The Knot Group

A knot is an embedding of the circle into $\mathbb{R}^3$ (or $S^3$),

$$k : S^1 \to \mathbb{R}^3.$$  

We shall assume our knots are tame, meaning the embedding can be extended to a solid torus, $K : S^1 \times D^2 \to \mathbb{R}^3$. The image is called a tubular neighborhood of the original knot. There exist knots that are not tame, these are called wild knots; see [https://en.wikipedia.org/wiki/Wild_knot](https://en.wikipedia.org/wiki/Wild_knot).

**Note:** If the embedding is smooth or piece-wise linear (PL) it can be shown that the knot is tame.

Two knots are regarded as equivalent if there is an ambient isotopy of $\mathbb{R}^3$ that takes the image of one onto the other. A knot with no crossings when projected onto a plane is called an unknot. The next simplest knot is the trefoil which has a plane projection with three crossings. (Any knot with just one or two crossings can be deformed into an unknot.) In fact, there are actually two trefoil knots that are mirror images of each other. They are called the left-handed and right-handed trefoils. See Figure 1.

![Figure 1. The left-handed and right-handed trefoils](image)

**Exercise.** Plot the system of parametric equations below with a computer for $t \in [0, 2\pi]$. Which trefoil is it? Find equations for the other. Write a formula for an extension to a tubular neighborhood.

$$x(t) = \sin t + 2 \sin 2t$$
$$y(t) = \cos t - 2 \cos 2t$$
$$z(t) = -\sin 3t$$

The knot group of a knot is the fundamental group of the complement on the knot. It is invariant under ambient isotopy. First we do the unknot. Recall that $S^3$ can be formed by gluing two unknotted solid tori together, with the meridian of each going to the longitude of the other. Thus, in $S^3$ the complement of an unknot can be homotoped to another to a solid torus. In $\mathbb{R}^3$ we just get a solid torus minus a point. Thus the fundamental group of the complement of an unknot is an infinite cyclic group.

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1An isotopy is a homotopy where each level is an homeomorphism. An isotopy of the whole space is called an ambient isotopy.
Next we do a trefoil knot. (Both versions give isomorphic groups.) The method generalizes to any knot. We will use a PL version of the trefoil and take as a tubular neighborhood, $N$, one with square cross sections, $1/3 \times 1/3$, that fits into $\mathbb{R}^2 \times [0,1]$ as in Figure 2. Since a tubular neighborhood is a deformation retract of the knot complement, the fundamental groups are isomorphic. A closeup view of one crossing is shown in Figure 3. We partition $N$ into six (easy!) pieces. The lower part is $N \cap \{z \leq 1/3\}$ and has three components that we call $B_1$, $B_2$ and $B_3$. The upper part is $N \cap \{z \geq 1/3\}$ also has three components that we call $A_1$, $A_2$ and $A_3$. The $A_i$’s all meet the $z = 1$ plane, while the $B_i$’s rest on the $z = 0$ plane.

![Figure 2. PL trefoil square tubular neighborhood](image)

We wish then to find the fundamental group of $\mathbb{R}^3 - N$. We shall decompose this space into two open sets and apply the Seifert-van Kampen theorem. Let

$$V_1 = \{(x, y, z) \mid z > 0\} - N$$

and

$$V_2 = \{(x, y, z) \mid z < 1/6\} - N.$$
Let $V_3 = V_1 \cap V_2$. Now we compute the fundamental groups of these three spaces. We let $p = (0,0,1/12)$ be the common base point, but leave this out of the notation for convenience. The obvious inclusion maps induce the following commutative diagram.

\[
\begin{array}{c}
\pi_1(V_1) \\
i \\
\pi_1(V_3) \\
\downarrow j \\
\pi_1(V_2) \\
\downarrow l \\
\pi_1(\mathbb{R}^3 - N)
\end{array}
\]

We start with $V_2$ because it is simplest. It is the $\{z < 1/6\}$ open lower half space with three trenches of depth $1/6$ dug out, that is,

$$V_2 = \{z < 1/6\} - (B_1 \cup B_2 \cup B_3).$$

There is a deformation retraction to the half space $\{z < 0\}$. This is homeomorphic to $\mathbb{R}^3$. Thus, $\pi_1(V_2)$ is the trivial group.

Next look at $V_3$. It is equal to $\{0 < z < 1/6\} - (B_1 \cup B_2 \cup B_3)$. This has a deformation retraction to a plane, say $z = 1/12$, with three disjoint rectangles cut out. This in turn has a deformation retraction to a wedge of three circles. (Prove this!) Thus, $\pi_1(V_3) \cong F_3$. Figure 4 illustrates a choice of generators, $\alpha$, $\beta$, and $\gamma$ for $V_3$. In the figure $B_1$, $B_2$ and $B_3$ are holes.

