Lines and Planes

1 Lines in the Plane

Every line of points $L$ in $\mathbb{R}^2$ can be expressed as the solution set for an equation of the form $Ax + By = C$. The equation is not unique for if we multiply both sides by any nonzero number the solution set is unchanged. Any line $L$ can also be expressed by a pair of parametric equations of the form:

$$
\begin{align*}
    x(t) &= at + b \\
    y(t) &= ct + d
\end{align*}
$$

These can be rewritten in vector form: $\langle x, y \rangle = \langle a, c \rangle t + \langle b, d \rangle$. The vectors $\langle a, c \rangle$ and $\langle b, d \rangle$ have a nice geometric/physical interpretation. Regard $t$ as time. Let $p(t) = \langle x(t), y(t) \rangle$ and call it the position vector. One can imagine a particle moving along $L$ in accordance with the given parametric equations. Then $p(0) = \langle b, d \rangle$ is the initial position. Notice,

$$
\frac{dp}{dt} = \langle x'(t), y'(t) \rangle = \langle a, c \rangle
$$

Thus, we call $\mathbf{v} = \langle a, c \rangle$ the velocity vector. It is parallel to $L$. It is customary to place its base point on $L$. See Figure 1(left side).

We now give a geometric interpretation for the “ABC” form of an equation of a line. First, suppose $C = 0$; this just means the line $L$ goes through the origin. Let $\mathbf{n} = \langle A, B \rangle$, and again set $\mathbf{p} = \langle x, y \rangle$. Then we have $\mathbf{n} \cdot \mathbf{p} = 0$. That is the vectors $\mathbf{n}$ and $\mathbf{p}$ are at right angles to each other. Thus, the line $L$ for $Ax + By = 0$ is the set of all points $(x, y)$ such that $\langle x, y \rangle$ is perpendicular to $\langle A, B \rangle$.

Now we consider the general case: $Ax + By = C$. Pick some particular point on the line and call it $(x_0, y_0)$. Then $C = Ax_0 + By_0$. Therefore, for

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any point \((x, y)\) on \(L\) we have \(Ax + By = Ax_0 + By_0\). We can rewrite this as

\[
\begin{align*}
Ax - Ax_0 + By - By_0 &= 0 \\
A(x - x_0) + B(y - y_0) &= 0 \\
\langle A, B \rangle \cdot \langle x - x_0, y - y_0 \rangle &= 0 \\
\mathbf{n} \cdot \langle x, y \rangle &= 0 \\
\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) &= 0
\end{align*}
\]

In the last line we have let \(\mathbf{p}_0 = \langle x_0, y_0 \rangle\). The vector \(\mathbf{p} - \mathbf{p}_0\) can be thought of as lying in \(L\) with its tail at \((x_0, y_0)\) and its head at \((x, y)\).

Thus, \(L\) is the unique line perpendicular to the vector \(\mathbf{n} = \langle A, B \rangle\) that passes through \((x_0, y_0)\). See Figure 1(right side). The vector \(\mathbf{n}\) is called a normal vector for the line \(L\). Given a vector to use as normal vector and a point we can easily find an equation for the corresponding line.

Figure 1: Left: A parametric line. Right: Normal vector to a line.

**Problem 1.** Consider the line determined by \(x(t) = 3t - 2\) and \(y(t) = -t + 7\). Find an equation for the line in ABC form.

**Problem 2.** Consider the line determined by \(4x - 7y = 2\). Find a pair of parametric equations for this line.

Note: Both of these problems have many correct answers.
2 Lines and Planes in 3-space

The three dimensional set $\mathbb{R}^3$ is the set of all triples $(x, y, z)$ where $x$, $y$, and $z$ are real numbers. Such a triple is called the $xyz$-coordinates of a point. These are also called rectilinear coordinates. The set $\{(x, 0, 0) \mid x \in \mathbb{R}\}$ is the $x$-axis. The $y$ and $z$ axes are defined similarly. They are clearly lines. The set $\{(x, y, 0) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$ is the $xy$-plane. The $yz$ and $xz$ planes are defined similarly. Visualizing structures in three dimensions takes practice.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{3D-space.png}
\caption{Three dimensional space: $\mathbb{R}^3$}
\end{figure}

Any line $L$ in $\mathbb{R}^3$ can expressed parametrically in the form:

\begin{align*}
x(t) &= at + b \\
y(t) &= ct + d \\
z(t) &= et + f
\end{align*}

or, in vector form, $\langle x, y, z \rangle = \langle a, c, e \rangle t + \langle b, d, f \rangle$. As with lines in $\mathbb{R}^2$ it is useful to think of $\langle a, c, e \rangle$ as a velocity vector and $\langle b, d, f \rangle$ as the position at $t = 0$.

