KNOT FACTORING

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1. Introduction

A *knot* is a closed loop embedded in a three dimensional space. Two knots are regarded as equivalent if one can be gradually deformed into the other. An *unknot* or *trivial knot* is any knot that is equivalent to the unit circle in the real plane.

In Figure 1 the first two knots are distinct, but the second two are equivalent. In fact, the first knot is an unknot. The middle and right knots are *trefoil* knots. There is a notion of combining knots called the *connected sum.* An example is in Figure 8. The connected sum of any knot with the unknot is itself, so the unknot serves as an identity. Although we call this operation a sum, it behaves more like multiplication in the following sense. There is a notion of *prime* knots and, just as with the positive integers there is a prime factorization theorem, Theorem 8, discovered by Horst Schubert in the 1940s [13].

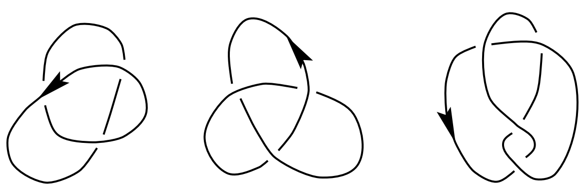

Figure 1. Some knots.

The goal of this paper is to present a proof of Schubert's theorem at a level accessible to advanced undergraduates, perhaps in the context of a supervised reading course. Before we proceed we develop some background material. The reader can probably accept much of this material as intuitively plausible. A rigorous development of low dimensional topology is beyond the scope of this paper, but our proof of Schubert's Theorem is complete.

Proofs of Schubert's Theorem can be found in the graduate level texts [4] and [11]; our proof follows the one in [4] for the most part. Background material on transversality and the theory of surfaces is rigorously developed in [7] and [12]. In the last few years several undergraduate texts on knot theory have come out: [1], [5], [8].

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After the proof of Schubert's Theorem we give an application of it in the study of knotted periodic orbits in flows. This uses a new area of mathematics called *template theory*.

2. Background

We now define some standard topological spaces and related notions.

Spheres: A one dimensional sphere, also called a 1-sphere or a circle, is any topological space homeomorphic to $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$, which is conventionally denoted by S^1. Likewise, two and three dimensional spheres are defined by $S^2 \cong \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ and $S^3 \cong \{(w, x, y, z) \in \mathbb{R}^4 \,|\, w^2 + x^2 + y^2 + z^2 = 1\}$, respectively. Two topological spaces are *homeomorphic* if there is an continuous bijection with the inverse continuous.

Balls or Disks: An n dimensional ball (or disk) is any topological space homeomorphic to $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1^2 + \cdots + x_n^2 \leq 1\}$, denoted by B^n (or D^n). The boundary of B^n is S^{n-1}, written as $\partial B^n = S^{n-1}$, where "∂" means "boundary of". Balls are closed, but open balls can be defined in the obvious way. The word *disk* means a 2-ball and the word *ball*, by itself, means a 3-ball.

Just as one can build S^2 by gluing two disks along their boundaries, one can form S^3 by gluing two balls along their boundaries. This gives a very useful way to view S^3. Imagine that one of the balls is a neighborhood of 0 and the other is a neighborhood of ∞. In a sense that can be made rigorous, S^3 is $\mathbb{R}^3$ plus a special point called ∞.

Surfaces: A *surface* is a topological space where every point has a neighborhood that is homeomorphic to an open disk. Thus a 2-sphere is a surface, and so is the plane. A 2-disk is not quite a surface by this definition. It is, however, a *surface with boundary*, where the *boundary* is the set of points with a neighborhood homeomorphic to $\{(x, y) \in \mathbb{R}^2 \,|\, x^2+y^2 < 1 \text{ and } y \geq 0\}$, an upper half disk. This definition of boundary is consistent with the boundary of a ball introduced above. The boundary of a surface S is denoted by ∂S. The *interior* of surface S is $S \backslash \partial S$ and is denoted by $\mathring{S}$; the same definition and notation are used for the interior of a ball. The following result is very important: *The boundary of every surface with boundary is a disjoint collection of circles.* Try to construct a counterexample! A closed, bounded surface without boundary is called a *closed surface.*

We are concerned only with *orientable* or *two-sided* surfaces. Nonorientable or *one-sided* surfaces include the Möbius band and the Klein bottle.

Every closed orientable surface is homeomorphic to one of the countable list of surfaces depicted in Figure 2. The first is of course the sphere. The second is the *torus*, then the *double-torus* and so on. The torus, denoted by T or T^2, plays a special role in knot theory. It is useful to view it as a rectangle with opposite sides identified. The number of "tunnels" in a surface is the *genus* of the surface. The word genus just means *type* or *kind.*

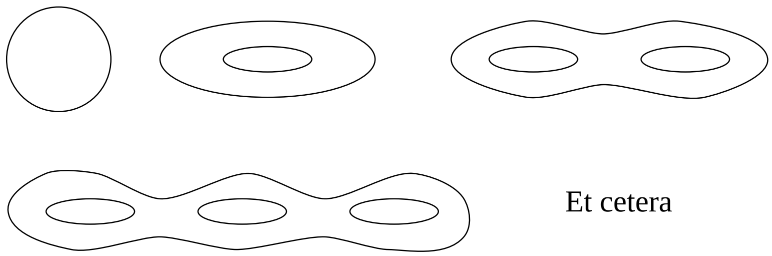

FIGURE 2. Standardly embedded surfaces.

For a surface with boundary, one can attach a disk along each boundary circle and form a closed surface. The genus of the resulting closed surface is regarded as the genus of the original surface with boundary. This definition is unambiguous, though this is by no means obvious.

Euler Characteristic: One way to make a surface is to glue together triangles along their boundaries. Call such a surface a *triangular surface*. The *Euler characteristic* of a triangular surface T is $\chi(T) = V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces. If two triangular surfaces are homeomorphic then they have the same Euler characteristic. For any surface S, it can be shown that there exists a triangular surface T that is homeomorphic to S. Thus, the Euler characteristic of a surface may be defined. The reader should try to show that $\chi(S^2) = 2$, $\chi(D^2) = 1$, and $\chi(T^2) = 0$.

The formula

$$g = 1 - \frac{\chi + b}{2}, \tag{1}$$

where b is the number of boundary components, connects the Euler characteristic of a surface with its genus. Check this for some examples.

Embeddings: An *embedding* is a homeomorphism of one space onto its image in another. We assume that all our embeddings are *smooth*, though we do not give a precise meaning to this term; rely on your intuition. If we embed a circle, S^1, into a 2-sphere, S^2, then $S^2 \backslash S^1$ consists of two open disks; this fact is the *Jordan Curve Theorem.* If we embed S^2 into S^3 then $S^3 \backslash S^2$ consists of two open balls. This fact is *Schonflies' Theorem.* Note: Schonflies Theorem requires the embedding to be smooth, but the Jordan Curve Theorem does not.

