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NONSINGULAR SMALE FLOWS IN THE 3-SPHERE WITH ONE ATTRACTOR AND ONE REPELLER

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Abstract. The aim of this paper is to show that any two knots can be realized as an attractor and repeller pair for some nonsingular Smale flow on S^3 with any linking number. We view this as progress, albeit limited, to the conjecture that all two component links can be realized as an attractor-repeller pair in a nonsingular Smale flow on S^3 with just one other basic set of saddle type.

1. Introduction

A nonsingular Smale flow (NSF) is a structurally stable flow with one dimensional invariant set that satisfies a transversality condition. For a nonsingular Smale flow on a compact 3-manifold the invariant set decomposes into a disjoint collection of compact *basic sets* which come in three types: attracting closed orbits, repelling closed orbits and saddle sets; this is a special case of a theorem of Smale's. The saddle sets may be isolated closed orbits, as the attractors and repellers have to be, or they could be suspensions of two-sided shifts of finite type. We will call these *chaotic saddle sets*. For details see [3, 2, 7].

Isolating invariant neighborhoods of isolated closed orbits are solid tori. The isolating invariant neighborhoods of chaotic saddles that we will work with are called *thickened templates*. To describe these we first define what we mean by a *template*. Templates are compact branched 2-manifolds with semi-flows. They are derived from three dimensional thickened templates by collapsing out the

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stable direction of the flow. Although many orbits are identified under the collapsing the closed orbits on a template correspond to closed orbits in the original flow. See [2].

The constructions that we will use are easier to perform on the two dimensional templates. We can then thicken them to construct the actual neighborhoods of chaotic saddle sets. Thus, we flip back and forth between the two points of view.

Templates are formed from a finite complex with two types of charts: *joining charts* and *splitting charts*, shown in Figure 1. In the joining charts the flow lines merge at a *branch line*. There are two entrance segments and one exit segment in the boundary. The semi-flow is tangent to the rest of the boundary. The splitting chart has one entrance segment, but its exit set is partitioned into three sub-segments, indicated by an inward curving of the middle sub-segment. The semi-flow is tangent to the two side segments. A template is formed by attaching exit sets to entrance sets. It is required that for a template the exit set consists of the middle portions of the splitting charts and that the entrance set be empty. It follows that the number of joining charts is equal to the number of splitting charts. For details see [2].

The *invariant set* of a template is the set of orbits of the semi-flow that never exit. The invariant set is the suspension (torus mapping) of a one-sided shift of finite type. (Its inverse limit is a suspended two-sided shift of finite type.) It is known that the invariant set contains infinitely many closed orbits which form infinitely many knot types [6].

We define the *split move* via Figure 2. It changes the topology of a template but does not effect the invariant set.

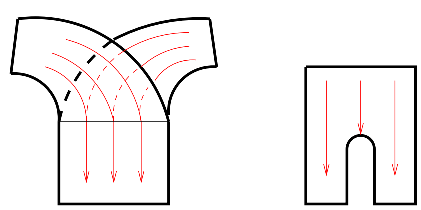

Figure 1. The charts

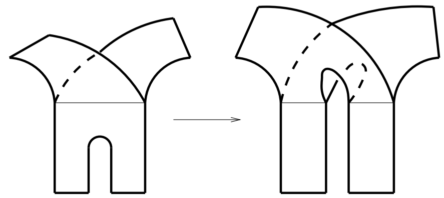

FIGURE 2. A split move

A complete template V is show in Figure 3 along with a periodic orbit. Rob Ghrist has shown that V realizes all knot types as periodic orbits [1]. We will use this fact in our construction.

