The wave equation with damping is $a^2u_{xx} = u_{tt} + \gamma u_t$. The boundary conditions are still $u(0, t) = u(L, t) = 0$ for all times $t$. The initial conditions are $u(x, 0) = f(x)$ for some given function $f(x)$, and $u_t(x, 0) = 0$.

We will use $a = \gamma = 1$ and $L = 2$ with

$$f(x) = \begin{cases} x & \text{for } x \in [0, 1) \\ 2-x & \text{for } x \in [1, 2] \end{cases}$$

Recall that in class we found the Fourier Sine Series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{8 \sin \left( \frac{n \pi x}{2} \right)}{n^2 \pi^2} \sin \left( \frac{n \pi x}{2} \right).$$

1. Verify that $a^2u_{xx} = u_{tt} + \gamma u_t$ is linear. That is, show that if $u_1$ and $u_2$ are solutions, then so is $u = C_1u_1 + C_2u_2$.

2. Suppose $u(x, t) = X(x)T(t)$, and show this leads to

$$X'' + \sigma X = 0 \quad \text{and} \quad T'' + T' + \sigma T = 0$$

$$X(0) = X(2) = 0 \quad \text{and} \quad T'(0) = 0$$

3. Clearly $X(x)$ is just as in class and must have $\sigma = \frac{n^2 \pi^2}{4}$, for $n = 1, 2, 3, \ldots$ in order to satisfy the boundary conditions. Show that for each $n$,

$$T_n(t) = e^{-t/2} \left( \sin \left( t \frac{\sqrt{n^2 \pi^2 - 1}}{2} \right) + \sqrt{n^2 \pi^2 - 1} \cos \left( t \frac{\sqrt{n^2 \pi^2 - 1}}{2} \right) \right),$$

solves the $T$ initial value problem.

4. By linearity (and some facts about limits) $u(x, t) = \sum_{n=1}^{\infty} c_n X_n(x)T_n(t)$ will be a solution to the PDE that satisfies the boundary conditions, provided it converges. Check that $u_t(x, 0) = 0$.

5. Use $u(x, 0) = f(x) = \text{the Fourier Series of the odd periodic extention of } f(x)$ that we found in class to show that,

$$c_n = \frac{8 \sin \left( \frac{n \pi}{2} \right)}{n^2 \pi^2 \sqrt{n^2 \pi^2 - 1}}$$

Rewrite this expression without using trig functions.

6. [Optional] Use Maple to animate your result. Either e-mail your Maple worksheet to me, or come by and show me.