Sets and Functions

1 Sets

A set is a collection of elements. The expression \( p \in S \) means \( p \) is an element of the set \( S \). A set may be defined in several ways: in ordinary English, \( e.g. \), let \( A \) be the set of positive even integers; by listing its elements within braces, \( e.g. \), let \( A = \{2, 4, 6, 8, \ldots\} \); or by using “set builder” notation, \( e.g. \), \( A = \{n \in \mathbb{Z} \mid n > 0 \text{ and } n \text{ is even}\} \), read: \( A \) is the set of all integers \( n \) such that \( n > 0 \) and \( n \) is even (\( \mathbb{Z} \) is the standard notation for the integers).

A set does not have an order. Thus \( \{a, b\} = \{b, a\} \). An ordered set is a set together with an ordering. When we want to stress that a set has been endowed with an ordering we will use parentheses instead of braces: \( (a, b) \) is an ordered set and is not equal to \( (b, a) \).

The following notations are standard:

- \( \emptyset = \{\} \), the empty set.
- \( A \subset B \): read \( A \) is a subset of \( B \), meaning, every element of \( A \) is an element of \( B \). Example: \( \{2, 5\} \subset \{1, 2, 3, 4, 5\} \).
- \( A \cup B \): \( A \) union \( B \), meaning, the set of all elements that are in \( A \) or in \( B \). Example: \( \{\$, *, !\} \cup \{\alpha, !, *, 17\} = \{\$, *, !, \alpha, *, 17\} \).
- \( A \cap B \): read \( A \) intersection \( B \), meaning, the set of all elements that are in \( A \) and in \( B \). Example: \( \{\$, *, !\} \cap \{\alpha, !, *, 17\} = \{!\} \).
- \( A - B \): read \( A \) minus \( B \), meaning, the set of all elements of \( A \) that are not elements of \( B \). Example: \( \{\$, *, !\} - \{\alpha, !, *, 17\} = \{\$, *\} \).
- \( A \times B \): read \( A \) cross (product) \( B \), meaning, the set of ordered pairs \( (a, b) \) where \( a \in A \) and \( b \in B \). Since there is a natural one-to-one correspondence between \( (A \times B) \times C \) and \( A \times (B \times C) \), \((a, b), c) \leftrightarrow (a, (b, c))\), we shall ignore the distinction between them and use the notation \( A \times B \times C \) for the set \( \{(a, b, c) \mid a \in A, b \in B, \text{ and } c \in C\} \). Other multiple cross products are defined similarly. Examples: \( \{1, 3\} \times \{0, 1, 2\} = \{(1, 0), (1, 1), (1, 2), (3, 0), (3, 1), (3, 2)\} \). \( \{\$, \#\} \times \{\%\} = \{(\$, \%), (\#, \%)\} \).

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\begin{itemize}
    \item $A^n = A \times \cdots \times A$, $n$ times. \textit{Example:} $\{2, 3\}^3 = \{(2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3), (3, 2, 2), (3, 2, 3), (3, 3, 2), (3, 3, 3)\}$.
\end{itemize}

Some standard sets are:
\begin{itemize}
    \item $\mathbb{Z}$ : the integers (from the German \textit{zummer})
    \item $\mathbb{Q}$ : the rationals (quotients)
    \item $\mathbb{R}$ : the reals
    \item $\mathbb{C}$ : the complex numbers
\end{itemize}

\textbf{Remark.} The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are normally given an ordering. Interestingly, $\mathbb{C}$ is not typically ordered.

\textbf{Interval Notation:}
\begin{align*}
    [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \quad [a, \infty) = \{x \in \mathbb{R} \mid a \leq x\} \\
    (a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \quad (a, \infty) = \{x \in \mathbb{R} \mid a < x\} \\
    (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \quad (-\infty, b) = \{x \in \mathbb{R} \mid x < b\} \\
    [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \quad (-\infty, b) = \{x \in \mathbb{R} \mid x < b\}
\end{align*}

\textbf{Remark.} The notation \(\{(a, b)\}\) is ambiguous; it could represent an interval or an ordered pair. One has to consider the context to understand the intended meaning. On behalf of mathematicians everywhere, I apologize for any in convenience this may cause.

\textbf{Examples:}
\begin{itemize}
    \item $\{x \in \mathbb{R} \mid x \leq -\sqrt{7}\} \cup \{x \in \mathbb{R} \mid x \geq \sqrt{7}\}$ is the solution set for $x^2 - 7 \geq 0$.
    \item $\mathbb{R} - \{0\}$ is the natural domain of $1/x$.
    \item $\mathbb{R}^2$ is the plane. $\mathbb{R}^3$ is 3-dimensional space. $\mathbb{R}^4$ is 4-dimensional space. And so on.
    \item $\phi \subset A$, $\phi = A \cap \phi$, and $A = A \cup \phi$ are true statements for all sets $A$.
    \item $\{x \in \mathbb{R} \mid -2 \leq x < 5\} = [-2, 5) = [-2, 7] \cap (-10, 5)$.
    \item $S = [0, 1] \times [0, 1]$ is the \textit{unit square} in the plane $\mathbb{R}^2$ with corners $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.
\end{itemize}
Problems:

1. Describe $[0,1] \times [0,2] \times [0,3]$.

2. Simplify $((1,3) \cap (2,5)) \cup [3,4]$.

3. Let $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$, $B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$, and $C = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. Graph $A - B$, $A \cap (\mathbb{R}^2 - B)$, $A \cap C$, and $A - C$.

4. Find the solution set in $\mathbb{R}^2$ of $\sin x \cos y = 0$.

5. Draw $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{R}$, and $((0,1] \cup \{2,3\}) \times ([-2,-1] \cup (2,3))$ as subsets of $\mathbb{R}^2$.