However, there is no way to express a line in $\mathbb{R}^3$ as a single equation in three variables. In fact, we will show that “typically” solution sets of equations of the form $Ax + By + Cz = D$ are planes and that every plane in $\mathbb{R}^3$ is the solution set of some equation in this form. Note: If $A = B = C = D = 0$, the solution set is all of $\mathbb{R}^3$; if $A = B = C = 0$ but $D \neq 0$ the solution set is empty.
Example 1. Convince yourself of the following:

- If $A = B = D = 0$ and $C \neq 0$ then $Ax + By + Cz = D$ is the $xy$-plane.
- If $A = C = D = 0$ and $B \neq 0$ then $Ax + By + Cz = D$ is the $xz$-plane.
- If $B = C = D = 0$ and $A \neq 0$ then $Ax + By + Cz = D$ is the $yz$-plane.

Let’s consider the case where $D = 0$. Let $n = \langle A, B, C \rangle$ and $p = \langle x, y, z \rangle$. Then the equation $Ax + By + Cz = 0$ becomes $n \cdot p = 0$. Thus, the solution set is the plane $P$, passing through the origin of $\mathbb{R}^3$ whose points, when regarded as vectors, are perpendicular to $n$.

We return to the general case: $Ax + By + Cz = D$. Let $p_0 = \langle x_0, y_0, z_0 \rangle$ be some fixed point that satisfies the given equation. We leave it to the reader to show that

$$n \cdot (p - p_0) = 0.$$  

Thus, the solution set of $Ax + By + Cz = D$ is the unique plane passing through $p_0$ and perpendicular to $n = \langle A, B, C \rangle$.

Example 2. Consider $x + y + z = 1$. The points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ satisfy the equation. We can connect them with line segments and visualize the triangle thus formed. This triangle sits in the plane. If we place the tail of $n = \langle 1, 1, 1 \rangle$ at any point of the triangle it is easy to see that it is perpendicular to the plane.

Example 3. Let $P_1$ be the plane given by $2x + 3y - z = 5$ and let $P_2$ be the plane given by $x + y + z = 1$. Find parametric equations for the line $L = P_1 \cap P_2$.

Solution.

\[
\begin{align*}
2x + 3y - z &= 4 \\
x + y + z &= 1
\end{align*}
\]  

$\implies y - 3z = 2$.

Let $z = t$. Then $y = 3t + 2$. Also, $x = 1 - y - z = -4t - 1$. We rewrite these as $\langle x, y, z \rangle = \langle -4, 3, 1 \rangle t + \langle -1, 2, 0 \rangle$.

Example 4. Find an equation for the plane passing through the three points $(1, 1, 1)$, $(1, 2, 3)$, and $(2, -1, 0)$.
Solution. We have three conditions and these give us three equations in four unknowns.

\[
\begin{align*}
A + B + C &= D \\
A + 2B + 3C &= D \\
2A - B &= D
\end{align*}
\implies
\begin{align*}
A + B + C &= D \\
B + 2C &= 0 \\
-3B - 2C &= -D
\end{align*}
\implies
\begin{align*}
A + B + C &= D \\
B + 2C &= 0 \\
C &= -D/4
\end{align*}
\implies
\begin{align*}
A &= 5D/4 \\
B &= D/2 \\
C &= -D/4
\end{align*}

Any nonzero value of \( D \) will do. Let \( D = 4 \). Then \( 3x + 2y - z = 4 \) is an equation for our plane.

Problem 1. Consider the three points \((1, 1, 1)\), \((2, 0, 2)\), and \((4, -1, 4)\). Show that they do not determine a unique plane because they lie on the same line. Find an equation for this line; write it in vector form.

Problem 2. Let \( P \) be the plane given by \( x + 2y - 3z = 1 \). Let \( L_{xy} \) be the intersection of \( P \) with the \( xy \)-plane. Define \( L_{xz} \) and \( L_{yz} \) similarly. Find equations for these three lines in “ABC” form.

