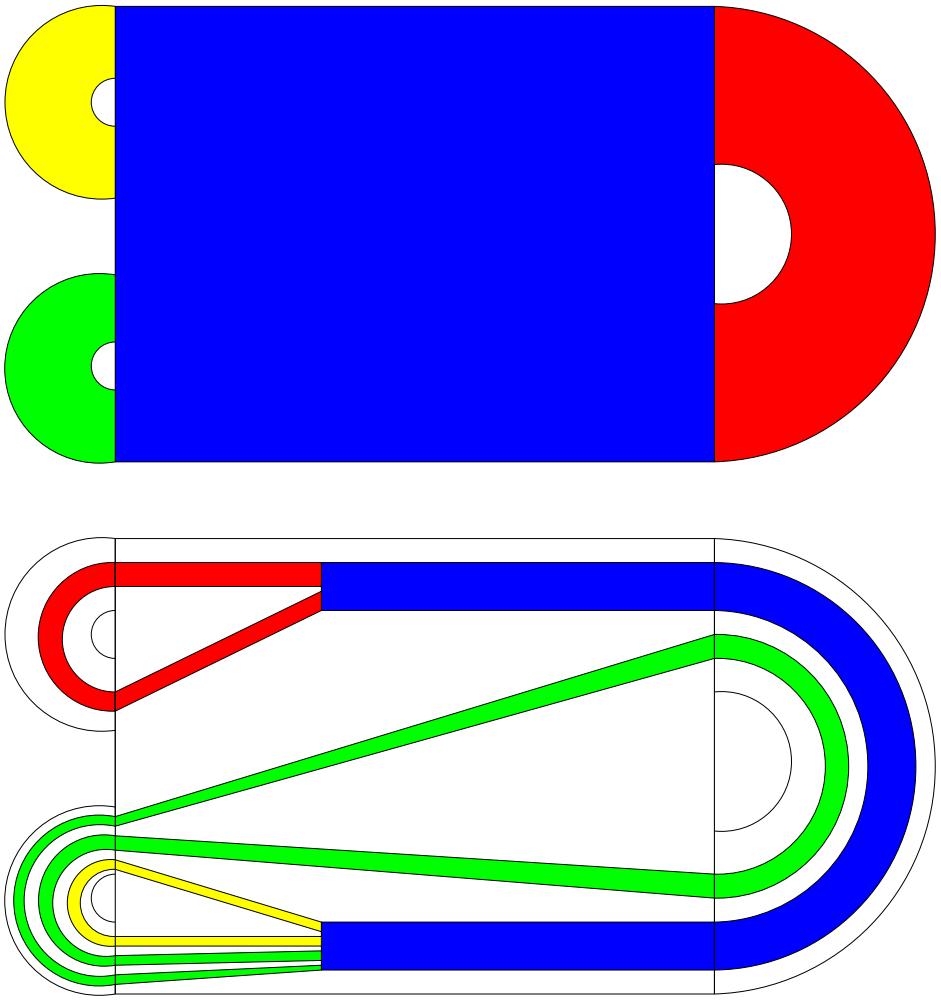
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Conjecture: This is likely true for templates with positive presentations (not necessarily braided).

Suspension of Plykin-like Attractors

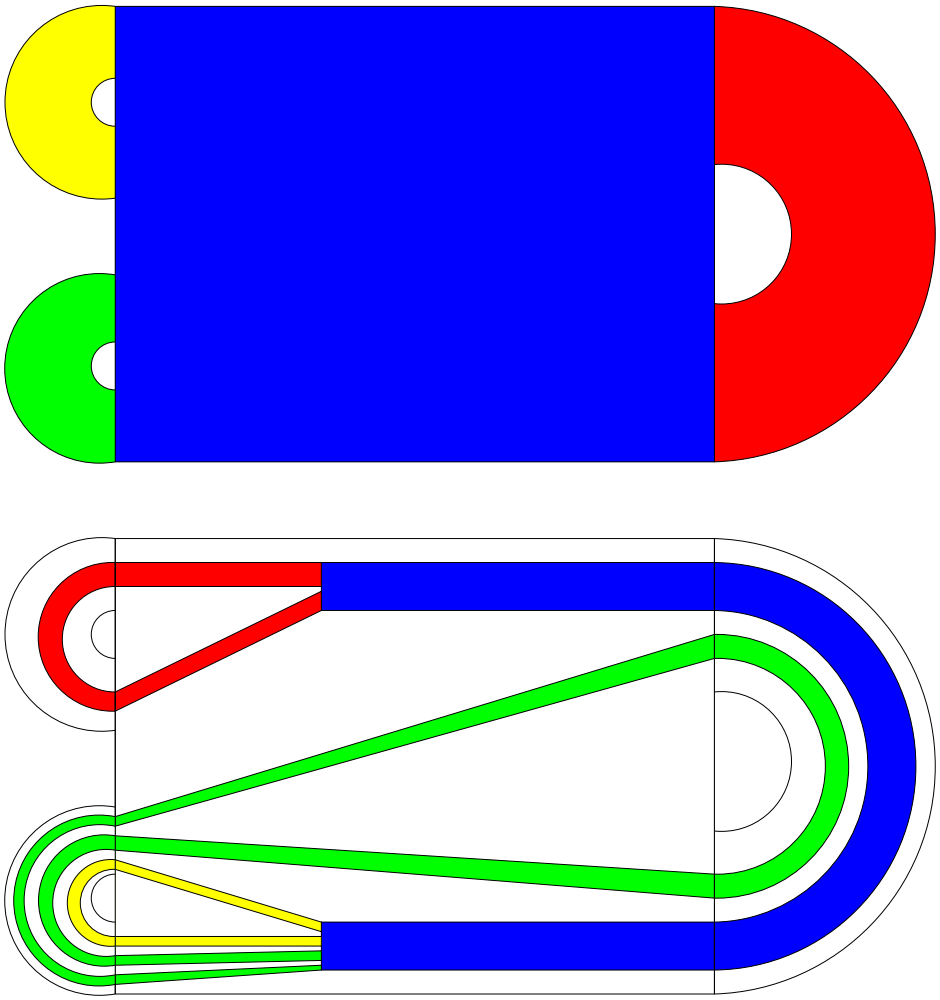


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Rob Ghrist's question:

Suspension of Plykin-like Attractors



Rob Ghrist's question: Do we always get universal templates?

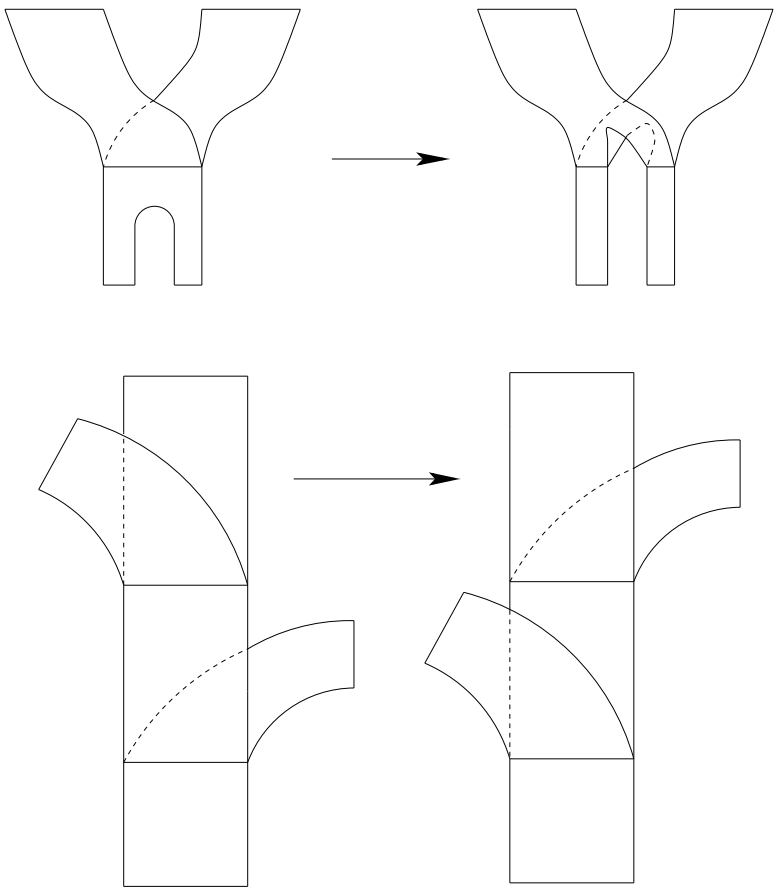
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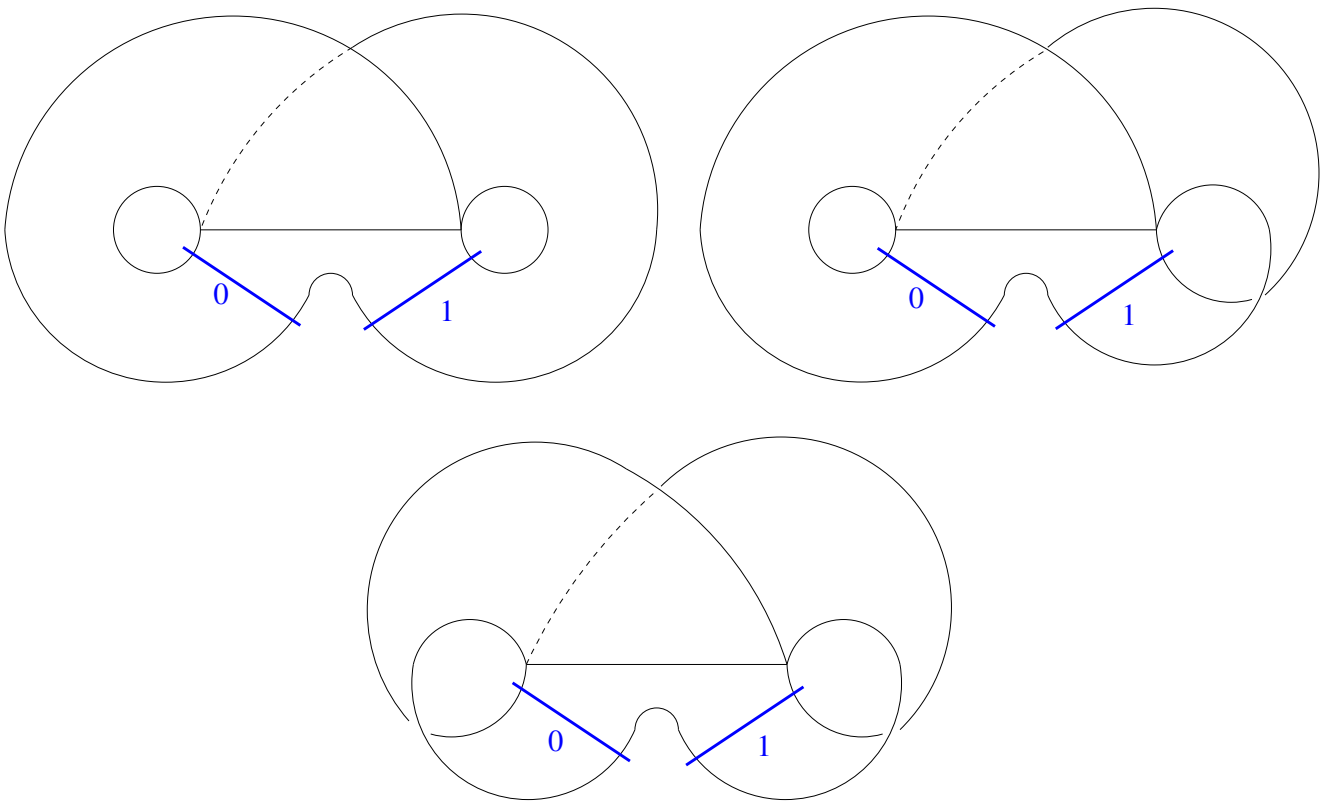
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- A graph theoretic approach due to Liu and Zhao. (2021)
- Quantum invariants due to Saito. (KSS, 2003)
- Twist-wise flow equivalence. I'll discuss this one.