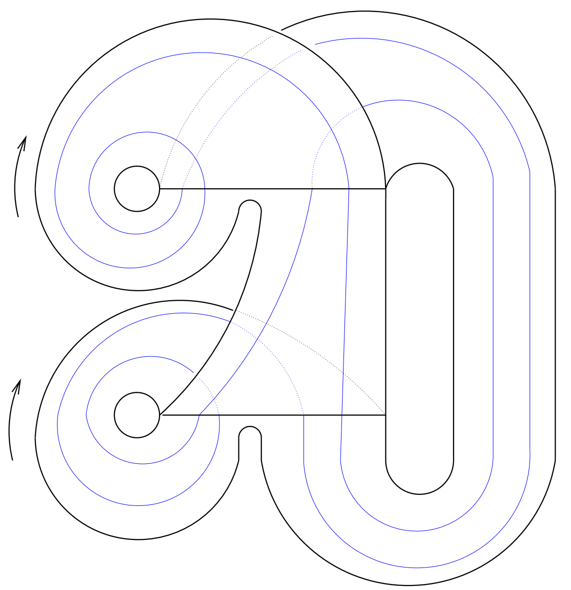

FIGURE 3. A template with all knot types

A thick template is a thickening of a template. Topologically a thick template is a handlebody; its boundary a closed surface. They support a flow whose invariant set is the inverse limit of the semi-flow of the template. In a Smale flow on a 3-manifold we use thick templates as isolating neighborhoods of chaotic saddle sets.

A template is recovered from a thick template by collapsing out the stable direction.

In Figure 4 we show the split move on a piece of a thickened template. We now describe the exit set of a thick template. The boundary of a template is a one dimensional branched manifold. A thickened version of a template's boundary is a two manifold whose interior will form the exit set of the flow on the thickened template. The boundary of the closure of the exit set consists of closed loops where the flow is tangent to the boundary of the thickened template. The complement of the closure of the exit set in the boundary of the thickened template is the entrance set for the flow. See [7] for more details.

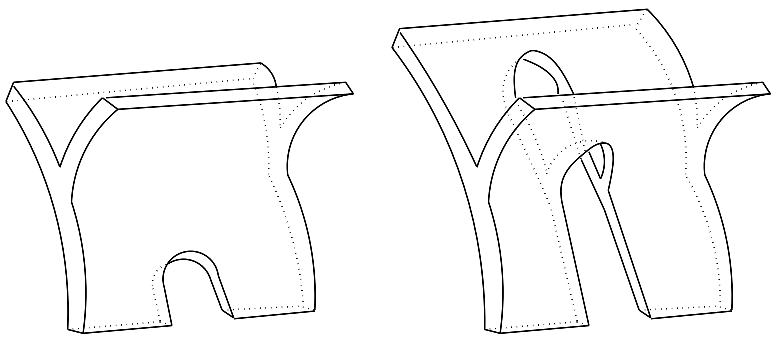

FIGURE 4. A split move on a portion of a thickened template

In Figure 5 we show the exit set of a thickened version of V. It will play an important role in our constructions.

The aim of this paper is to show that any two knots can be realized as an attractor and repeller pair for some NSF on S^3 with any linking number. We view this as progress, albeit limited, to our conjecture that all two component links can be realized as an attractor-repeller pair in an NSF on S^3 with just one other basic set of saddle type [8]. Our construction will produce three basic sets of saddle type.

2. The construction

Theorem 2.1. *Let K_1 and K_2 be knots and let n be a nonnegative integer. Then there exists a nonsingular Smale flow on S^3 with*

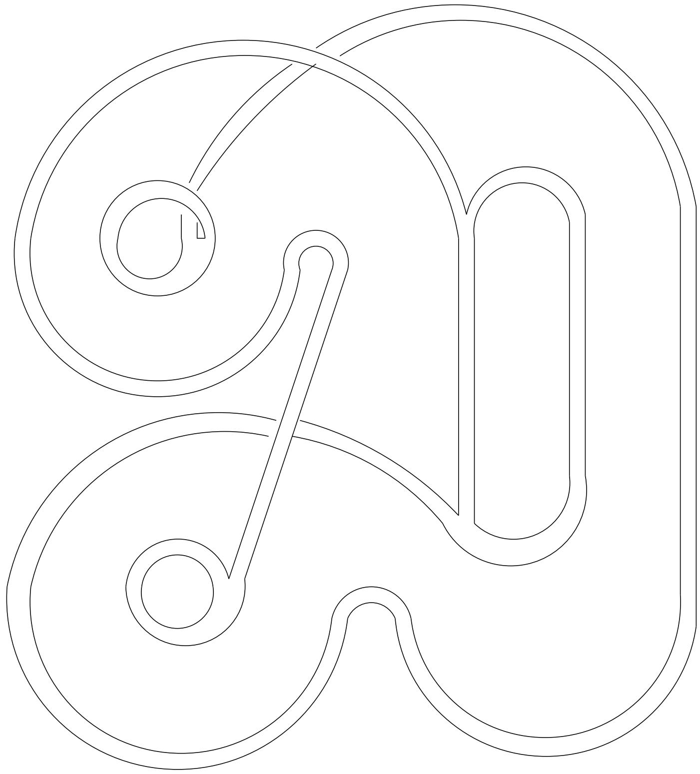

FIGURE 5. The exit of the thickened template V_T

attracting set K_1*, repelling set* K_2 *and the unsigned linking number of* K_1 *and* K_2 *is* n*.*