The surfaces show in Figure 2 are the *standard* embeddings. If we tied a knot in one tube of the double-torus we would have a different embedding of the same surface type.

Knots and Arcs: A *knot* is an embedding $K : S^1 \to S^3$ or $\mathbb{R}^3$. Often the symbol K is used both for the map and for the image $K(S^1)$. We normally assign an *orientation* to our knots. This is just a preferred direction and is indicated by placing an arrowhead on a drawing of a knot; see Figure 1. Two disjoint knots are *linked* if they cannot be pulled apart, or, equivalently, if there is no 2-sphere that separates them. In Figure 3 the pair of knots to the left is unlinked while the other pair is linked.

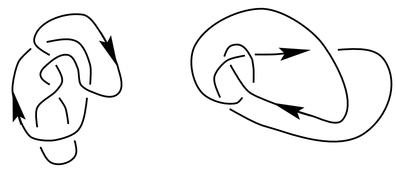

FIGURE 3. Links.

An *arc* is an embedding of $D^1 \cong [-1,1]$ into any higher dimensional space. Figure 4 shows two embeddings of arcs in 3-balls such that the end points of the arcs are in the boundary of the balls and no other points of the arcs are in the boundary of the balls. Now suppose we glue the two 3-balls together with a homeomorphism between their boundaries, which takes the end points of the each arc to the other arc's end points. The result is an unoriented knot in a 3-sphere.

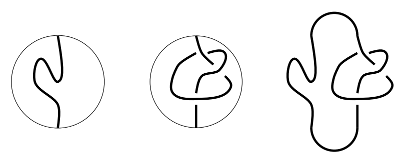

FIGURE 4. A "sum" of two arcs.

It can be shown that for any given knot K there is an orientable surface S with boundary K, called a *Seifert surface* of K. For example, if K is an unknot we could take S to be a disk. Of course we could also use any closed surface with an open disk removed. For the other knots the situation is not quite so easy. Figure 5 shows an orientable surface whose boundary is a trefoil. One can show, though not easily, that the surface is a torus with a disk removed. However, the attaching of the disk is abstract in the sense that the resulting closed surface is not embedded is any three-dimensional space. In general, given any knot K, one finds a Seifert surface of least genus that bounds the knot. The genus of this surface is the *genus of the knot*. While this idea is a hard one to grasp, it is explained well in the undergraduate texts [1], [5], and [8]. The unknot has genus zero while the trefoil has genus one. The unknot is the only knot with genus zero because it is the only knot that bounds a disk.

Transversality: The intersection of two arcs in the plane is *transverse* if the qualitative features of the intersection set are preserved under small perturbations. Perhaps the best way to understand transversality is through examples. Figure 6 shows pairs of arcs embedded in the plane. The first pair of arcs is not transverse, while the pair in the middle is transverse.

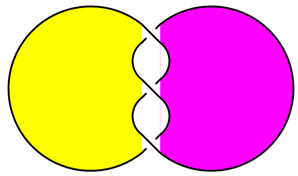

FIGURE 5. Surface with trefoil as boundary.

What about the third pair? If the middle pair were to be regarded as being embedded in $\mathbb{R}^3$ the arcs would no longer meet transversely. Why?

FIGURE 6. Non-transverse arcs (left), transverse arcs (center), disjoint arcs (right).

The concept of transversality is of immense importance in several areas of mathematics. It can be extended to intersections of surfaces: see [7, p. 30]. The reader should think hard about the following claim: *If the intersection of two closed surfaces is transverse then it is a finite disjoint collection of circles.* Is the converse true?

One can also apply transversality to the intersection of knots or arcs and surfaces. If an arc meets a surface in a finite number of points at which it passes clear through the surface then most very small deformations do not change the number of intersection points. In this case the intersection is transverse. If at an intersection point the arc does not pass from one side of the surface to the other then it can be removed by a small deformation. Thus, such an intersection is not transverse. Also, if an arc meets a surface in infinitely many points a small deformation can be found that removes all but finitely many of them. Again, such an intersection is not transverse. Note: this last case holds only for arcs of finite length and surfaces of finite area.

We give an application. A *torus knot* is a nontrivial knot K embedded in a torus T that has been embedded in S^3 in the usual manner.

Lemma 1. *Every torus knot is equivalent to a torus knot that wraps around the torus p times meridianally (short way around) and q times longitudinally (long way around), where p and q are relatively prime. Such a knot is called a (p, q)-torus knot.*

Proof. Represent the torus as a rectangle in the plane with opposite edges identified. We may assume without loss of generality that K is transverse to these edges. Why? Thus, K may be divided into at most finitely many arcs. Again, why? We give a procedure for deforming K to a (p, q)-torus knot. In Figure 7 we show a example giving a (5,3)-torus knot.

Step 1. Push each arc that connects an edge to itself through that edge, starting with innermost arcs and working out as needed.
Step 2. If there is only one "arc" left, it is really a loop and hence the knot is trivial. Assume this is not so. Straighten each remaining arc into a line segment.
Step 3. Choose an edge point close to the origin (the lower left corner) and slide it over so that its segment starts at the origin. Adjust other segments as needed.
Step 4. Space the end points of the segments out evenly.

The reader should ponder why p and q must be relatively prime and what would happen if they were not. □