How to Classify?

The three templates below all have the same underlining dynamics. They are all suspensions of a full shift map on two symbols.



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Matrices that give topologically equivalent suspension flows are flow equivalent.
Theorem (Franks). Let A and B be matrices representing SFTs. Their suspension flows are topologically equivalent iff

$$\det(I - A) = \det(I - B) \qquad \text{(Parry-Sullivan number)},$$

and

$$\mathbb{Z}^n/(I - A)\mathbb{Z}^n \cong \mathbb{Z}^m/(I - B)\mathbb{Z}^m \qquad \text{(Bowen-Franks group)}.$$

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$$L(0,0) \leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad L(0,1) \leftrightarrow \begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix} \quad L(1,1) \leftrightarrow \begin{bmatrix} t & t \\ t & t \end{bmatrix}.$$

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Definition. Two templates are twist-wise flow equivalent if their invariant sets are flow equivalent and the diffeomorphism can be extended into the normal bundle.

Invariants of Twist-wise Flow Equivalence

Let G be the group-ring $\mathbb{Z}[\mathbb{Z}/2] = \{a + bt \mid a \,\&\, b \in \mathbb{Z}\}$ mod $t^2 = 1$.

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Theorem (S.) Let $A(t)$ and $B(t)$ be a twist matrices over G for templates T_A and T_B. If the templates are twist-wise flow equivalent, then

- $\det(I - A(\pm 1)) = \det(I - B(\pm 1))$
- $\mathbb{Z}^n/(I - A(\pm 1))\mathbb{Z}^n \cong \mathbb{Z}^m/(I - B(\pm 1))\mathbb{Z}^m$
- $\mathbb{Z}^{2n}/(I - A(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})\mathbb{Z}^{2n} \cong \mathbb{Z}^{2m}/(I - B(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})\mathbb{Z}^{2m}$
- $O(A(t)) = O(B(t))$, where $O(A(t))$ equals "orientable" if $\mathrm{tr}\,(A^k(t))$ has no t's for all k, and equals "nonorientable" otherwise.

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These are not complete.

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Theorem (Mike Boyle and S.). Let A and B be nontrivial essentially irreducible matrices over $\mathbb{Z}^+[\mathbb{Z}/2]$ and assume they are nonorientable. Then A and B are twist-wise flow equivalent if and only if there is an $SL(\mathbb{N}, \mathbb{Z}[\mathbb{Z}/2])$ equivalence from $(I - A)_\infty$ to $(I - B)_\infty$.

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A normal form is only known to exist for PIDs. But, $(1 - t)(1 + t) = 1 - t^2 = 1 - 1 = 0$. This is a big open question.

Smale flows and templates

Definition. A flow ϕ_t on a manifold M is called a *Smale flow* provided

(a) the chain-recurrent set $\mathcal{R}$ of ϕ_t has a hyperbolic structure,

(b) the basic sets of $\mathcal{R}$ are one-dimensional, and

(c) the stable manifold of any orbit in $\mathcal{R}$ has transversal intersection with the unstable manifold of any other orbit of $\mathcal{R}$.

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Most references allow for fixed points but we will be working primarily with nonsingular flows.

Smale flows and templates

Definition. A flow ϕ_t on a manifold M is called a *Smale flow* provided

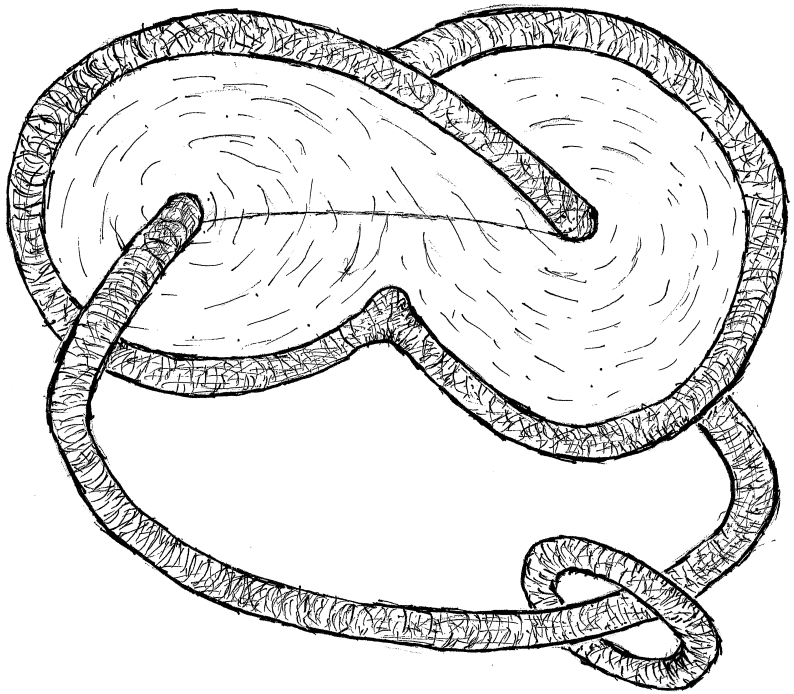
(a) the chain-recurrent set $\mathcal{R}$ of ϕ_t has a hyperbolic structure,

