PERIODIC PRIME KNOTS AND TOPOLOGICALLY TRANSITIVE FLOWS ON 3-MANIFOLDS

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Abstract. Suppose that \( \varphi \) is a nonsingular (fixed point free) \( C^1 \) flow on a smooth closed 3-dimensional manifold \( M \) with \( H_2(M) \neq 0 \). Suppose that \( \varphi \) has a dense orbit. We show that there exists an open dense set \( N \subseteq M \) such that any knotted periodic orbit which intersects \( N \) is a nontrivial prime knot.

1. Introduction

We need some standard terminology from knot theory. For presentation of knots in dynamical systems see the book [5] by Ghrist, Holmes, and Sullivan. Let \( \Gamma \subseteq M \) denote a knot. By this we mean that \( \Gamma \) is the image of a continuous injective function from the circle to a 3-dimensional manifold \( M \). We shall say that \( \Gamma \) is a trivial knot if it bounds a disk. We say that \( \Gamma \) is a composite knot if there exists a 2-sphere \( S \) in \( M \) such that \( S \cap \Gamma \) is two points, \( z \) and \( w \), and the intersection of each component of \( \Gamma - \{ z, w \} \) together with a segment in \( S \) from \( z \) to \( w \) is a nontrivial knot. We shall say that \( \Gamma \) is a prime knot if it is neither composite or trivial. When the knot is of class \( C^1 \) and

\[
\Theta : \Gamma \times \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1 \} \to M
\]

is a \( C^1 \) embedding such that, for all \( x \in \Gamma \), \( \Theta((x, 0, 0)) = x \), the concepts of trivial, composite, and prime extend to the solid torus which is the image of \( \Theta \).

Our main theorem is Theorem 1. As a consequence of this theorem, for any topologically transitive \( C^1 \) nonsingular flow on \( S^3 \), there is an open dense set \( N \subseteq S^3 \) such that any periodic orbit intersecting \( N \) is a nontrivial prime knot.

Theorem 1. Let \( M \) be a smooth closed (compact, no boundary) 3-dimensional manifold with \( H_2(M) = 0 \). Suppose \( \varphi \) is a \( C^1 \) nonsingular (fixed point free) topologically transitive (\( \varphi \) has a dense orbit) flow on \( M \). There exists an open dense set \( N \subseteq M \) such that if \( \gamma \) is a periodic orbit with \( \gamma \cap N \neq \emptyset \) then \( \gamma \) is a nontrivial prime knot.

Remark: It is possible that some periodic orbits are trivial. As an example, Harrison and Pugh in [7] define a nonsingular flow on \( S^3 \) with a dense orbit by Birkhoff suspending Katok diffeomorphisms of a disk. The flow has a dense orbit but the diffeomorphism of the disk has a fixed point which corresponds to a trivial knot in the flow.

For the rest of this paper, let \( M \) be a smooth closed 3-dimensional manifold with \( H_2(M) = 0 \), and let \( \varphi \) be a \( C^1 \) nonsingular topologically transitive flow on \( M \).

2000 Mathematics Subject Classification. Primary: 57N12; Secondary: 57M25, 37C10, 57S25.

Key words and phrases. dense orbit; minimal flow; global cross section; knots; prime knots; transitive flow.
Our motivation for this result is a Theorem 2 below, which appears as Theorem 1 from [3]. Let \( p \) be any point in the dense orbit of \( \varphi \). Let \( D \) be a compact disk containing \( p \) which is transverse to the flow. That is, \( D \) is a compact disk and there is an open disk \( E \) containing \( D \) that is transverse to the flow. We call such a disk a transverse disk, and if \( D \) is in addition a global cross section we will call it a global transverse disk. Let \( q \in D \) be a point in the forward orbit of \( p \) and let \( \overline{pq} \) denote the orbit segment beginning at \( p \) and ending \( q \). Let \([pq]\) denote a compact segment in \( D - \overline{pq} \cap D \) connecting \( p \) to \( q \). Let \( \Gamma = \overline{pq} \cup [pq] \).

