# Vector Spaces [1]

**Definition 1.** A *Vector Space* is a nonempty subset $V$ of some $\mathbb{R}^n$ such that the two conditions hold:

1. If $\mathbf{v}$ and $\mathbf{w}$ are both in $V$ then their vector sum $\mathbf{v} + \mathbf{w}$ is in $V$.

2. If $\mathbf{v} \in V$ and $r \in \mathbb{R}$ then their scalar product $r\mathbf{v}$ is in $V$.

**Example 1.** The solution set $P$ of $y = x^2$ is not a vector space. *Proof:* $(1, 1) \in P$ and $2 \in \mathbb{R}$ but $2(1, 1) = (2, 2)$ is not in $P$.

**Example 2.** The unit circle, denoted $U$, in $\mathbb{R}^2$ is not a vector space.*Proof:* The points $(1, 0)$ and $(-1, 0)$ are both in $U$, but $(1, 0) + (-1, 0) = (0, 0)$ is not in $U$.

**Example 3.** Any line in $\mathbb{R}^2$ that goes through the origin is a vector space. *Proof:* Let $L$ be the points of a line passing through $(0, 0)$. Then $L$ is either the solution set of an equation of the form $y = mx$ or $L$ is the $y$-axis. In the first case we pick any two points $(a, b)$ and $(c, d)$ in $L$ and let $r \in \mathbb{R}$. Now, $(a, b) + (c, d) = (a + c, b + d)$ and we must show that $(a + c, b + d)$ is in $L$. Since $(a, b)$ and $(c, d)$ are in $L$ we know that $b = ma$ and $d = mc$. Thus, $b + d = ma + mc = m(a + c)$, which implies $(a + c, b + d)$ is in $L$. We must also show that $r(a, b) = (ra, rb)$ is in $L$. But, since $rb = r(ma) = m(ra)$ we get that $(ra, rb) \in L$. In the case that $L$ is the $y$-axis, we know that $b = d = 0$. Thus $(a, b) + (c, d) = (a, 0) + (c, 0) = (a + c, 0)$ which is on the $y$-axis. Lastly, $r(a, b) = r(a, 0) = (ra, 0)$ is also on the $y$-axis. This completes the proof. (Vector Spaces are sometimes called *Linear Spaces*).

**Problem 1.** Show that the solution set of $y = 2x + 1$ fails to be a vector space.

**Fact 1.** Every vector space contains the origin. *Proof:* Let $V$ be a vector space. Since a vector space is nonempty we can pick a $\mathbf{v} \in V$. Then $0\mathbf{v} = \mathbf{0}$, so the origin, $\mathbf{0}$, is in $V$.

**Example 4.** The subset containing just the origin, $\{\mathbf{0}\} = \{(0, \ldots, 0)\} \subset \mathbb{R}^n$, is a vector space. Check this.

**Problem 2.** Let $A$ be a given $m\mathbf{x}n$ matrix. Show that the solution set of $A\mathbf{x} = \mathbf{0}$ is a vector space, but that the solution set of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \neq \mathbf{0}$ is not. (Remember, the empty set is not a vector space.)

**Problem 3.** Use Problem 2 to show that any line passing through the origin of $\mathbb{R}^3$ is a vector space.

**Problem 4.** Use Problem 2 to show that any plane passing through the origin of $\mathbb{R}^3$ is a vector space.

**Problem 5.** What are all of the vector spaces in $\mathbb{R}^3$? What about $\mathbb{R}^n$? (You do not have to prove your claims.)

**Example 5.** Let $\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^n$. Let $W = \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \,|\, r_1, r_2 \in \mathbb{R}\}$. Then $W$ is a vector space. ($W$ is called the *span* of $\{\mathbf{v}_1, \mathbf{v}_2\}$.) *Proof:* Every element of $W$ is a vector in $R^n$. Let $\mathbf{u}$ and $\mathbf{v}$ be in $W$ and let $r$ be any real number. We can write $u = a\mathbf{v}_1 + b\mathbf{v}_2$ and $v = c\mathbf{v}_1 + d\mathbf{v}_2$. Then $r\mathbf{u} = ra\mathbf{v}_1 + rb\mathbf{v}_2$ which is in $W$ since $ra$ and $rb$ are real numbers. Also, $\mathbf{u} + \mathbf{v} = (a + c)\mathbf{v}_1 + (b + d)\mathbf{v}_2$ is in $W$. Thus, $W$ is a vector space.