Problem 3. Graph, separately, each of the planes determined by these three equations: \( 2x + 2y - 3z = 1 \), \( x + 2y + 4z = -1 \), and \( 3x - 2y - 2z = 7 \).

Problem 4. Find the point of intersection of the three planes determined by these three equations: \( 2x+2y-3z = 1 \), \( x+2y+4z = -1 \), and \( 3x-2y-2z = 7 \).

Problem 5. Show that the two planes determined by \( 2x + 2y - 3z = 1 \) and \( 4x + 4y - 6z = 0 \) do not intersect and are thus parallel.

Problem 6. Let \( P \) be the plane given by \( 2x + 3y - 2z = 1 \). Let \( L \) be the line given by \( \langle x, y, z \rangle = \langle 1, 1, 1 \rangle t + \langle 1, 0, 1 \rangle \). Find the point where they meet.

Problem 7. Show that these four points lie in the same plane: \((1, 1, -1)\), \((-1, 0, 0)\), \((-1, 1, -\frac{1}{2})\), and \((1, -1, 0)\). Find an equation for this plane.
3 Parametric Equation for a Plane

There is another form for equations of planes in $\mathbb{R}^3$ that is the analog of the parametric form for equations of a line. The difference is we will need two parameters, $r$ and $s$, instead of one. Of course, the time metaphor is no longer useful.

Let $P$ be a plane given by $Ax + By + Cz = D$. Assume that $C \neq 0$. Then we can solve for $z$ and get $z = D/C - A/Cx - B/Cy$. (If $C = 0$ solve for $x$ or $y$ instead.) Think of $z$ as the height above the $xy$-plane. Now let $x = r$ and $y = s$, and think of $r$ and $s$ as free parameters. We can now write

$$\langle x, y, z \rangle = \langle r, s, D/C - A/Cr - B/Cs \rangle$$

$$= \langle 0, 0, D/C \rangle + r \langle 1, 0, -A/C \rangle + s \langle 0, 1, -B/C \rangle$$

This equation is far from unique. We can start with any point $(x_0, y_0, z_0) \in P$, regard it as a vector $p_0 = \langle x_0, y_0, z_0 \rangle$ and add multiplies of $\langle 1, 0, -A/C \rangle$ and $\langle 0, 1, -B/C \rangle$ to it and stay in the plane. Furthermore, if we let $v_1$ and $v_2$ be nonzero multiplies of $\langle 1, 0, -A/C \rangle$ and $\langle 0, 1, -B/C \rangle$, respectively then

$$p = p_0 + rv_1 + sv_2$$

gives the same plane $P$. Indeed, we could use any pair of vectors in $P$ with tails at $p_0$ as long as they point in different directions.

We will use this formulation to place a coordinate system on $P$. Take a point $(x_0, y_0, z_0)$ on $P$ and call it the origin of $P$. Then any point on $P$ can be gotten to by adding multiplies of $v_1$ and $v_2$ to $p_0$. Thus, for any point on $P$ we can think of it as having coordinates $(r, s)$. See Figure 3.

**Example 1.** Define a plane $P$ by

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + r \langle 1, 1, 0 \rangle + s \langle 0, 1, 1 \rangle$$

Show that the point $(0,2,4)$ is on $P$ and find its $rs$-coordinates.

**Solution.** We have three equations and two unknowns.

$$\begin{align*}
0 &= 1 + 1r + 0s \\
2 &= 2 + 1r + 1s \\
4 &= 3 + 0r + 1s
\end{align*}$$

$$\implies \begin{cases} r = -1 \\ s = 1 \end{cases}$$

Thus, $(0,2,4) \in P$ and it has $rs$-coordinates $(-1,1)$ relative to the given parametric equation. \qed
Problem 1. Using the same plane $P$ in Example 1, find the $rs$-coordinates of $(3, 3, 2)$.

Problem 2. Show that the point $(1, 2, -1)$ is not on the plane $P$ of Example 1.

Problem 3 (Hard). The equation $2r + 3s = 1$ determines a line $L$ in the plane $P$ of Example 1, using $rs$-coordinates. Find a parametric equation for $L$ in $xyz$-coordinates.

Figure 3: Coordinates for a plane: The dark lines are the $r$ and $s$-axes