![Figure 4](image)

**Figure 4.** $V_3$ with a choice of generators

Finally, consider $V_1$. It can be thought of as the open half space $\{z > 0\}$ with three tunnels dug out. This is homeomorphic to an open rectangular box with three tunnels dug. We can use an isotopy to straighten the tunnels as shown in Figure 5. This space can be homotoped to an open rectangle with three disks removed, which in turn can be homotoped to a wedge of three circles. Thus, $\pi_1(V_1) \cong F_3$. 

We will call the generators $a$, $b$, and $c$. Each loops once around one of the tunnels. So, now we have the following commutative diagram.

Since $\pi_1(V_2)$ is trivial, it follows that $\pi_1(\mathbb{R}^3 - N)$ is generated by the images of the generators of $\pi_1(V_1)$ under the homomorphism induced by inclusion. Now the tricky part is to find the relations.

To do this we first study one crossing. We have drawn four loops in Figure 6(left). Each is homotopic to a generator. Two of them are actually the same generator. We label them $a$, $b$, and $c$. We will write $ab$ for $a * b$ and so on and use the same symbols for the loops and the corresponding group elements. Study the loop $cbc^{-1}a$. Notice that it is homotopic to the $\alpha$ generator of $\pi_1(V_3)$. That might be easier to see in Figure 6(right). But this generator is mapped to the identity in $\pi_1(V_2)$. Thus, we have the relation $cbc^{-1}a = 1$ in $\pi_1(\mathbb{R}^3 - N)$.

Each crossing induces a similar relation. By the Seifert-van Kampen theorem, we arrive at a presentation for $\pi_1(\mathbb{R}^3 - N)$. We use the stylized diagram in Figure 7 to do the computation for our trefoil knot. This gives

$$
\pi_1(\mathbb{R}^3 - N) \cong \langle a, b, c \mid aba^{-1}c = 1, c^{-1}acb = 1, bc^{-1}b^{-1}a^{-1} = 1 \rangle.
$$
This is called the Wirtinger presentation of the knot group. The method can be applied to any knot. Often it is useful to derive different presentations. For the trefoil group we show one of the two most commonly used presentations.

\[
\langle a, b, c \mid aba^{-1} c = 1, c^{-1}acb = 1, bc^{-1}b^{-1}a^{-1}\rangle
\]

\[
= \langle a, b, c \mid c^{-1} = aba^{-1}, c^{-1}acb = 1, c = b^{-1}a^{-1}b \rangle \quad T1, T2, T1, T2
\]

\[
= \langle a, b, c \mid c^{-1} = aba^{-1}, c = b^{-1}a^{-1}b \rangle \quad T2
\]

\[
= \langle a, b, c \mid c^{-1} = aba^{-1}, c = b^{-1}a^{-1}b, aba^{-1} = b^{-1}ab \rangle \quad T1
\]

\[
= \langle a, b, c \mid c = b^{-1}a^{-1}b, aba^{-1} = b^{-1}ab \rangle \quad T1
\]

\[
= \langle a, b \mid aba^{-1} = b^{-1}ab \rangle \quad T4
\]

\[
= \langle a, b \mid aba = bab \rangle \quad T1, T2
\]
We can abelianize this group by adding the relation $ba = ba$. You can check this implies $a = b$ and thus the abelianization gives an infinite cyclic group. It turns out that the abelianization of any knot group is infinite cyclic.

Finally, we prove that the trefoil group itself is not infinite cyclic. From this we can conclude that the trefoil is not equivalent to the unknot. From an earlier handout we know $\langle a, b \mid aba = bab \rangle \cong \langle x, y \mid x^2 = y^3 \rangle$. Call this group $G$. Let $Q = \langle \langle x^2 \rangle \rangle$. There is a homomorphism from $G$ onto $G/Q$. But

$$G/Q = \langle x, y \mid x^2 = y^3, x^2 = 1 \rangle = \langle x, y \mid x^2 = 1, y^3 = 1 \rangle \cong \mathbb{Z}/2 \ast \mathbb{Z}/3,$$

which is not abelian. But any homomorphic image of an infinite cyclic group must be abelian.

**Exercise.** Find a presentation for the knot group of the figure-8 knot; see Figure 8(left). Show that it is equivalent to

$$\langle p, q \mid p^{-1}qpq^{-1}pq = qp^{-1}qp \rangle.$$ 

Show that it is not infinite cyclic and that it is not isomorphic to the trefoil group. Show that its abelianization is infinite cyclic.

![Figure 8. The figure-8 knot and the Hopf link](image)

**Exercise.** By the way, the figure-8 knot is ambient isotopic to its mirror image. See if you can show this by drawing the mirror image and then gradually deforming it into the original.

**Exercise.** These ideas can be extended to links. Figure 8 also shows the Hopf link. Find the fundamental group of its complement. Show that the result is isomorphic to $\mathbb{Z}^2$.

**References.**

(1) Knots and Links, by Dale Rolfsen, AMS Chelsea Publishing.


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