**Example 6.** A set of matrices can sometimes be thought of as a vector space. For example the set of $3 \times 3$ matrices is really just $\mathbb{R}^9$, since matrix addition behaves like vector addition and multiplying a scalar and a matrix is just like multiplying a scalar and a vector.

**Example 7.** Now consider the set of $3 \times 3$ symmetric matrices. The sum of two symmetric matrices is still symmetric and scalar multiplication also preserves the symmetry. We can think of this set as a 6 dimensional vector space sitting inside of $\mathbb{R}^9$. Why?

## Vector-like Spaces

It often happens that a set that is not a subset of any $\mathbb{R}^n$ still has properties very much like those of a vector space.

**Example 8.** Adding polynomials is a lot like adding vectors and we can think of multiplying a polynomial by a real constant as an analog of scalar multiplication. Let $C$ be the set of cubic polynomials and let $P_3$ be the set of all polynomials of degree 3 or less. If $p(x) = x^3 + 2$ and $q(x) = x - x^3$ then $p(x) + q(x)$ is not a cubic polynomial. Thus, $C$ does not behave like a vector space. But $P_3$ does act very much like a vector space. It is "closed" under addition and multiplication by scalars. (Although, $P_3$ is not "closed" when we multiply one polynomial by another.)

**Example 9.** The solution set $S$ of the differential equation $y'' = -y$, that is the set of all real functions $f(x)$ such that $f''(x) = -f(x)$, is a like a vector space. If $f$ and $g$ are both in $S$ and $r \in \mathbb{R}$ then the reader should be able to check that $f(x) + g(x)$ and $rf(x)$ are also in $S$. Thus, the algebraic structure of $S$ is much like that of a vector space. We challenge the reader to figure out what the set $S$ is.

**Example 10.** The set of convergent infinite series can be thought as vector space. Suppose $r \in \mathbb{R}$ and that $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$ both converge to finite limits. Then, it can be shown that $\Sigma_{i=1}^{\infty} a_i + b_i$ and $\Sigma_{i=1}^{\infty} r a_i$ both converge to finite limits.

**Definition 2.** An *Vector-like Space* is any set $V$ together with two operations called *vector addition* and *scalar multiplication* that obey the following axioms.

I. Closure axioms:

  a. Vector addition is closed: $\mathbf{v} + \mathbf{w} \in V$ for all $\mathbf{v}$ and $\mathbf{w}$ in $V$.

  b. Scalar multiplication is closed: $r\mathbf{v} \in V$ for all $\mathbf{v} \in V$ and $r \in \mathbb{R}$.

II. Algebraic axioms:

  a. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, $\forall\, \mathbf{v}, \mathbf{w} \in V$ (commutativity)

  b. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$, $\forall\, \mathbf{v}, \mathbf{w}, \mathbf{u} \in V$ (associativity)

  c. $\exists\, \mathbf{z} \in V$ such that $\mathbf{z} + \mathbf{v} = \mathbf{v}$, $\forall\, \mathbf{v} \in V$ (additive identity: zero)

  d. $\forall\, \mathbf{v} \in V$, $\exists\, \bar{\mathbf{v}} \in V$ such that $\bar{\mathbf{v}} + \mathbf{v} = \mathbf{z}$ (additive inverses)

  e. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$, $\forall r \in \mathbb{R}, \mathbf{v}, \mathbf{w} \in V$ (distributivity)

  f. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$, $\forall\, r, s \in \mathbb{R}, \mathbf{v} \in V$ (distributivity)

  g. $r(s\mathbf{v}) = (rs)\mathbf{v}$, $\forall\, r, s \in \mathbb{R}, \mathbf{v} \in V$ (associativity of scalar multiplication)

  h. $1\mathbf{v} = \mathbf{v}$, $\forall\, \mathbf{v} \in V$ (scalar multiplicative identity)

Vector-like spaces are often referred to simply as vector spaces when no confusion can arise.