Proof. **Step 1.** There exists an NSF flow on S^3 with three basic sets, an attracting closed orbit A, a repelling closed orbit R and a saddle set S such that A and R are unknots with linking number $n \geq 2$; just generalize Example A.2.7 of [2] - this can also be seen abstractly from John Franks' paper [4].

Step 2. Let ϕ be a NSF on S^3 with attracting periodic orbit K. Let K' be any knot and let $p \in K$. Then we will show there exists a NSF ϕ' on S^3 identical to ϕ outside a small enough flow box about p such that the connect sum $K\#K'$ is a attracting periodic orbit. (By reversing the flow directions the same construction below allows for K and $K\#K'$ to be repellers.)

The proof is by construction. The template V in Figure 3 contains all knots as periodic orbits [2]. We will call the upper branch line β_1 and the lower branch line β_2. Let $b \in \beta_2$ be the midpoint. The flow line starting at b quickly exits the template.

Let V_T be a thickened version of V. It gives a neighborhood of an invariant set of saddle type. It has a cross section whose first return map is topologically conjugate to a suspension of the full 2-shift. The closure of the exit set of the flow is shown in Figure 5 as mentioned above. It is the union of three annuli and two strips and is topologically a closed disk with three open disks removed from its interior. The closure of its complement in ∂V_T is the closure of the entrance set and has the same topology. (This is not true of all templates.)

We perform some split moves that are easier to see in the 2-dimensional template V.

Select a periodic orbit in V whose knot type is K' - we will call it K'. Select a point q of $K' \cap \beta_2$ that is closest to the splitting point b so that we can draw a line from q to b without crossing K' again. Now take the flow line that exits the template at b and trace it backwards until we arrive at the point q'. See Figure 6. Now cut open the template V along the flow line γ joining b to q' and call the new template V'. This is just performing a series of split moves and does not change the periodic orbits. See Figure 7. Let V'_T be the thickened version of V'. It is a smaller or tighter neighborhood of the invariant saddle set than V_T.

Note: The orbit K' shown in the figures is actually an unknot. This was done to make the illustrations easier to follow. We also remark that for any knot type there is a periodic orbit of V that does meet the lower branch line's left half at least once.

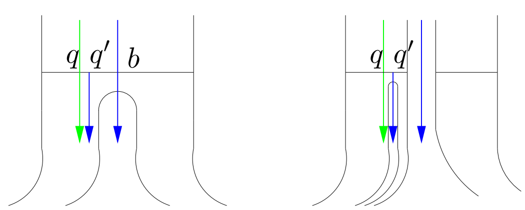

FIGURE 6. Splittings

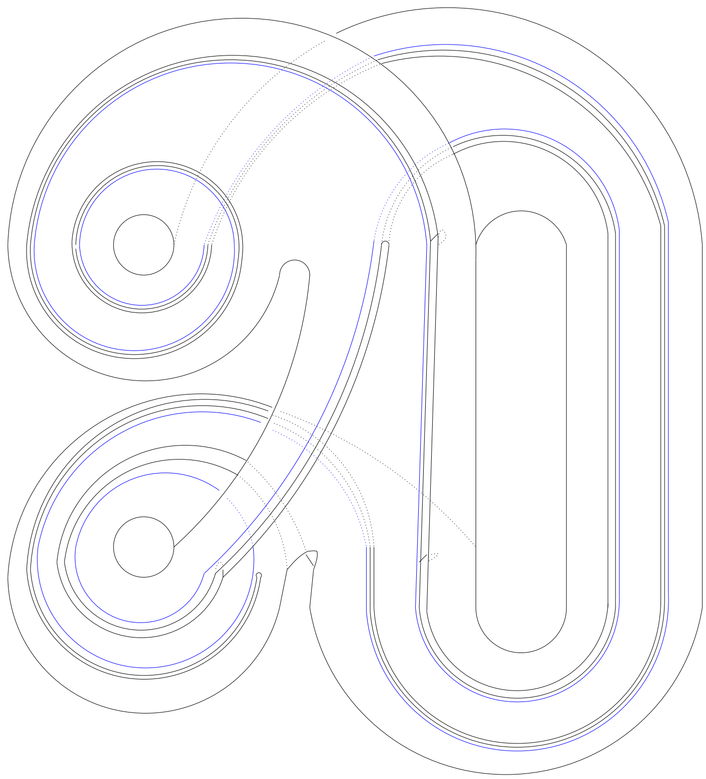

FIGURE 7. The template V'