6. Let $A$ be a set. What is $A \times \phi$?

7. [Hard] Let $A$, $B$, and $C$ be sets. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. (You can draw pictures to “see” this, but you need to reason from the definitions to prove it.)

2 Functions

Intuitively, a function $f$ from a set $A$ to a set $B$ assigns to each element of $A$ one element of $B$. Formally, $f$ is a subset of $A \times B$ such that for every $a \in A$ there is one and only one $b \in B$ with $(a, b) \in f$. We normally write $f : A \rightarrow B$, and express $(a, b) \in f$ by $b = f(a)$.

A function $f : A \rightarrow B$ is onto if for every $b \in B$ there is at least one $a \in A$ such that $(a, b) \in f$, i.e., such that $f(a) = b$. A function $f : A \rightarrow B$ is one-to-one if for every $b \in B$ there is at most one $a \in A$ with $f(a) = b$.

Let $f : A \rightarrow B$, $A' \subset A$, and $B' \subset B$. Then we define,

- $f(A') = \{b \in B \mid b = f(a) \text{ for at least one } a \in A'\}$ and is called the image of $A'$ under $f$. We call $f(A)$ the range of $f$.

- $f^{-1}(b) = \{a \in A \mid b = f(a)\}$.

- $f^{-1}(B') = \{a \in A \mid a \in f^{-1}(b) \text{ for at least one } b \in B'\}$
If $f$ is one-to-one and onto then $f^{-1}(b)$ always consists of a single element and we regard $f^{-1}$ as a function from $B$ to $A$. In this case we say $f$ is invertible.

A **binary operation** is a function from the cross product of two sets to a third set. For example, the adding of two numbers is a binary operation from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. So is multiplication. For any binary operation $f : A \times B \to C$, if $a_1 = a_2 \in A$ and $b \in B$ then $f(a_1, b) = f(a_2, b)$. For multiplication this means for real numbers $a$, $b$, and $c$, if $a = b$ then $ac = bc$. Note that we have written $f(a, b)$ instead of $f((a, b))$ since this shorthand is customary.

**Example 1.** Let $S = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit, \square, \bigcirc, \star\}$ and let $L = \{\alpha, \theta, \phi, \pi, \zeta\}$. Let $f : S \to L$ be defined as indicated by Figure 1. But what *is* $f$ really? It is the set of arrows. But each arrow is a pictorial representative of an ordered pair. Thus $(\clubsuit, \alpha) \in f$ but $(\diamondsuit, \zeta) \notin f$. Or, equivalently, $f(\clubsuit) = \alpha$ while $f(\diamondsuit) \neq \zeta$. This function is not one-to-one since, for example, $f(\clubsuit) = f(\bigcirc)$. It is not onto since there is no $x \in S$ such that $f(x) = \zeta$, that is, for every $x \in S$, $(x, \zeta) \notin f$. Or, we could say $\zeta$ is not in the range of $f$.

![Figure 1: A function](image)

If we order the elements of $S$ and $L$ then we can **graph** $f$. This is shown in Figure 2. We can see that the graph of $f$ is a subset of $S \times L$. Notice that the familiar **horizontal line test** shows that $f$ is not one-to-one, while the **vertical line test** confirms that $f$ is indeed a function.

**Additional Examples:**
Figure 2: A graph of the function in Figure 1

- Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 \). Then \( f^{-1}(4) = \{-2, 2\} \), \( f^{-1}([0, 1]) = [-1, 1] \), and \( f^{-1}([1, 9]) = [-3, -1] \cup [1, 3] \).

- Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \sin \pi x \). Then \( f^{-1}(0) = \mathbb{Z} \), and
  \[
  f^{-1}([0, 1]) = \cdots \cup [-4, -3] \cup [-2, -1] \cup [0, 1] \cup [2, 3] \cup \cdots
  \]

- Let \( A = \{1, 2, 3, \ldots\} \). Then the set \( \{(1, 2), (2, 3), (3, 4), \ldots\} \subset A \times A \), is the function \( f : A \to A \) produced by adding a one: \( f(n) = n + 1 \). It is one-to-one but not onto. But if we let \( B = A - \{1\} \) and let \( g : A \to B \) be addition by one, then \( g \) is onto.

- The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is the set \( \{(x, y) \in \mathbb{R}^2 | y = x^2\} \). Thus, you can think of the function \( f \) as the graph in the plane \( \mathbb{R}^2 \).

- Let \( A = \{2, 3\} \). Let \( f = \{(2, 3), (3, 3)\} \), \( g = \{(2, 3), (3, 2)\} \), and \( h = \{(2, 2), (2, 3)\} \). Then, \( f \) is a function from \( A \) to \( A \) that is not one-to-one or onto, \( g \) is a one-to-one onto function from \( A \) to \( A \), while \( h \) is not be a function. Check that \( g^{-1}(f(3)) = 2 \) and that \( f(g(f(x))) = g(f(g(x))) \) for all \( x \in A \).

**Problems:**

1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = x^2 + y^2 \). Draw a picture of \( f^{-1}([4, 9]) \). **Recall:** \([4, 9] \subset \mathbb{R} \) is the closed interval from 4 to 9. **Hint:** What is \( f^{-1}(4) \)?
2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sin x \cos y$. Find $f^{-1}(0)$ and $f^{-1}(1)$. Draw pictures of them.

3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x, x^2)$. Show that $f$ is one-to-one but not onto.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + y, x + y)$. Show that $f$ is neither one-to-one nor onto. Describe the range of $f$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (3x + 2y, x - y)$. Show that $f$ is one-to-one and onto. Find $f^{-1}$. What is the image of $\{(x, y) \in \mathbb{R}^2 \mid x = y\}$?