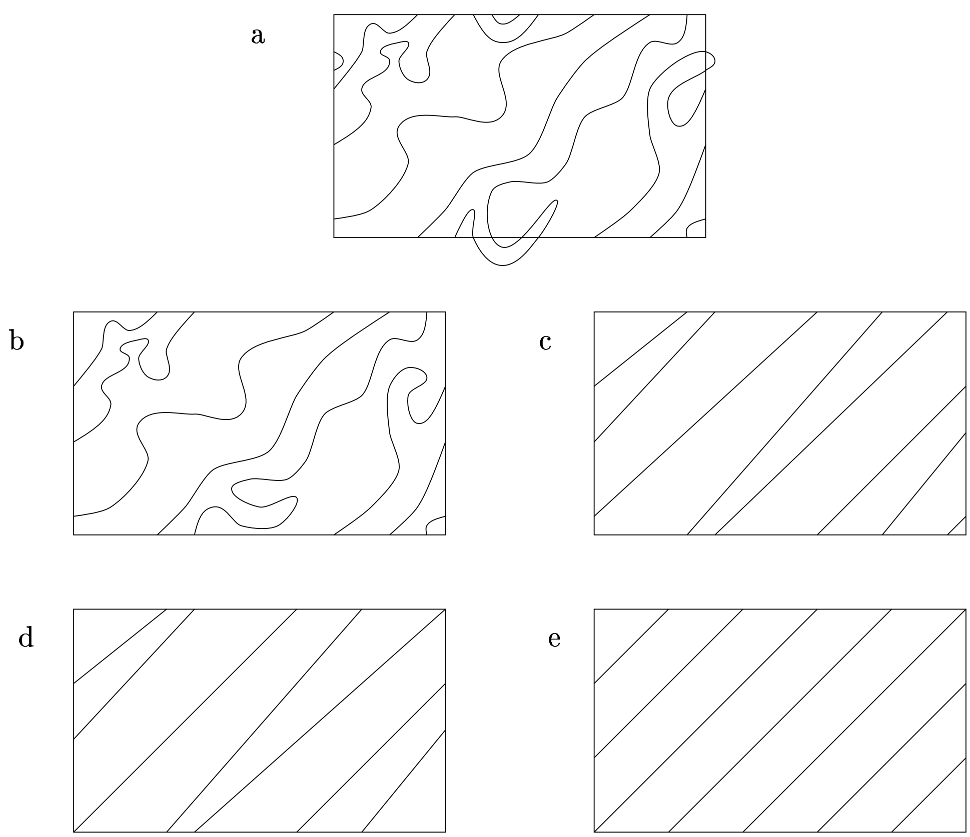

FIGURE 7. Straightening a torus knot.

3. Factoring Knots

Now we are ready for our main topic. In Figure 8 we show how to "add" two knots. Our first task is to formalize this idea.

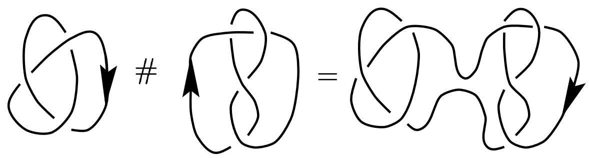

FIGURE 8. The connected sum of two knots.

Definition 2 (Connected Sums). Let K_1 and K_2 be knots in distinct 3-spheres S_1^3 and S_2^3, respectively. Pick points $a_1 \in K_1$ and $a_2 \in K_2$. Choose small balls, B_1 and B_2, centered at a_1 and a_2, respectively, such that $B_i \cap K_i$ can be deformed to an axis of B_i, for $i = 1, 2$. Form a union $S_1^3 \setminus \mathring{B}_1 \cup S_2^3 \setminus \mathring{B}_2$, using a gluing homeomorphism that matches $K_1 \cap \partial B_1$ to $K_2 \cap \partial B_2$ with the exiting end points going to the entering end points. Thus, we have a new 3-sphere containing a new knot called the *connected sum* of K_1 and K_2, which we denote by $K_1 \# K_2$.

The reverse operation is called *factoring.*

Definition 3 (Factoring). Let K be a knot is S^3. Let S^2 be a 2-sphere that meets K, transversely, in exactly two points. Then S^2 divides S^3 into two open balls whose closures we call B_1 and B_2. Let α be any arc on S^2 that joins the two point of $K \cap S^2$. Let $K_i = (B_i \cap K) \cup \alpha$, for $i = 1, 2$. Then we say that K has been *factored* into K_1 and K_2; see Figure 9. By Definition 2 we have $K = K_1 \# K_2$.

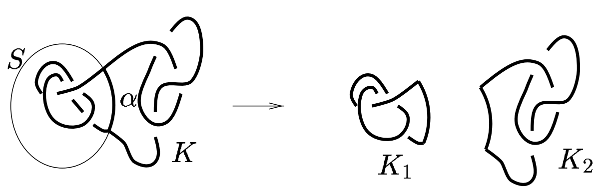

FIGURE 9. Factoring a knot.

We claim that Definition 2 is independent of the choice of the points a_1 and a_2, and that the knot types of K_1 and K_2 are independent of the choice of α in Definition 3. It is clear that $K \# U = K$. Commutativity, $K1 \# K2 = K2 \# K1$, is demonstrated in Figure 10. The idea is to make one factor very small and then pass it through the other. Associativity, $(K_1 \# K_2) \# K_3 = K_1 \# (K_2 \# K_3)$, is also easy.

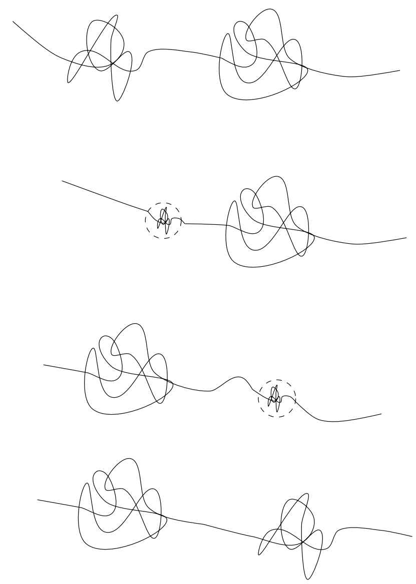

FIGURE 10. Commutativity of Connected Sums.

Definition 4. A nontrivial knot is *prime* if its only factors are itself and the unknot. Otherwise, a nontrivial knot is *composite*. The unknot is neither prime or composite.

Theorem 5. *Torus knots are prime.*

Outline of Proof. Let K be a torus knot. Suppose that K is composite and that S^2 is a cutting sphere for K. We may assume that S^2 and T^2 are transverse. Why? Thus $S^2 \cap T^2$ is a finite union of disjoint circles.

Let $\lambda \subset S \cap T$ be a circle. Suppose λ is a meridian or a longitude. Then if we orient the knot K it is clear from Lemma 1 that K passes into and out of S at least twice. Since this is plainly impossible we must assume that λ is neither a meridian nor a longitude of T. Suppose λ is some other (p,q) curve on T ($|p| > 1$ or $|q| > 1$). Form a "ribbon" R in S^2 whose core is λ. One can check that the two boundary components of R are linked, but a linked pair of curves cannot be embedded in a sphere. So, it must be that λ is a *trivial loop* in T: it bounds a disk in the surface T; this is more restrictive than being a trivial knot. Without loss of generality, we may assume K meets λ. But then we see that the factor "inside" λ is an unknot; see Figure 11. This is a contradiction. □

Remark 6. It is known that the genus of a (p,q)-torus knot is

$$g = (p-1)(q-1)/2.$$

Thus, there are prime knots of any genus.

Theorem 7. *Let* $K = K_1 \# K_2$. *Then* $g(K) = g(K_1) + g(K_2)$.

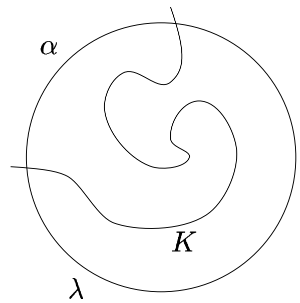

FIGURE 11. Factor is unknotted.

The proof uses the following result due to Neuwirth, which we only illustrate. Let F be a Seifert surface for a knot K. Let λ be a loop in $\mathring{F}$, the interior of F. Suppose that there is no disk $\Delta \subset F$ with $\partial\Delta = \lambda$, but that there is a disk D in the 3-sphere with $\partial D = \lambda$ and $\mathring{D} \cap F = \phi$. Then the surface F can be replaced by another Seifert surface with smaller genus. Figure 12 gives an example showing how to reduce the genus of a Seifert surface of a trefoil.