(b) the basic sets of $\mathcal{R}$ are one-dimensional, and

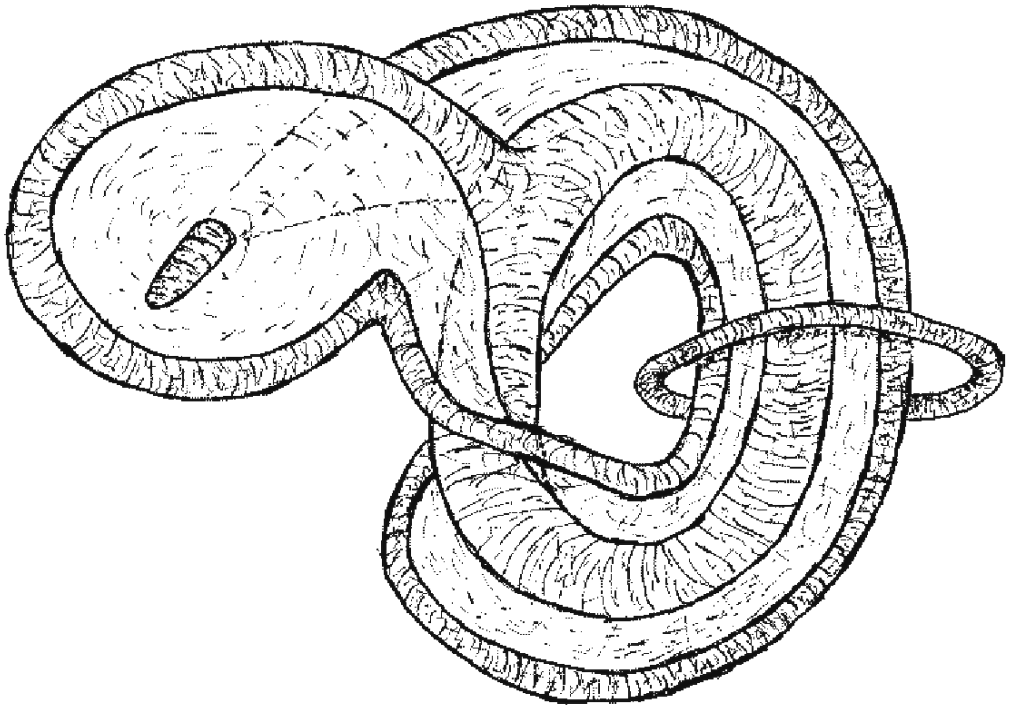
(c) the stable manifold of any orbit in $\mathcal{R}$ has transversal intersection with the unstable manifold of any other orbit of $\mathcal{R}$.

Most references allow for fixed points but we will be working primarily with nonsingular flows. Nontrivial saddle sets are modeled by templates [Birman & Williams].

A Lorenz-Smale flow on S^3



Another Lorenz-Smale flow on S^3



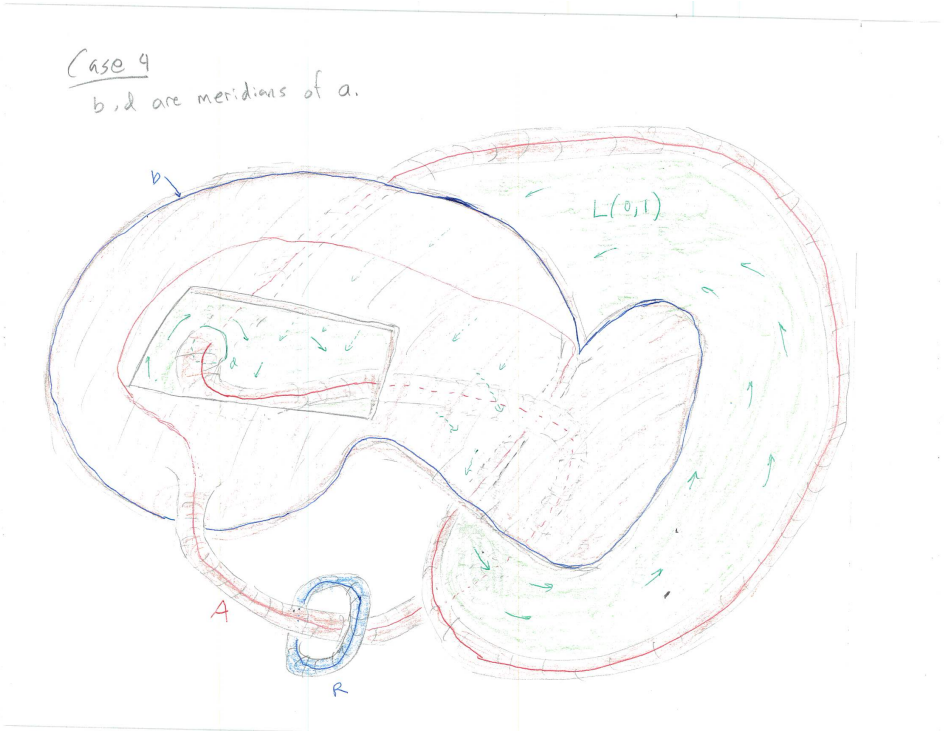
L(even,even)

