Part I: NO CALCULATORS

1. [20 points] Match the differential equation with its direction field. (You get 4 points for each correct match, -1 for each wrong match.)

   (1) $y' = y - x$, (2) $y' = 2 - y$, (3) $y' = |y - 2|$, (4) $y' = y + x$, (5) $y' = x - y$. 
2. [20 points] Solve the initial value problem, \((x^2 + 1)y' + 3xy = 6x\), with initial condition \(y(0) = 2\).

3. [20 points] Consider the equation
   \[ y' = \frac{y}{x - 1}. \]
   Find the general solution. Draw the integral “curves” for the following initial values: \(y(0) = \pm 2, \pm 1, 0\), and \(y(2) = \pm 2, \pm 1, 0\). Hint: be careful with your absolute value signs.

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Part II: CALCULATORS ALLOWED

4. [20 points] A body of mass \(m\) falls from rest in a medium offering resistance proportional to the square of the velocity. Find the relation between the velocity \(v\) and the time \(t\). Find the limiting velocity, \(v_l\).
   
   Hint:
   \[
   \int \frac{1}{a^2 - b^2 x^2} \, dx = \frac{1}{2ab} \ln \left| \frac{a + bx}{a - bx} \right| + C,
   \]
   where \(a\) and \(b\) are positive constants.

5. [20 points] Find the general solution of \(y'' + 2y' + y = \cos(\alpha x)\).

6. [20 points] Suppose that \(y = f(x)\) and \(y = g(x)\) are linearly independent solutions of \(y'' + p(x)y' + q(x)y = 0\). Suppose further that it is known that the Wronskian of \(f\) and \(g\) is 1 for all values of \(x\). Find \(p(x)\). Hint: Use Abel’s formula.

7. [20 points]
   a. A 20g mass stretches a spring 5 cm. Find the spring constant \(K\), in g/sec^2. \([g = 980 \text{ cm/sec}^2]\)
   b. Let \(\gamma = 40\) dyne-sec/cm be the damping constant. We pull the mass down 2 cm more and then let go (\(u(0) = 2, u'(0) = 0\)). Find \(u(t)\).
   c. Graph \(u(t)\). About how many oscillations will there be until the amplitude is below .5 cm?

8. [20 points] Consider the 3rd order differential equation, \(y'''' + (x+1)y'' + (\sin x)y' + y = 0\), with initial conditions \(y(0) = 0, y'(0) = 1,\) and \(y''(0) = 2\). Apply the series method and find the first 5 terms of the Taylor series for \(y(x)\), centered about zero.

9. [20 points] Consider a metal rod, 1 foot long. Let the initial temperature distribution be given by \(f(x) = 0\). Now suppose the ends are somehow set to be
   \[ u(0, t) = 10^\circ \text{ and } u(1, t) = 20^\circ, \]
   for \(t > 0\). Write down all of the integrals you would need to solve this problem AND show how you would put the results together to obtain \(u(x, t)\). DO NOT EVALUATE ANY OF THE INTEGRALS!
10. [20 points] Let

\[ f(x) = \begin{cases} 
1 & x \in [0, 1] \\
x & x \in [1, 2] \\
2 & x \in [2, 3] 
\end{cases} \]

Graph the even and odd extensions of \( f(x) \). Find the first two terms in of the Fourier series of the even extension.

Hints:

\[
\int x \cos(ax) \, dx = \frac{\cos(ax) + ax \sin(ax)}{a^2} + C
\]

\[
\int x \sin(ax) \, dx = \frac{\sin(ax) - ax \cos(ax)}{a^2} + C
\]