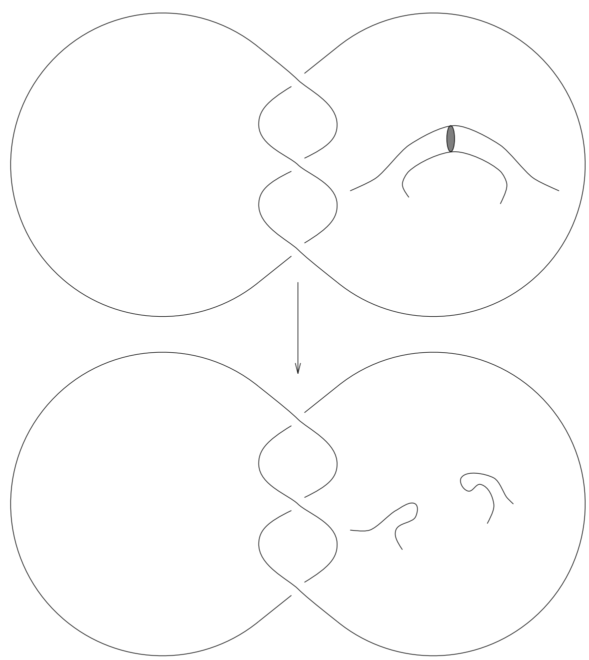

FIGURE 12. Reducing the genus of a Seifert surface.

Proof of Theorem 7. Let $g_1 = g(K_1)$, $g_2 = g(K_2)$, and $g = g(K)$.

Part 1: First we show that $g_1+g_2 \geq g$. Let F_1 and F_2 be Seifert surfaces for K_1 and K_2, respectively. Assume that they have minimal genus and have been triangulated. Figure 13 shows how to construct a Seifert surface F for K by connecting F_1 and F_2 with a strip.

Problem 1. Use the Euler characteristic and formula (1) to show that $g(F) = g_1 + g_2$.

Since we do not know that the surface F has minimal genus we can conclude only that $g_1 + g_2 \geq g$.

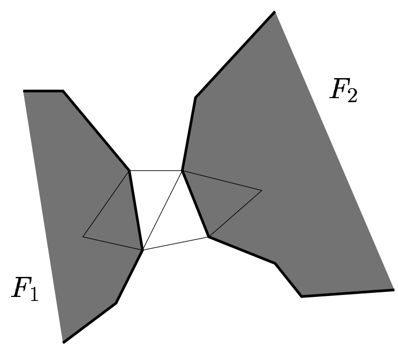

FIGURE 13. Joining two Seifert surfaces.

Part 2: Let F be a Seifert surface for $K = K_1 \# K_2$ with minimal genus g. Let S be a 2-sphere that factors K into K_1 and K_2. We study $F \cap S$, which we take to be transverse. Clearly $F \cap S$ contains an arc that connects the two points of $K \cap S$, but $F \cap S$ may also contain some loops. Let n be the number of loops in $F \cap S$.

Suppose $n = 0$. The sphere S divides the 3-sphere into two open balls whose closures we denote by B_1 and B_2, where $K_i \subset B_i$, for $i = 1, 2$. Let $F_i = F \cap B_i$. Then F_i is a Seifert surface for K_i. Each of the F_i has minimal genus by Part 1. By Problem 1, $g_1 + g_2 = g$.

Now suppose $n > 0$ and assume that the result holds for Seifert surfaces with fewer than n loops. Let λ be a loop in $F \cap S$ that is innermost. Thus there is a disk D in S with boundary λ that does not meet F anywhere except λ. By Neuwirth's result we know that λ must also bound a disk Δ in F. Thus we can form a new surface F' by removing Δ from F, gluing in the disk D, and then pushing off S; see Figure 14. Thus, F' is a Seifert surface for K with minimal genus that has fewer than n loops in $F' \cap S$. □

Theorem 8 (Schubert). *Let K be a nontrivial knot. Then K can be factored into primes and any two prime factorings of K are the same up to order.*

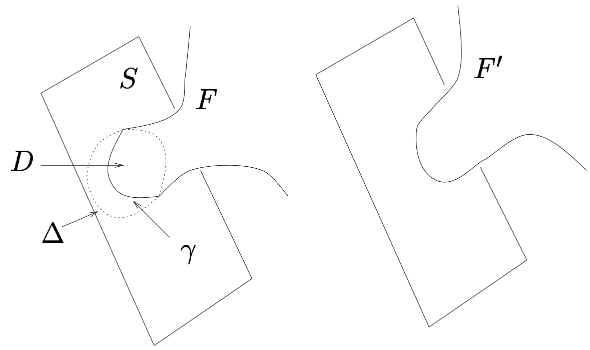

FIGURE 14. Reducing the number of intersection loops.

We leave the proof of the first claim as an exercise. It is essentially a corollary of Theorem 7. Think about the genus of K and of any of its factors.

The proof of uniqueness is accomplished in a series of lemmas. We first define a *decomposing sphere system* for a knot K. This is a rather complicated-looking definition but the example that follows should help make the ideas clear.

Definition 9. Let $\mathbf{S} = \{S_1, \ldots, S_m\}$ be a collection of disjoint spheres in S^3 such that $K \cap S_j$ is two points (and is transverse) for $j = 1, \ldots, m$. We use these spheres to factor K. If each factor is prime we say $\mathbf{S}$ is a *decomposing sphere system* or *dss* for K.

To show that this definition is precise, we show how the factoring is to be done. Let B_j^{in} and B_j^{out} be the respective closures of the two components of $S^3 \backslash S_j$. We also renumber these balls as $B_1, \ldots, B_{2m}$, and let $c(j)$ be the index of the ball $B_{c(j)} = S^3 \backslash \mathring{B}_j$. Now, for each $j = 1, \ldots, m$ choose an arc $\alpha_j \subset S_j$ that joins the two points of $K \cap S_j$. Let $R_j = B_j \setminus \left(\bigcup_{B_i \subset B_j} \mathring{B}_i \right)$. Let $K_j' = K \cap R_j$. Let K_j be the knot formed from the union of the arcs K_j' and the arcs $\{\alpha_i | S_i \subset B_j\}$. We say B_j or R_j *determines the factor* K_j.

Note that different balls can determine the same factors. If fact this must happen if $m > 1$ as there are $2m$ balls and only $m+1$ factors; see Lemma 11. It is therefore convenient to renumber the regions, $(R_1, \ldots, R_{m+1})$, and the corresponding factors, $(K_1, \ldots, K_{m+1})$.