**Some Quick Facts:** The following properties will be derived for vector spaces form the above axioms and the axioms of arithmetic:

a. Zero elements are unique. *Proof:* Suppose $\mathbf{z}$ and $\mathbf{u}$ are both zero elements of a vector space. Then $\mathbf{u} = \mathbf{z} + \mathbf{u} = \mathbf{u} + \mathbf{z} = \mathbf{z}$. So, $\mathbf{u}$ and $\mathbf{z}$ must be the same.

⋄ From now on we shall denote the zero element by $\mathbf{0}$.

b. For any $\mathbf{v}$, $0\mathbf{v}$ is the zero element. *Proof:* $\mathbf{v} = 1\mathbf{v} = (1+0)\mathbf{v} = 1\mathbf{v} + 0\mathbf{v} = \mathbf{v} + 0\mathbf{v}$. Now $\mathbf{v}$ has an inverse $\bar{\mathbf{v}}$. Add this to both sides: $\mathbf{v} + \bar{\mathbf{v}} = \mathbf{v} + \bar{\mathbf{v}} + 0\mathbf{v}$. This gives $\mathbf{0} = \mathbf{0} + 0\mathbf{v}$, which implies $\mathbf{0} = 0\mathbf{v}$.

c. For any $r \in \mathbb{R}$, $r\mathbf{0} = \mathbf{0}$. *Proof:* Exercise!

d. If $r\mathbf{v} = \mathbf{0}$, then either $r = 0$ or $\mathbf{v} = \mathbf{0}$. *Proof:* Exercise!

e. Inverses are unique. *Proof:* Let $\mathbf{v}$ be in a vector space. Suppose that $\bar{\mathbf{v}}$ and $\tilde{\mathbf{v}}$ are both inverses of $\mathbf{v}$. Then $\bar{\mathbf{v}} = \bar{\mathbf{v}} + \mathbf{0} = \bar{\mathbf{v}} + \mathbf{v} + \tilde{\mathbf{v}} = \mathbf{v} + \bar{\mathbf{v}} + \tilde{\mathbf{v}} = \mathbf{0} + \tilde{\mathbf{v}} = \tilde{\mathbf{v}}$. So, $\bar{\mathbf{v}}$ and $\tilde{\mathbf{v}}$ are the same.

f. For any $\mathbf{v}$, $-1 \cdot \mathbf{v} = \bar{\mathbf{v}}$. *Proof:* $\mathbf{v} + -1\mathbf{v} = 1\mathbf{v} + -1\mathbf{v} = (1-1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$. Thus, $-1\mathbf{v} = \bar{\mathbf{v}}$ by uniqueness.

⋄ From now on we shall denote the inverse of $\mathbf{v}$ by $-\mathbf{v}$.

The foregoing may have struck the reader as unnecessary. Are not these results obvious? The answer is that they are not. Further, the mental discipline needed to construct the proofs is of value in itself. Thus, you may be tested on these proofs. It may also seem that the last axiom in Definition 2, IIh, is obvious. But in fact it does not follow from the previous axioms as you will show in Problem 8 below.

**Example 11.** The set $P_3$ of polynomials of degree three or less, is a vector space. We shall check each axiom of Definition 2.

Ia. $(ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + h) = (a+e)x^3 + (b+f)x^2 + (c+g)x + (d+h) \in C$

Ib. $r(ax^3 + bx^2 + cx + d) = rax^3 + rbx^2 + rcx + rd \in C$

IIa. $(ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + h) = (a+e)x^3 + (b+f)x^2 + (c+g)x + (d+h) = (e+a)x^3 + (f+b)x^2 + (g+c)x + (h+d) = (ex^3 + fx^2 + gx + h) + (ax^3 + bx^2 + cx + d)$

IIb. $[(ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + h)] + (ix^3 + jx^2 + kx + l) = ([a+e]+i)x^3 + ([b+e]+j)x^2 + ([c+g]+k)x + ([d+h]+l) = (a+[e+i])x^3 + (b+[e+j])x^2 + (c+[g+k])x + (d+[h+l]) = (ax^3 + bx^2 + cx + d) + [(ex^3 + fx^2 + gx + h) + (ix^3 + jx^2 + kx + l)]$

IIc. Use $0 = 0x^3 + 0x^2 + 0x + 0 \in C$ as the zero elemenmt.