Theorem (S.). For a simple Smale flow in S^3 with it saddle set modeled by $L(\text{even}, \text{even})$ the following and only the following configurations are realizable. The link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the saddle set is modeled by a standardly embedded Lorenz template, both bands are unknotted, untwisted, and unlinked, with the core of each band a meridian of the trefoil component of $a \cup r$. In the former case there are three possibilities: (1) The saddle set is standardly embedded. (2) One band is twisted with n full-twists for any n, but remains unknotted and unlinked to the other band, which must be unknotted and untwisted. (3) One band is a (p, q) torus knot, for any pair of coprime integers, with $p + q - 1$ full twists. The other band is unknotted, untwisted and unlinked to the knotted one.

L(even,odd)

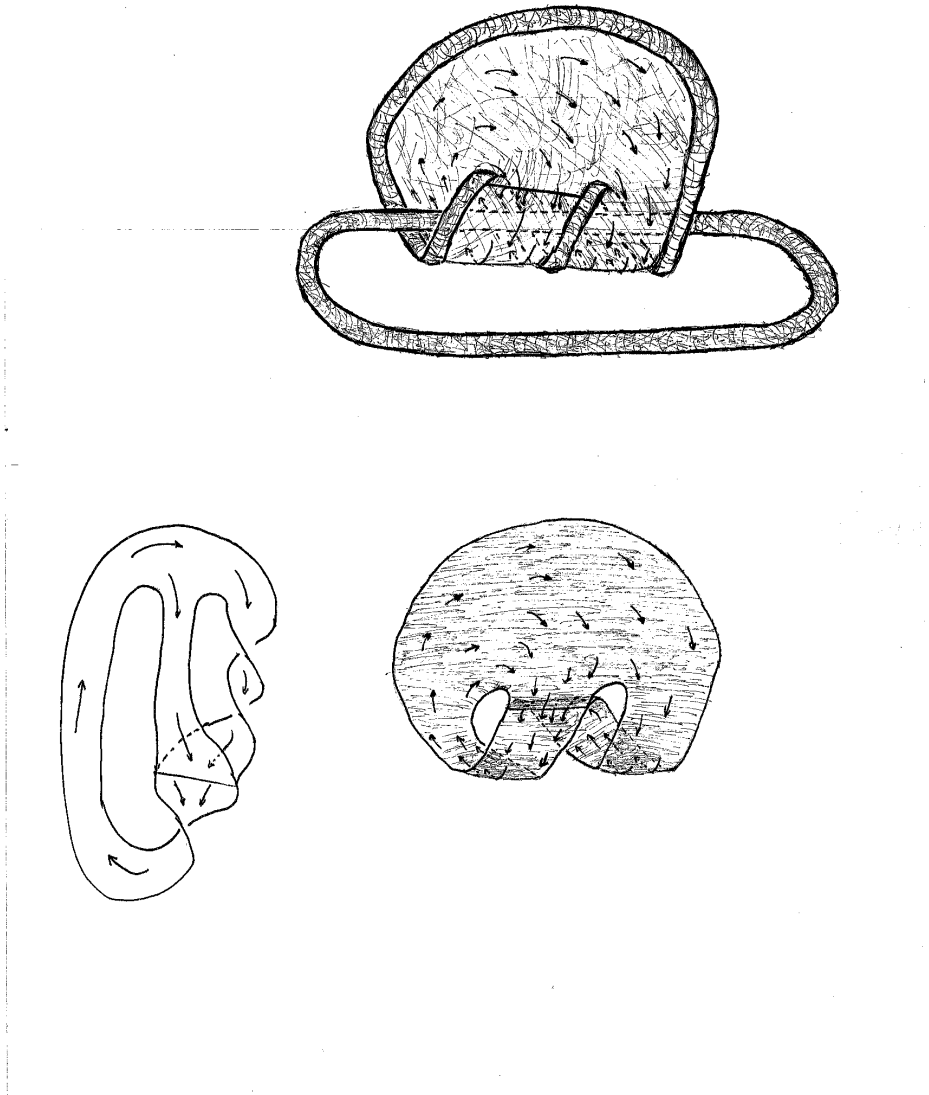
Theorem (Bin Yu). For a Lorenz-like Smale flow in S^3 with template $L(\text{even}, \text{odd})$ the following and only the following configurations are realizable. The link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the saddle set is modeled by a standardly embedded $L(0,1)$ template, i.e. both bands are unknotted, untwisted, and unlinked, with the core of each band a meridian of the trefoil component of $a \cup r$. In the former case there are three possibilities: (1) The saddle set is standardly embedded. (2) The saddle set is modeled by a $L(2p+2q-2, 2p+2q-1)$ and the cores of the two bands are parallel (p,q) torus knots. (3) The saddle set is modeled by $L(0, 2p+2q-1)$ and the core of the twisted band is a (p,q) torus knot, while the core of the other band is unknotted and unlinked with the twisted band.