Example 10. Figure 15 shows a dss with three spheres for a knot K with four prime factors. The following list shows which ball determines which factor.

$$
\begin{array}{lll}
B_1^{\text{in}} & \longrightarrow R_1 \supset K_1 & \\
B_1^{\text{out}} & \longrightarrow R_4 \supset K_4 & \text{redundant} \\
B_2^{\text{in}} & \longrightarrow R_2 \supset K_2 & \\
B_2^{\text{out}} & \longrightarrow R_1 \supset K_1 & \\
B_3^{\text{in}} & \longrightarrow R_3 \supset K_3 & \text{redundant} \\
B_3^{\text{out}} & \longrightarrow R_4 \supset K_4 &
\end{array}
$$

We also give, in Figure 16, a set of spheres that do not form a dss for the knot shown. In fact there are three thing wrong with Figure 16. Find them, and then construct a valid dss.

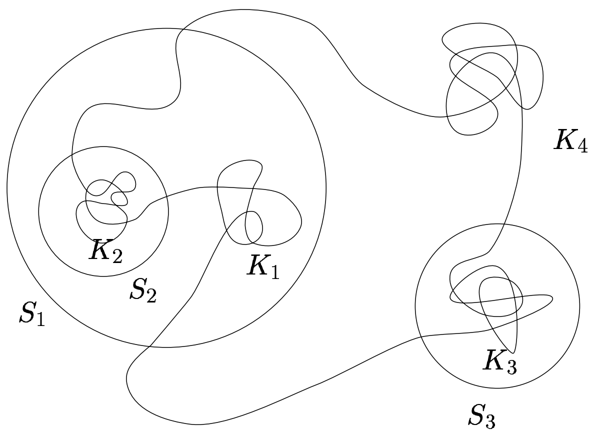

FIGURE 15. A decomposing sphere system.

Lemma 11. *Let* $\mathbf{S}$ *be a dss for* K *with* m *spheres. Then the factoring has* $m+1$ *prime knots:* $K = K_1 \# \cdots \# K_{m+1}$.

Proof. The result is clear for $m = 0$ and $m = 1$. Suppose this is so for all $n < m$. Let B_l be an innermost ball, i.e., B_l contains none of the other balls. Replace the arc $K_l' = K \cap B_l$ with the arc $\alpha_l \subset S_l = \partial B_l$, forming a new knot $\hat{K}$, i.e., $\hat{K} = (K \backslash K_l) \cup \alpha_l$. By Definition 3 we have $K = \hat{K} \# K_l$. Let $\hat{\mathbf{S}} = \mathbf{S} \backslash \{S_l\}$; $\hat{\mathbf{S}}$ is a dss for $\hat{K}$. By the induction hypothesis, $\hat{K}$ is decomposed by $\hat{\mathbf{S}}$ into m factors, $\hat{K} = \hat{K}_1 \# \cdots \# \hat{K}_m$. Thus $\mathbf{S}$ gives the factorization $K = \hat{K}_1 \# \cdots \# \hat{K}_m \# K_l$, which has $m+1$ terms, as required. $\square$

Definition 12. Let $\mathbf{S}$ and $\mathbf{S}'$ be dss's for the knot K. Then $\mathbf{S} \sim \mathbf{S}'$ if they determine the same factorizations of K.

Lemma 13. *Let* $\mathbf{S} = \{S_1, \ldots, S_m\}$ *be a dss for* K*. Let* B_k *be an outermost ball within* B_i*. Then* $B_{c(j)}$ *and* B_i *determine the same knot.*

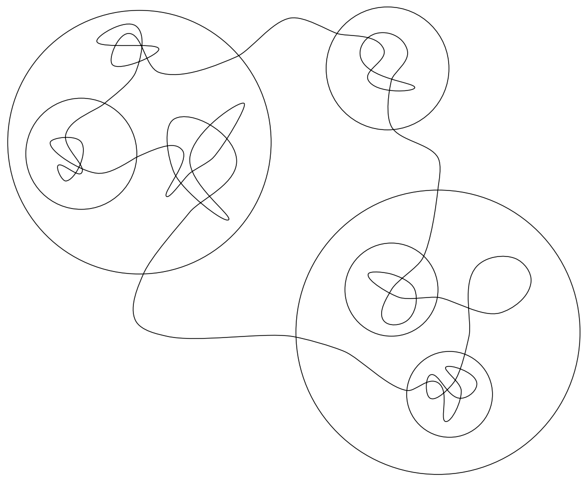

FIGURE 16. Not a valid dss.

We leave the proof to the reader. The point is to show that the associated regions are the same in each case. When we say that B_k is outermost in B_i, we mean that there is no other ball B_q such that $B_k \subset B_q \subset B_i$.

The next lemma gives us an operation that allows us to change one dss into an equivalent one.

Lemma 14 (Switching Move)**.** *Let $\mathbf{S} = \{S_1, \ldots, S_m\}$ be a dss for K. Let $\hat{S}_j$ be another 2-sphere in S^3, disjoint from each $S_i \in \mathbf{S}$, that bounds $\hat{B}_j$. Let $\hat{\mathbf{S}} = (\mathbf{S} \backslash \{S_j\}) \cup \{\hat{S}_j\}$. Suppose B_j is outermost in $\hat{B}_j$ and that $\hat{B}_j$ determines the same knot K_j (relative to $\hat{\mathbf{S}}$) as B_j does (relative to $\mathbf{S}$). Then $\hat{\mathbf{S}} \sim \mathbf{S}$.*

Proof. We suggest that the reader draw the extra sphere, $\hat{S}_j$, in Figure 15 and refer to it while reading the proof.

By hypothesis B_j and $\hat{B}_j$ determine the same factor K_j. Hence the region $\hat{R}_j = \hat{B}_j \backslash \bigcup_{B_i \subset \hat{B}_j} \overset{\circ}{B}_i$ determines an unknot. Thus, $B_{c(j)}$ and $\hat{B}_{c(j)}$ determine the same factor $K_{c(j)}$.

Now let $i \neq j$ and suppose $B_i \subset \hat{B}_j$; the other case, $B_i \subset \hat{B}_{c(j)}$, is similar. If B_i is not outermost within $\hat{B}_j$, i.e., there is a k such that $B_i \subset B_k \subset \hat{B}_j$, then both B_i and $B_{c(i)}$ determine the same factors, K_i and $K_{c(i)}$, with respect to both systems $\mathbf{S}$ and $\hat{\mathbf{S}}$.

If B_i is outermost in $\hat{B}_j$ then B_i still determines K_i in both systems. But $B_{c(i)}$ determines K_j with respect to $\hat{\mathbf{S}}$. But Lemma 13 ensures that $K_{c(i)} = K_{c(j)}$, which is determined by $\hat{B}_{c(j)}$. Thus, the factorizations are the same and so $\mathbf{S} \sim \hat{\mathbf{S}}$. $\square$

Lemma 15. *Let $\mathbf{S}$ and $\mathbf{S}'$ be dss's for K. Let B_i (or B_i') be innermost with respect to $\mathbf{S}$ and $\mathbf{S}'$. Suppose $S_i \cap \mathbf{S}' = \phi$ (or $S_i' \cap \mathbf{S} = \phi$). Then $\mathbf{S} \sim \mathbf{S}'$.*

Note. We abuse notation a bit by regarding $\mathbf{S}$ and $\mathbf{S}'$ both as finite sets of spheres and as the union of the sets of points that form the spheres.