IId. Given $\mathbf{v} \in C$ use $-1$ times $\mathbf{v}$ as the inverse.

IIe. $r[(ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + h)] = r[(a+e)x^3 + (b+f)x^2 + (c+g)x + (d+h)] = r(a+e)x^3 + r(b+f)x^2 + r(c+g)x + r(d+h) = (ra+re)x^3 + (rb+rf)x^2 + (rc+rg)x + (rd+rh) = (rax^3 + rbx^2 + rcx + rd) + (rex^3 + rfx^2 + rgx + rh) = r(ax^3 + bx^2 + cx + d) + r(ex^3 + fx^2 + gx + h)$

IIf. $(r+s)(ax^3+bx^2+cx+d) = (r+s)ax^3+(r+s)bx^2+(r+s)cx+(r+s)d = (ra+sa)x^3+(rb+sb)x^2+(rc+sc)x+(rd+sd) = (rax^3+rbx^2+rcx+rd)+(sax^3+sbx^2+scx+sd) = r(ax^3+bx^2+cx+d)+s(ax^3+bx^2+cx+d)$

IIg. $r(s(ax^3+bx^2+cx+d)) = r(sax^3+sbx^2+scx+sd) = r(sa)x^3+r(sb)x^2+r(sc)x+r(sd) = (rs)ax^3+(rs)bx^2+(rs)cx+(rs)d = (rs)(ax^3+bx^2+cx+d)$

IIh. $1(ax^3+bx^2+cx+d) = ax^3+bx^2+cx+d$

**Problem 6.** Prove that the sets in examples 9 and 10 are vector spaces.

**Problem 7.** Let $Z$ be the set of continuous functions that are zero at zero. Let $T$ be the set of continuous functions that are two at zero. It each case we define vector addition and scalar multiplication in the obvious way. Convince yourself that $Z$ is indeed a vector space. However, $T$ is not a vector space. For example, $T$ is not closed under vector addition or scalar multiplication. For each axiom of Definition 2 either give an example showing it fails for $T$ or prove that it does hold for $T$. (Note: To prove $T$ is not a vector space it is enough to find one example where one axiom fails.)

**Problem 8.** Consider the set $\mathbb{R}^2$ with the usual vector addition but with scalar multiplication defined as follows: $r(x, y) = (rx, 0)$. Show that axioms Ia-b and IIa-g hold but that IIh is false.

**Example 12.** Which of the following do you think are vector spaces?

a. The set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

b. Continuous functions.

c. Polynomials.

d. The set $E$ of even functions.

e. The set $O$ of odd functions.

f. The set of functions that are even or odd, that is $E \cup O$.

g. The set of functions that are even and odd, that is $E \cap O$. What is this set?

h. Positive functions.

i. Functions with integral zero over some fixed interval.

j. Functions with integral one over some fixed interval.

k. All solutions to $y'' + 3xy' + 4y = 0$.

l. All solutions to $y'' = x^2$.

## Subspaces

**Definition 3.** If $V$ is a subset of a vector space $W$ and $V$ also has a vector space structure (using the same operations as on $W$) then we say $V$ is a *subspace* of $W$.

**Theorem 1.** If $V$ is a subset of a vector space $W$ then $V$ is a subspace of $W$ if the two closure axioms of Definition 2 hold.

Before giving a proof we shall do an application.

**Example 13.** The set $P_3$ of polynomials of degree three or less is a subset of the set of functions, which is a vector space. Thus, we only needed to check axioms Ia and Ib.

*Proof.* Let $V \subset W$, where $W$ is a vector space. Assume axioms Ia and Ib hold for $V$. Thus, we need to check axioms IIa-h.

IIa. Since IIa holds for all vectors in $W$ in holds for all vectors in $V$.

IIb. The same argument works.

IIc. Take any element $\mathbf{v} \in V$. Then by Ib $0\mathbf{v} \in V$. But, $\mathbf{0} = 0\mathbf{v}$, and $\mathbf{0} + \mathbf{v} = \mathbf{v}$ in $W$ so it holds in any subset.

IId. Similar to IIc.

IIe. Same as IIa.

IIf. Same as IIa.

IIg. Same as IIa.

IIh. Same as IIa.

□

**Problem 9.** For each set listed in Example 12 prove whether or not it is a vector space.