**Theorem 2.** If \( q \) is close enough to \( p \) then \( \Gamma \) is a nontrivial prime knot. The result holds in the case \( H_2(M) \neq 0 \) if the flow has no periodic orbits.

For a point \( x \in M \) we use \( \gamma_x \) to denote the orbit through \( x \). Theorem 3 below is proven as Theorem 2.1 in [6]. We use it to prove a periodic orbit forms a prime knot under our specified conditions.

**Theorem 3.** A solid torus \( T \) contained in \( M \) is a (nontrivial) prime knot if there exists a transversely orientable bidimensional \( C^2 \) foliation \( \mathcal{F} \) on \( \mathcal{V} = M - T \) such that:

1. \( \mathcal{F} \) is transversal to \( \partial \mathcal{V} \). Moreover, every leaf of \( \mathcal{F} \) has nonempty intersection with \( \partial \mathcal{V} \).
2. The one-dimensional foliation \( \mathcal{F}|_{\partial \mathcal{V}} \) on \( \partial \mathcal{V} \) contains a meridian \( \sigma \) as a leaf. Moreover, \( \mathcal{F}|_{\partial \mathcal{V}} \) contains no Reeb components.
3. If \( \mathcal{F} \) has a compact leaf \( K \), there are finitely many discs \( D_1, D_2, \ldots, D_s \) contained in \( T \) such that the union of \( K \) with \( \cup_{i=1}^s D_i \) is a torus \( L \). The union \( \cup_{i=1}^s D_i \) satisfies \( L \cap \partial T = K \cap \partial T = \bigcup_{i=1}^s \partial D_i \).
4. Let \( B = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 \leq 9 \text{ and } x \leq 2 \} \) and decompose its boundary \( \partial B \) as the union of \( B_1 = \{(x, y) \in B | x^2 + y^2 = 1 \}, B_2 = \{(x, y) \in B | x = 2 \} \) and \( B_3 = \{(x, y) \in B | x^2 + y^2 = 9 \} \). There exists an embedding \( \lambda : B \times [-1, 1] \to \mathcal{V} \) such that
   a. \( \lambda : (B_1 \cup B_2) \times [-1, 1] \) is precisely the intersection of \( \partial \mathcal{V} \) with the image \( \operatorname{Im}(\lambda) \) of \( \lambda \).
   b. The complement of \( \lambda(B_1 \times (-1/2, 1/2)) \) in \( \partial \mathcal{V} \) is a union of meridians of \( \partial \mathcal{V} \) which are leaves of \( \mathcal{F}|_{\partial \mathcal{V}} \).
   c. For all \( p \in B \), the segments \( \lambda \{p\} \times [-1, 1] \) are transversal to \( \mathcal{F} \).
   d. Let \( H \) be a half straight line of \( \mathbb{R}^2 \) starting at the origin. Then, for all \( z \in [-1, 1], \lambda((H \cap B) \times \{z\}) \) is contained in a leaf of \( \mathcal{F} \). Also, for all \( z \in [-1, -1/2] \cup (1/2, 1], \lambda(B \times \{z\}) \) is a plaque of \( \mathcal{F} \).

**Proof.** (of Theorem 1)

Let \( p \) be any point in the dense orbit. We will prove that there is a neighborhood \( N_p \) of \( p \) such that if \( a \in N_p \) and \( \gamma_a \) is periodic then \( \gamma_a \) is a nontrivial prime knot. Once this is proven for every \( p \) in the dense orbit, the set \( N = \cup_p N_p \) is the open (it is the union of open sets) dense (it contains the dense orbit) set required in the theorem.