Proof. Use induction on $m+m'$. The theorem is clear for $m+m'=0$, since then K is prime and $\mathbf{S}=\mathbf{S}'=\phi$. Suppose that $m+m'=M$ and that the theorem is true for $m+m'<M$.

The sphere $S_i=\partial B_i$ is outermost within some S'_j with respect to the $\mathcal{S}'$ dss; draw some pictures to see this. In $\mathbf{S}'$, replace S'_j with S_i, call this $\mathbf{S}''$. Lemma 14 then implies that $\mathbf{S}''\sim\mathbf{S}'$.

Let $\hat{K}=(K\backslash B_i)\cup\alpha_i$. Then $K=\hat{K}\#K_i$. Let $\hat{\mathbf{S}}=\mathbf{S}\backslash S_i$, and $\hat{\mathbf{S}}''=\mathbf{S}''\backslash S_i$. These are dss's for $\hat{K}$. By the induction hypothesis $\hat{\mathbf{S}}\sim\hat{\mathbf{S}}''$, and thus they give the same factors $\hat{K}=\hat{K}_1\#\cdots\#\hat{K}_m$. Then by Definition 3, $\mathbf{S}$ and $\mathbf{S}''$ give $K=\hat{K}_1\#\cdots\#\hat{K}_m\#K_i$. Hence, $\mathbf{S}\sim\mathbf{S}''\sim\mathbf{S}'$. □

Theorem 16. *Let* $\mathbf{S}$ *and* $\mathbf{S}'$ *be dss's for* K. *Then* $\mathbf{S}\sim\mathbf{S}'$.

Proof. If $\mathbf{S}\cap\mathbf{S}'$ has zero components use Lemma 15. Let n be the number of components of $\mathbf{S}\cap\mathbf{S}'$ and assume the theorem is true if there are fewer components.

Let B'_j be an innermost ball with respect to $\mathbf{S}$ and $\mathbf{S}'$. Let λ' be an innermost curve of $S'_j\cap\mathbf{S}$; if $S'_j\cap\mathbf{S}=\phi$ use Lemma 15. Thus there exists a disk $D'\subset S'_j$ with $\partial D'=\lambda'$ and $\overset{\circ}{D'}\cap\mathbf{S}=\phi$.

Now $\lambda'\subset S_i$ for some $S_i\in\mathbf{S}$. Either D' is in B_i^{in} or B_i^{out}, so just write $D'\subset B_i$. Because λ' is innermost, $\overset{\circ}{D'}\subset\overset{\circ}{B_i}$, so D' divides B_i into two balls, B_{i1} and B_{i2}. One of these determines the factor K_i and the other determines an unknot.

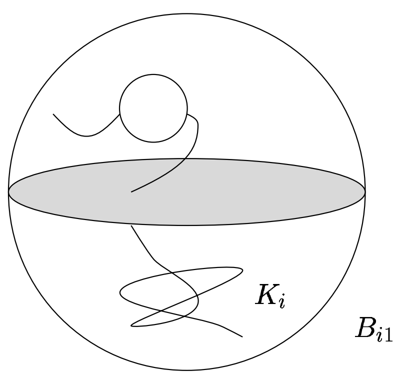

FIGURE 17. Replace B_i with B_{i1}.

Without loss of generality, let B_{i1} determine K_i. Let S^* be the boundary sphere of an slightly shrunken copy of B_{i1} that is transverse to the other spheres. Replace S_i with S^* forming $\mathbf{S}^*=(\mathbf{S}\backslash S_i)\cup S^*$. By Lemma 14, $\mathbf{S}^*\sim\mathbf{S}$. But $S^*\cap S'_j$ does not contain λ', so $\mathbf{S}^*\cap\mathbf{S}'$ has fewer components. □

4. Application: Knots on Templates

A template is like a surface with boundary but we allow "branching". That is, sheets can merge together along *branch lines*. Figure 18 shows what is perhaps the simplest example of a template, known as the *Lorenz template*. A template also has a *flow* on it. The flow on the Lorenz template goes downward from the branch line and then wraps around along the two bands and then comes back in to the branch line again. The flow of a template contains closed orbits, which may be regarded as knots. We sometimes abuse notation and use the name of a template to denote the set of knot types on it. For more on template theory see [10].

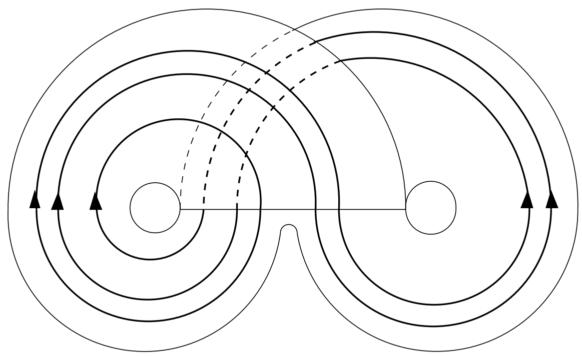

Figure 18. The Lorenz template with trefoil orbit.

The study of templates grew out of a desire to understand flows associated with the solutions of differential equations in three dimensions. The Lorenz template was proposed as a model of the Lorenz system of differential equations. It is conjectured that every knotted closed orbit in the Lorenz system is contained on the Lorenz template. Thus, it is natural to ask just what knots appear on the Lorenz template. In 1983 Williams showed that the Lorenz knots (i.e., knots on the Lorenz template) are prime. Earlier, Birman and Williams [3] conjectured that for any template $\mathcal{T}$ there is an upper bound on the number of prime factors for closed orbits in $\mathcal{T}$. For the Lorenz template the bound is one. They proposed this bound as a measure of the complexity of the system being modeled.

We call a template *Lorenz-like* if it has two bands, which may have twists in them. We denote by $\mathcal{L}(m,n)$ the template with m half twists in the left band and n half twists in the right band; see Figure 19. Thus the Lorenz template is $\mathcal{L}(0,0)$, which at times we denote by $\mathcal{L}$. The Lorenz-like templates model other differential equations. What can we say about the prime factors of Lorenz-like knots?

Williams showed in [18] for $n \geq 0$ that $\mathcal{L}(0,n)$ has only prime knots. However, he found an example of a composite knot on $\mathcal{L}(0,-1)$, which is shown in Figure 20. As a graduate student working under Williams, I studied

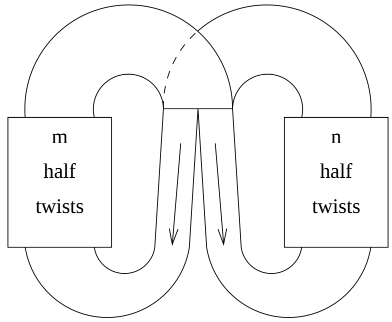

FIGURE 19. Lorenz-like templates.