$L(0,1)$ with trefoil attractor

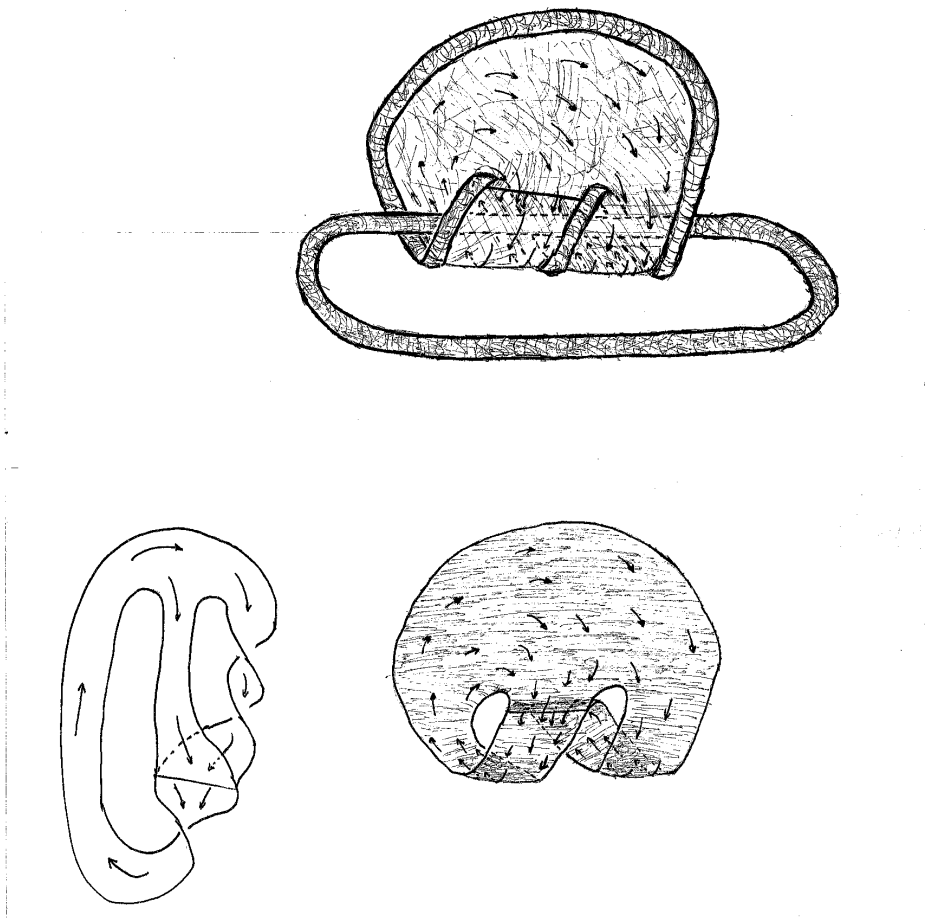


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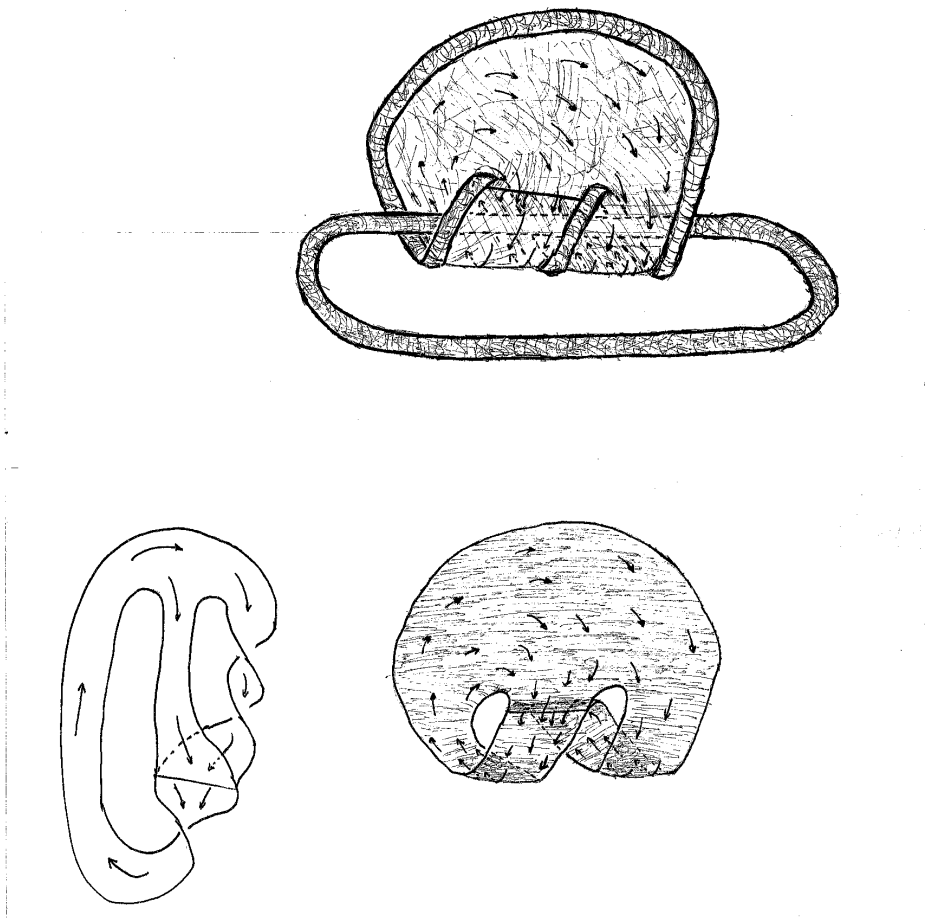


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Linking number is 3. Discuss general case.