The idea of the proof is simple. In [3], Theorem 2 is proven by showing that there exists a solid torus neighborhood of \( \Gamma = [pq] \cup \overline{pq} \) and a foliation satisfying the criteria of Theorem 3 proving that this solid torus is a prime knot, and hence \( \Gamma \) is a prime knot. We show that for any periodic point \( a \) in a small neighborhood of \( p \), this foliation can be moved by a small amount so that a torus neighborhood of \( \gamma_a \) is a prime knot, and hence that \( \gamma_a \) itself is a prime knot.
Let $D$ be a global transverse disk containing $p$. In [2] it is proven that any non-singular $C^1$ flow on a manifold of dimension greater than 2 has a global transverse disk. We can assume that the disk contains $p$, for if $D$ is any global transverse disk and $t_p$ is any time such that $\varphi(t_p, p) \in D$ then, $\varphi(-t_p, D)$ is a global transverse disk containing $p$.

It is proven in [3] that there is a disk $D_1 \subset D$ containing $p$, a foliation $\mathcal{F}$ on $M$, a solid torus neighborhood $T$ of $\overline{pD} \cup [p]$, and an imbedding $\lambda$ satisfying the conditions of Theorem 3, proving that $T$ is a prime solid torus. (See Figure 3 of [3] and Figure 1.) This can be chosen so that the embedding $\lambda : B \to M$ has its image in a flowbox $W$ whose base is $D_1$, whose top is a disk $U \subset D$, and such that $W \cap D = D_1 \cup U$ and $D_1 \cap U = \emptyset$. Moreover, we can assume that $T \cap W$ is a pair of cylindrical flow boxes $T_1$ and $T_2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{flowbox.png}
\caption{The imbedding $\lambda(B)$ inside the flowbox $W$.}
\end{figure}

Let $V$ denote the interior of the base of $T_1$. Note that $V$ is an open disk. Let $a$ be any periodic point in $V$. Then the orbit beginning at $a$ follows the orbit beginning at $p$ through the cylinders $T_1$ and $T_2$. Define $p_1$, $p_2$, and $p_3$ by

- $p_1 = \varphi(t_1, p)$, where $t_1 = \min\{t > 0 : \varphi(t, p) \in U\}$
- $p_2 = \varphi(t_2, p)$, where $t_2 = \min\{t > t_1 : \varphi(t, p) \in D_1\}$
- $p_3 = \varphi(t_3, p)$, where $t_3 = \min\{t > t_2 : \varphi(t, p) \in U\}$
Define $a_1$, $a_2$, and $a_3$ in the same manner. (See Figure 1.) Perturb the foliation $\mathcal{F}$ from [3] so that it is defined on $M - \overline{aa_1}$ instead of $M - \overline{pp_1}$. Specifically, there is a homeomorphism $\phi$ of $T_1$ that fixes the vertical boundary, is constant on the vertical coordinate, and takes $\overline{aa_1}$ to $\overline{pp_1}$. Define the new foliation $\mathcal{F}'$ to be equal to $\mathcal{F}$ on $M - T_1$ and to be the pullback by $\phi$ of $\mathcal{F}$ on $T_1$. Then define $T'$ to be a small tubular neighborhood of $\gamma_0$.

By reducing the size of $D_1$ so that $\gamma_0 \cap D_1$ is two points $a$ and $a_2$ if necessary, if $T'$ is chosen small enough (with $T'$ a torus neighborhood of $\gamma_0$) then $T' \cap W$ has two components. Let $T'_1$ be the component containing $\overline{aa_1}$ and $T'_2$ be the other component. As in [3], we can then define $\lambda : B \rightarrow B$ satisfying the criteria of Theorem 3 and the solid torus $T'$ is a prime knot. Hence the periodic orbit through $a$ is a prime knot.

Let $\epsilon > 0$ and define $N_p = \varphi((\epsilon, \epsilon), V)$. If $\epsilon$ is small enough then $N_p$ is an open neighborhood of $p$ and any periodic orbit which intersects $N_p$ intersects $V$ and hence is a nontrivial prime knot. □

We conclude with two questions:

- Under the assumptions of Theorem 1, is it true that every orbit is either prime or trivial?
- Can the assumption that $H_2(M) = 0$ be removed?

REFERENCES


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