We are now going to attach some tubes and balls to the exit set of V'_T. These will form a solid cylinder $C' = D \times I$. The core of C' will be isotopic to K' minus a short arc; that is we can attach a short arc to the core of C' to form a loop with knot type K'. The ends of the cylinder will, eventually, be attached to a cylindrical neighborhood C of K minus a short arc, to form a tubular neighborhood of $K \# K'$. The union of C' and V'_T will be a ball with $D \times \{0, 1\}$ in its boundary and the rest of the boundary will be a entrance set for the flow we are constructing.

Now we describe C'. Let A be a solid cylinder that is attached to the part of the exit set of V'_T that we just carved out, that is the part of the exit set that does not correspond to any part of the exit set of V_T. We need for A to "pop out" away from V'_T at the narrow end, the location corresponding to the last split move. This is depicted, somewhat schematically, in Figure 8. This end of A will be an end of C'. The other end of A will be attached to three tubes, B, E and F (one might visualize it as splitting into three tubes). One, E, will go away from V'_T and its other end will be an end of C'. The other two will snake around the exit set of V'_T. See Figure 9.

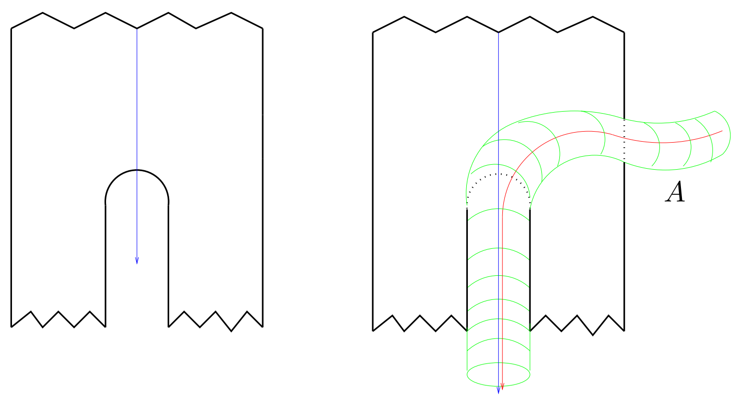

FIGURE 8. Attaching part of tube A to a splitting chart of V'_T.

Now B will attach to a ball G that contains one of the annuli of the exit set of V'_T that was inherited from the original V_T. Also attached to G will be a tube H which snakes along the exit set of V'_T until it in turn attaches to a ball J that contains another annulus of the exit set of V'_T. The tube F snakes along the exit set until it attaches to a ball L that contains the third annulus of the exit set of V'_T. This is illustrated schematically in Figure 10; it does not show the middle part of A for clarity. The union of tubes A, B, E and H and the balls G, J and L form C'.

We endow the sets tubes A, B, E, F, H and the balls J, L and G with vector fields depicted in Figure 11. The attachings are to induce a smooth vector field on C' and thence on $C' \cup V'_T$.

The reader can check via an Euler characteristic calculation that $C' \cup V'_T$ is a ball. We distinguish two disks in its boundary that