$\mathcal{L}(0,-2)$ for a time. At one point I was *sure* I had a proof that all its knots are prime. But no, I eventually found that it, too, had composite knots. One is shown in Figure 21; it is the connected sum of two trefoils [14].

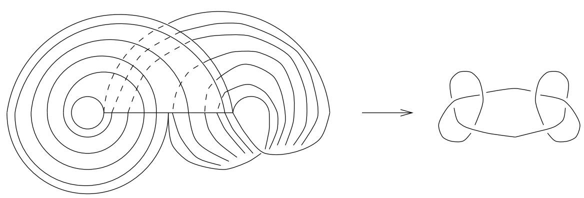

FIGURE 20. A composite knot on $\mathcal{L}(0,-1)$.

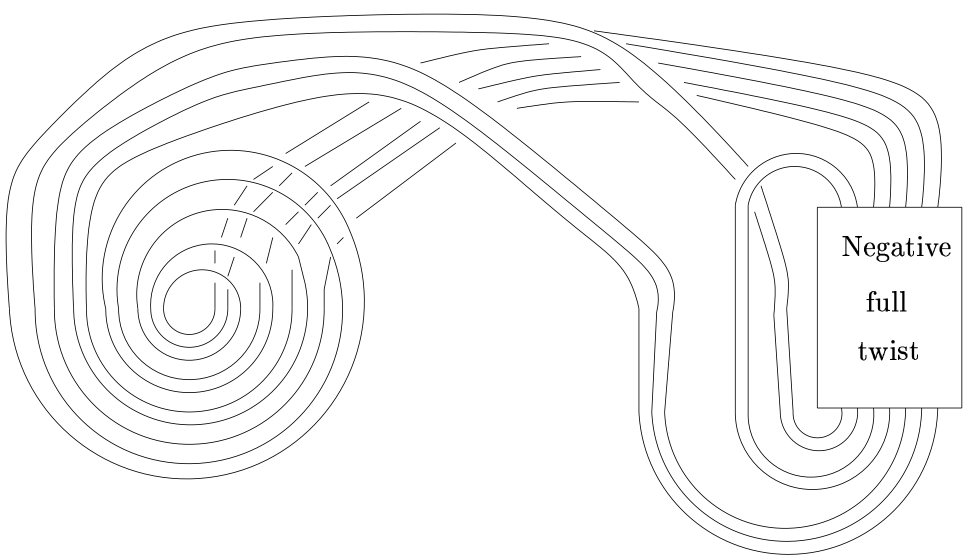

FIGURE 21. A composite knot on $\mathcal{L}(0,-2)$.

We now show that for all $n < 0$ the templates $\mathcal{L}(0,n)$ contain composite knots [14]. The proof of the next lemma is given in Figure 22. The boxes mean, "insert the indicated number of half twists."

Lemma 17. *As sets of knots, $\mathcal{L}(0,n) \subset \mathcal{L}(0,n-2)$.*

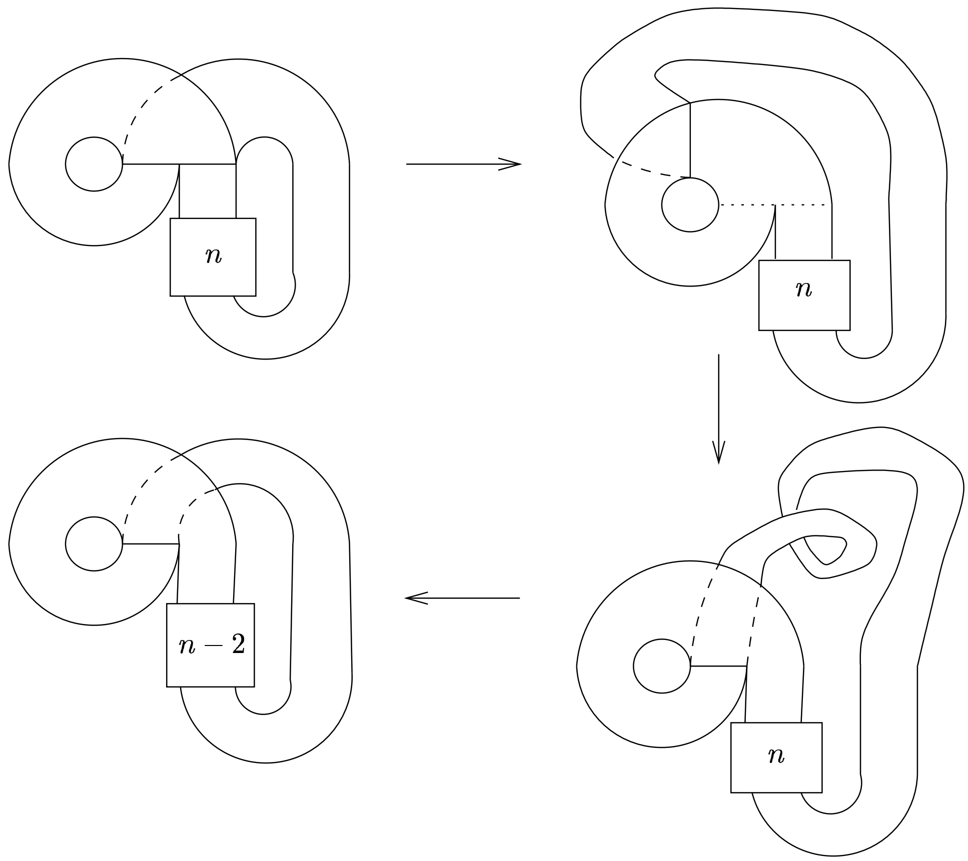

FIGURE 22. Proof!

Theorem 18. *For $n < 0$, the template $\mathcal{L}(0,n)$ has composite knots.*

Proof. Use Lemma 17 and the examples of Figures 20 and 21. □

Problem 2. Show that $\mathcal{L}(0,-4) \subset \mathcal{L}(0,-1)$. Hint: cut $\mathcal{L}(0,-1)$ along the closed orbit that makes just one trip around on the twisted branch. This gives a new template, which contains the same knots as $\mathcal{L}(0,-1)$. (The linking of the new boundary orbit with the other orbits is different, but the knots are unchanged.) Now, show that $\mathcal{L}(0,-4)$ fits inside the "surgeried" $\mathcal{L}(0,-1)$; see [14]. [Remark: My 7th grade math teacher showed us what happens when you cut a Möbius band down the middle. This problem shows that you actually do use what you learn in school!]