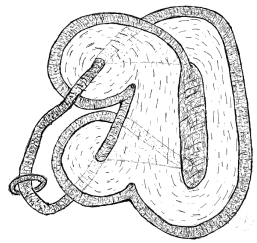
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Elizabeth Haynes studied the realizations of the template U. The attractor-repeller links can be a Hopf link, a figure-8 and meridian or a trefoil and meridian.

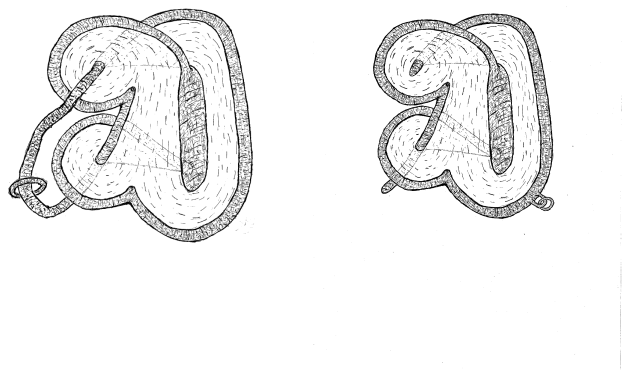
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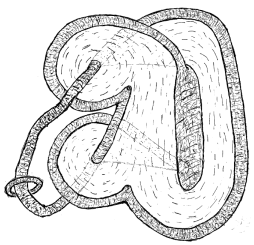
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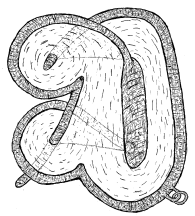
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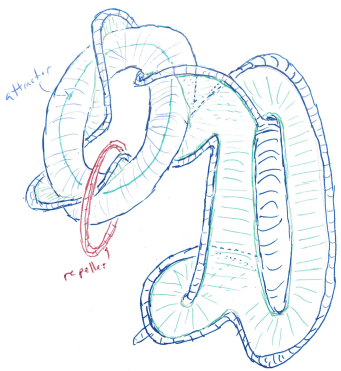


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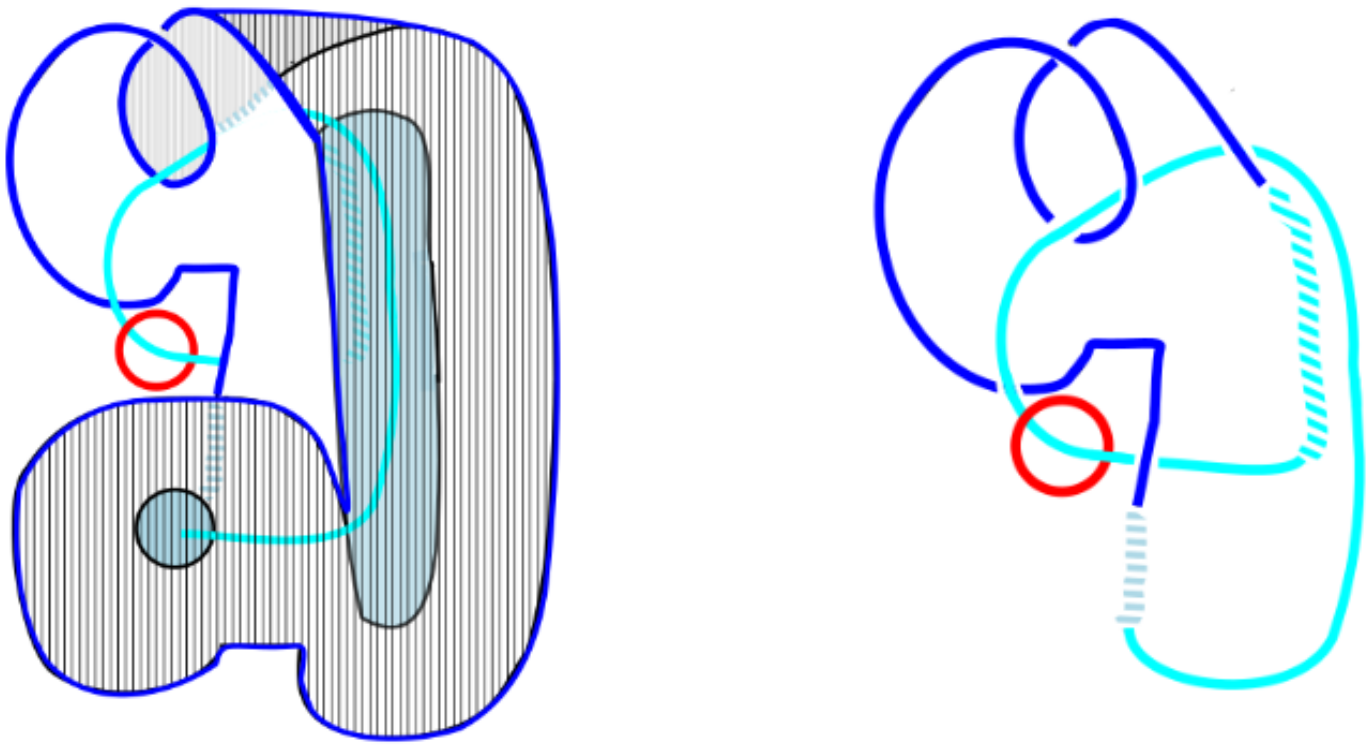
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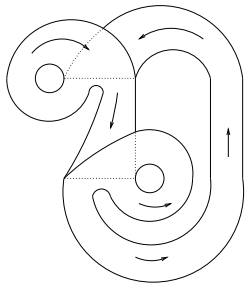
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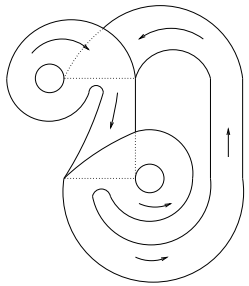
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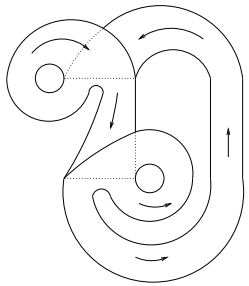


