The Principle of Mathematical Induction

The Principle of Mathematical Induction (PMI) is just the following observation. Let \( P(n) \) be a statement for each positive integer \( n \). If \( P(1) \) is true and if \( P(k) \implies P(k+1) \) for all positive integers \( k \), then \( P(n) \) is true for all positive integers \( n \). In other words, if \( P(1) \) and \( P(k) \implies P(k+1) \) then \( P(1) \implies P(2) \implies P(3) \implies P(4) \implies \cdots \).

We will give three examples of proofs that use the Principle of Mathematical Induction.

Example 1 (The power rule). We will take the the product rule for derivatives as given: \((fg)' = f'g + fg'\). Also assume that \( x' = 1 \) is given. We will prove that \((x^n)' = nx^{n-1}\) for all positive integers \( n \). But first we will check the power rule for some small values of \( n \).

\[
\begin{align*}
(x^1)' &= x' = 1 = 1x^0 \\
(x^2)' &= (xx)' = x'x + xx' = 1x + x1 = 2x = 2x^1 \\
(x^3)' &= (x^2x)' = (x^2)'x + x^2x' = 2xx + x^2 = 3x^2 \\
(x^4)' &= (x^3x)' = (x^3)'x + x^3x' = 3x^2x + x^3 = 4x^3
\end{align*}
\]

So far so good. But this could take a very long time. To get around the fact that we cannot possibly check a statement for every positive integer mathematicians invented the PMI. Let \( n = k \) be a fixed but arbitrary positive integer and suppose that \((x^k)' = kx^{k-1}\) is true. Then let \( n = k + 1 \) and compute:

\[
(x^{k+1})' = (x^kx)' = (x^k)'x + x^kx' = kx^{k-1}x + x^k = (k+1)x^k
\]

Thus, the conditions for the PMI are in place. The power rule is true for \( n = 1 \) and if it is true for \( n = k \) it is true for \( n = k + 1 \). Thus, by the PMI the power rule \((x^n)' = nx^{n-1}\) is true for all positive integers \( n \).

Example 2. Prove that

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]
for all positive integers \( n \). \textit{Proof:} The formula works for \( n = 1 \) since it reduces to \( 1 = 1 \). Suppose that for an arbitrary fixed positive integer \( n = k \) it is true that
\[
\sum_{i=1}^{k} i = \frac{k(k+1)}{2}
\]
Let \( n = k + 1 \). Now
\[
\sum_{i=1}^{n} i = \sum_{i=1}^{k+1} i = \left( \sum_{i=1}^{k} i \right) + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{n(n+1)}{2}.
\]
Thus, by PMI the hypothesized summation formula holds true for all positive integers \( n \).

\textbf{Example 3.} Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f(ab) = f(a)f(b) \). Then \( f(a^n) = (f(a))^n \). \textit{Proof:} The equation is true for \( n = 1 \) since it reduces to \( f(a) = f(a) \). Suppose that for an arbitrary fixed positive integer \( n = k \) it is true that \( f(a^k) = (f(a))^k \). Then if we let \( n = k + 1 \) and \( b = a^k \) we get \( f(a^{k+1}) = f(aa^k) = f(a)f(a^k) = f(a)(f(a))^k = (f(a))^{k+1} \). Thus, by the PMI, \( f(ab) = f(a)f(b) \) implies \( f(a^n) = (f(a))^n \) for all positive integers \( n \).

\textbf{Problem 1.} Let \( (\frac{1}{x})' = -\frac{1}{x^2} \) and \( (fg)' = f'g + fg' \) be given. Prove that \( \left( \frac{1}{x^n} \right)' = -\frac{n}{x^{n+1}} \), for all positive integers \( n \).

\textbf{Problem 2.} Prove that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \), for all positive integers \( n \).
Problem 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f(ab) = f(a) + f(b) \).
Prove that \( f(a^n) = nf(a) \) for all positive integers \( n \).

Problem 4. Prove that \( 7^n + 2 \) is divisible by 3 for all integers \( n \geq 0 \).

Problem 5. Consider a list of \( n \) symbols where \( n \geq 2 \). We wish to switch
the first symbol with last symbol leaving the others unchanged. But, we can
only switch two adjacent symbols at a time; call this operation an adjacent
switch move. Prove that the number of adjacent switch moves needed to
switch the first and last symbol, leaving the others unchanged, is always an
odd number. We will use this result later.

Problem 6. Prove that the sum of the first \( n \) odd positive integers is \( n^2 \).
(For example, \( 1 + 3 + 5 + 7 = 16 \).)

Problem 7. Prove that if \( n \) is a positive integer, then \( x - y \) divides \( x^n - y^n \).

Problem 8. The goal of this problem is to prove that \( 2^n \) is greater than \( n^2 \)
for all integers \( n \) greater than or equal to five. We have broken the proof into
three steps.
(a) Prove that \( 4n + 2 > 2n + 3 \) for \( n \geq 1 \).
(b) Prove that \( 2^n > 2n + 1 \) for all \( n \geq 3 \).
(c) Prove that \( 2^n > n^2 \) for all \( n \geq 5 \).

Problem 9. a. Prove that \( \sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r - 1} \), for all integers \( n \geq 1 \)

b. Suppose that \( -1 < r < 1 \). What is \( \lim_{n \to \infty} \sum_{i=0}^{n} ar^i \)?

c. Compute \( \sum_{i=0}^{7} (1/3)^i \). d. Compute \( \sum_{i=0}^{\infty} (1/3)^i \).

Problem 10. Prove that \( \cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n \) for all
integers \( n \geq 0 \), where \( i \) is the imaginary unit, \( i^2 = -1 \). What happens when
\( n \) is a negative integer?

Problem 11. Find a formula for \( \cos\left(\frac{\pi}{2^n}\right) \) for all integers \( n \geq 2 \). Use
induction to prove it is correct.

Problem 12. Prove that for all positive integers \( n \), \( \cos nx = p_n(\cos x) \) for
some polynomial \( p_n \) of degree \( n \).

Problem 13. Prove that \( \int_{0}^{\infty} x^n e^{-x} \, dx = n! \) for all integers \( n \geq 0 \).