It is now natural to ask if we can find a bound on the number of prime factors of knots on the templates $\mathcal{L}(0, n)$ for negative n. Thus far all our examples have had just two prime factors. Before answering, we state a theorem of this type but for a different template, $\mathcal{H}$, shown in Figure 23a.

Theorem 19. *The knots in the template $\mathcal{H}$ have at most two prime factors.*

The proof is in [16]. There is some evidence for the following.

Conjecture 20. For m and n both greater than zero $\mathcal{L}(m, n)$ has at most two prime factors.

On the other hand, if we switch the crossings of one of the loops in $\mathcal{H}$, producing the template $\mathcal{U}$ in Figure 23b, we get a very different result, see [15].

Theorem 21. *For every $n > 0$ there is a knot on the template $\mathcal{U}$ with n prime factors.*

Rob Ghrist was able to push this result considerably further [9].

Theorem 22. *The template $\mathcal{U}$ contains all knots and all links!*

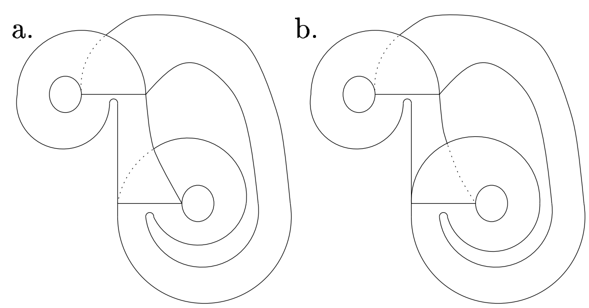

FIGURE 23. The templates $\mathcal{H}$ and $\mathcal{U}$.

A template is *universal* if it contains all knots and all links; there are, however, no known examples of templates that contain all knots that do not contain all links. It is shown in [16] that $\mathcal{U} \subset \mathcal{L}(0, -2)$. Thus, for $n < 0$ it is easy to show that the $\mathcal{L}(0, n)$ templates are universal. Just apply Problem 2 and Lemma 17.

We have seen that the Birman-Williams conjecture has failed. However, the purpose behind making a conjecture is not to show that mathematicians have crystal balls, but to stimulate research. In this sense the Birman-Williams conjecture has succeeded.

There is a common feature in the examples where the Birman-Williams conjecture holds. In each case, all the crossings on the template are of the same type. Crossings of an oriented knot (or link) come in two types, left-handed (or positive) and right-handed (or negative). In Figure 24 if you point your left thumb in the direction of the over crossing arc of the left crossing with the palm of hand down facing the page, your fingers, if you curl them under your palm, point in the direction of the under-crossing arc. For the crossing on the right, you can do this with your right hand. No matter how it is rotated or even flipped over, the handedness of a crossing is the same. Thus, crossings are either left-handed or right-handed.

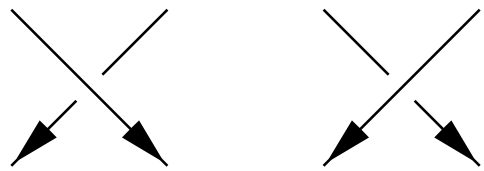

FIGURE 24. Crossing types.

In all of the templates we examined, where there is a bound on the number of prime factors, all of the crossings are of the same type. We call such templates *positive templates*, though it would make more sense to call them *uniform templates*. None of the universal templates are positive. And, every template where a bound on the number of prime factors has been established is positive. Perhaps, this is just a coincidence. However, in Williams' proof that Lorenz knots are prime, the positivity of the templates is used, though not explicitly. In trying to construct a proof of Theorem 19, one of the more important revelations came when I saw that Williams really was, perhaps unknowingly, using the positiveness of the templates. These observations form the basis for the following conjecture.

Conjecture 23. Let $\mathcal{T}$ be a *positive template*, meaning that all the crossings of the orbits are the same type. Then there is an integer n such all the knots on $\mathcal{T}$ have n or fewer prime factors.

5. Factoring Positive Braids

A *braid presentation* is a knot diagram in which any arc in the knot goes counter-clockwise with respect to a common central point called the *braid axis*; Figure 25 gives two examples. It has long been known that every knot has a braid presentation that is easy to construct; see [4]. If a knot k has a braid presentation in which all of the crossings are of the same type, then we say that k is a *positive braid*. Peter Cromwell has developed a simple method for factoring positive braids [6], which we now describe.

Note: There exist knots with diagrams with all positive crossings but for which any braid presentation has mixed crossings. In fact, there is an example of such a knot with just five crossings. The problem of proving that a knot cannot be presented as a positive braid is nontrivial; see [17].

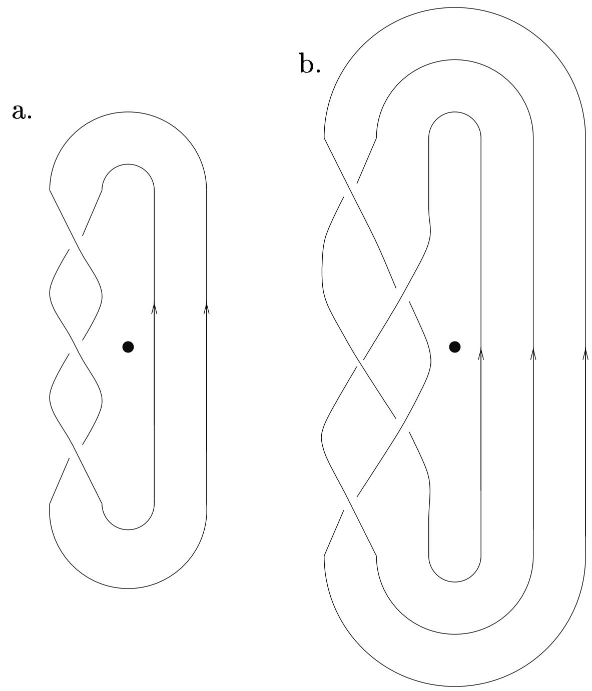

FIGURE 25. Two braided knots.

Every braid can be described by a *braid word* as follows. If the braid has n strands, number the gaps between them $1, \ldots, n-1$. Now follow the braid counter-clockwise around the braid axis and list the number of the gap in which each crossing occurs. But, if the crossing is negative put a bar over the number. Thus, the braid word of the braid in Figure 25a is 111 while that of the braid in Figure 25b is $1\bar{2}1\bar{2}$.

Cromwell's procedure goes as follows. Assuming we have a positive braid diagram, write down its word. If every number appears more than once then the knot is prime. If a number appears only once, delete that crossing and thus factor the knot. It is easy to detect if a positive braid is the unknot. (Figure out a clear criterion for this.) If a factor is trivial throw it away; if both are trivial the original braid was the unknot. We repeat the procedure with each nontrivial factor until we have only prime factors left.

Problem 3. Show that Cromwell's produce can fail for braids with mixed crossing types.

Problem 4. Use Cromwell's procedure to give an easy proof of Williams' theorem that Lorenz knots are prime.

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