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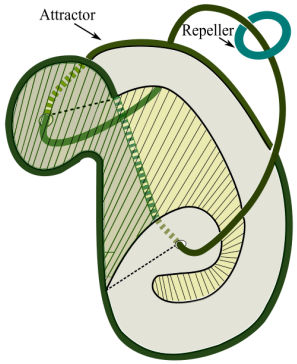


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Theorem (Franks 1981). For a Smale flow on S^3 with one attractor a, one repeller r, and one chaotic saddle set with structure matrix S, $|\mathrm{lk}(a, r)| = |\det(I - S)|$.

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Theorem (Franks 1981). For a Smale flow on S^3 with one attractor a, one repeller r, and one chaotic saddle set with structure matrix S, $|\mathrm{lk}(a, r)| = |\det(I - S)|$.

Corollary (S.) For such a flow if the the saddle set if a full n-shift, then $\mathrm{lk}(a, r)$ is odd is n is even and even if n is odd.

Proof of Corollary

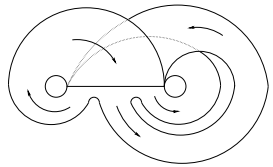
Assume a Markov partition that is fine enough that the incidence matrix A is a 0-1 matrix and that the structure matrix S is well defined. The incidence matrix is strong shift equivalent to the 1×1 matrix $[n]$. Hence,

$$\det(I - A) = \det([1 - n]) = 1 - n.$$

The structure matrix S is mod 2 equivalent to A. The result follows.

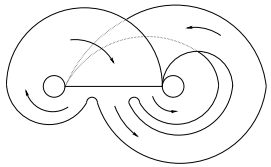
Full 3-shift

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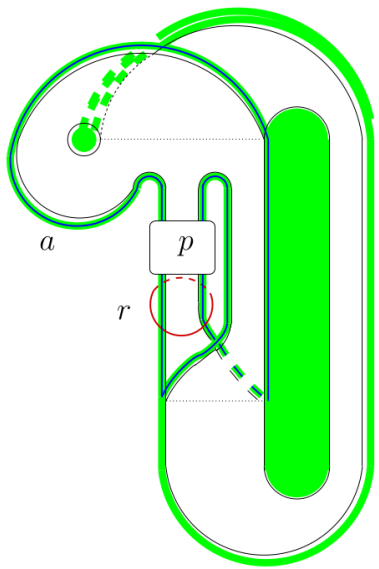
This template is model for a suspension of a full 3-shift map.

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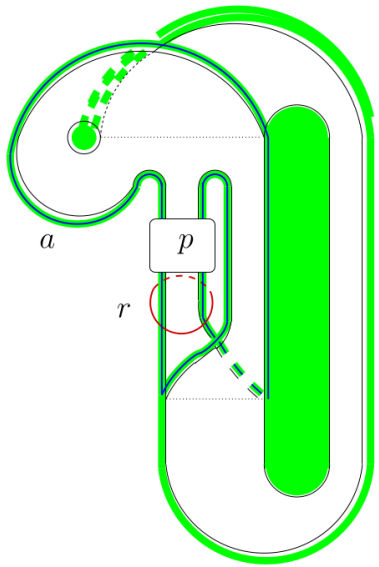
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Here are some realizations...



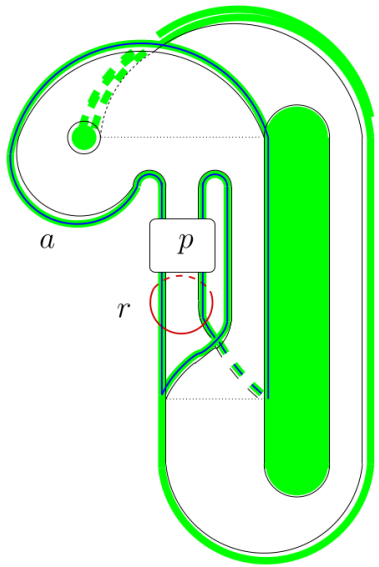
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Mohammed M. Alshamrani and myself are attempting to show these are all the realizations.

Open Problems

- Characterize which templates are universal.
- Characterize which templates are fiber only.
- Characterize links on $L(0, n > 0)$ as T-links.
- Efficient algorithms for finding knots on templates.
- Extend bounded factor theorem to all positive templates.
- Templates as attractors for other ODEs.
- Templates for Plykin-like attractors.
- Can other templates, besides $L(0, 0)$, be found in other geodesic flows?
- Do $L(-1, -1)$ knots have negative signature? What's up with $L(1, -1)$?
- Is twist-wise flow equivalence decidable?
- Are there other realizations for $L(odd, odd)$?

References

See separate file for references.