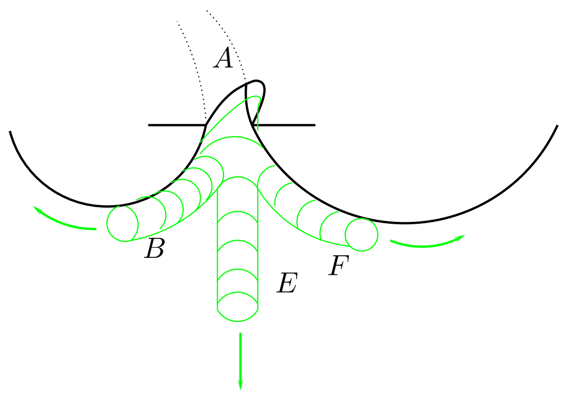

FIGURE 9. Tube A attaches to tubes B, E and F

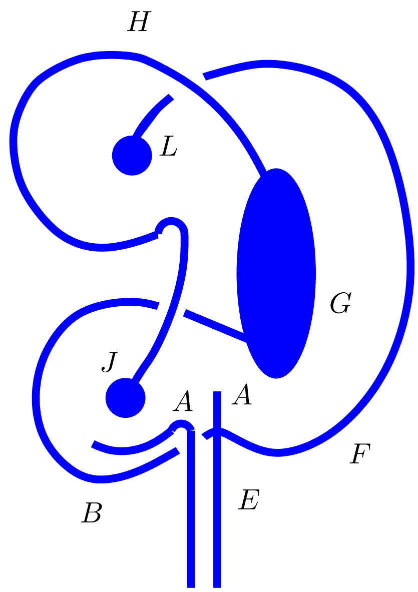

FIGURE 10. The tubes and balls of C' attached to the exit set of V'_T

where the ends of C' and think of $C' \cup V'_T$ as a cylinder with these

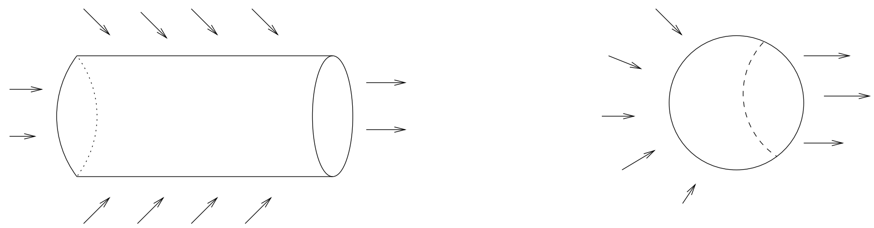

FIGURE 11. Vector fields on tubes and balls used to make C'

two disks as ends. See Figure 12 which also indicates the vector field.

Now we go back to the original flow ϕ with attractor K. Next select a small closed arc α of the orbit K containing p in its interior. We remove a small cylinder from S^3 whose core is α. The core of the tube A is equivalent to K' minus a small arc α'. Thus if we glue in C', deformed into a cylinder as in Figure 12, to fill the hole we just created in S^3, we can arrange the gluing so that the closures of $K - \alpha$ and $K' - \alpha'$ form the connected sum $K \# K'$. Figure 12 also indicates how the vector fields can be matched; that this can be done smoothly is well known.

Step 3. We can now finish the proof. There exists an NSF with basic sets an unknotted attractor A, an unknotted repeller R and single orbit saddle set S all unlinked [3]. Apply Step 2 to A and R. This takes care of the case for the linking number $n = 0$. Recall that a Hopf flow on S^3 has two and only two closed orbits and that they are unknotted and have linking number one. Thus for $n = 1$ apply Step 2 to each component of a Hopf flow. For $n \geq 2$ apply Step 2 to Step 1 twice. □

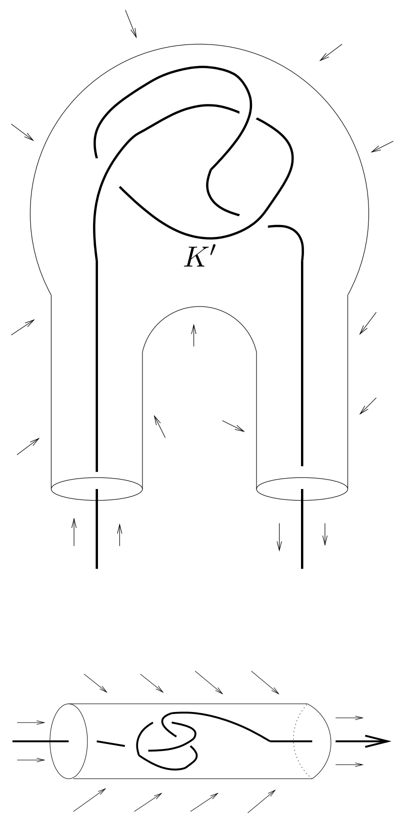

FIGURE 12. Two versions of $C' \cup V'_T$

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