ELEMENTARY SURGERY ALONG A TORUS KNOT

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In this paper a classification of the manifolds obtained by a \((p, q)\) surgery along an \((r, s)\) torus knot is given. If \(|\sigma| = |rsp + q| \neq 0\), then the manifold is a Seifert manifold, singularly fibered by simple closed curves over the 2-sphere with singularities of types \(\alpha_1 = s\), \(\alpha_2 = r\), and \(\alpha_3 = |\sigma|\). If \(|\sigma| = 1\), then there are only two singular fibers of types \(\alpha_1 = s\), \(\alpha_2 = r\), and the manifold is a lens space \(L(|q|, ps^t)\). If \(|\sigma| = 0\), then the manifold is not singularly fibered but is the connected sum of two lens spaces \(L(r, s) \# L(s, r)\). It is also shown that the torus knots are the only knots whose complements can be singularly fibered.

1. DEFINITIONS. A knot \(K\) is a polygonal simple closed curve in \(S^3\) which does not bound a disk in \(S^3\). A solid torus \(T\) is a 3-manifold homeomorphic to \(S^1 \times D^2\). The boundary of \(T\) is a torus, a 2-manifold homeomorphic to \(S^1 \times S^1\). A meridian of \(T\) is a simple closed curve on \(\partial T\) which bounds a disk in \(T\) but is not homologous to zero on \(\partial T\). A meridional disk of \(T\) is a disk \(D\) in \(T\) such that \(D \cap \partial T = \partial D\) and \(\partial D\) is a meridian of \(T\). A longitude of \(T\) is a simple closed curve on \(\partial T\) which is transverse to a meridian of \(T\) and is null-homologous in \(\overline{S^3 - T}\). A meridional longitude pair for \(T\) is an ordered pair \((M, L)\) of curves such that \(M\) is a meridian of \(T\) and \(L\) is a longitude of \(T\) transverse to \(M\). \(\pi_1(\partial T) \cong \mathbb{Z} \times \mathbb{Z}\) with generators \(M\) and \(L\). \(qM + pL\) is the homotopy class of a simple closed curve on \(\partial T\) if and only if \(p\) and \(q\) are relatively prime.

A torus knot of type \((r, s)\), denoted \(K(r, s)\), is defined as follows. Let \(T\) be a standardly embedded solid torus in \(S^3\), that is, \(T\) is isotopic to a regular neighborhood of a polygonal curve in the \(x-y\) plane. Then \(\overline{S^3 - T}\) is a solid torus. Let \(J_1\) and \(J_2\) be oriented simple closed curves on \(\partial T\) such that \(J_1\) bounds a disk in \(T\) and \(J_2\) bounds a disk in \(\overline{S^3 - T}\), that is \(J_1\) is meridional and \(J_2\) is longitudinal. Identifying \(J_1\) with \((1, 0)\) and \(J_2\) with \((0, 1)\), let \(r\) and \(s\) be relatively prime integers, \(r > s > 0\), and let \(K(r, s)\) be a simple closed curve in \((r, s)\). Then \(K(r, s)\) is a torus knot of type \((r, s)\). By Van Kampen’s theorem \(\pi_1(S^3 - K(r, s)) \cong \langle a, b \mid a^r = b^s \rangle\).

A space is a lens space if it contains a solid torus such that the closure of its complement is also a solid torus. Hence one way to view a lens space is as the space obtained by identifying two solid tori by a homeomorphism on the boundary.

Basic Construction: Elementary surgery along a knot. Let \(N\)
be a regular neighborhood of $K$, $M$ an oriented meridional curve for $N$ on $\partial N$, and $L$ an oriented curve on $\partial N$ which is transverse to $M$ and bounds an orientable surface in $S^3-N$. Consider $M \cap L$ as a base point for $\pi_1(S^3-N)$. Let $T$ be a solid torus and $h : T \to N$ be a homomorphism. Then $S^3 \cong S^3-N \cup_{h|\partial T} T$. Now let $h : \partial T \to \partial T$ be a homeomorphism with the property that $h^{-1} \cdot h : \partial T \to \partial T$ does not extend to a homeomorphism of $T$ onto $T$. Let $\mathcal{M}^3 = \bar{S^3-N} \cup_{h|\partial T} T$, then we say $\mathcal{M}^3$ is obtained from $S^3$ by performing an elementary surgery along $K$. The fundamental group of $\mathcal{M}^3$ is obtained by adjoining a relation of the form $L^p = M^q$ where (1) $pL-qM$ is the image under $h_i$ of the boundary of a meridional disk of $T$, (2) $p$ and $q$ are relatively prime, (3) $p \neq 0$ since we have performed an elementary surgery and we may assume that $p > 0$ since $\mathcal{M}^3 (p, q) \cong \mathcal{M}^3(-p, -q)$. If $K$ is unknotted, then an elementary surgery along $K$ will yield a lens space, since the complement of the interior of a regular neighborhood of $K$ is a solid torus and the effect of the surgery is a manifold which can be obtained by identifying two solid tori along their boundaries.

A solid torus fibered by $u$, $v$, denoted by $sT^u(v/u)$, is gotten from $D^2 \times I$ by rotating the top $2\pi v/u$ where $(u, v) = 1$, $0 \leq v \leq u/2$, and then identifying top and bottom. A fiber is denoted by $F$. A cross-circle $Q$ is a simple closed curve meeting each $F$ in one point. A singularly fibered manifold $\mathcal{M}^3$, in the sense of Seifert, is a topological 3-manifold partitioned into subsets homeomorphic to $S^1$, the fibers, such that each fiber has a closed neighborhood preserving homeomorphic to some $sT^u(v/u)$.

$\mathcal{M}^3$ is obtained as follows. Let $B$ be a sphere with $g > 0$ handles $(k$-crosscaps), cut $B$ along a set of loops based at $x_0$ to get a $4g$-gon $(2k$-gon) $P$ with sides $A_1^{-1}B_1^{-1}A_1B_1 \cdots A_k^{-1}B_k^{-1}A_kB_k(C_iC_i' \cdots C_iC_i')$ to be identified in pairs, and remove a disk $D_0$ around $x_0$ to get $\bar{P}$. $\bar{P} \times S^1$ is a 3-manifold on which we make some identifications. Let $\chi : \pi_1(B, x_0) \to \text{Aut} \pi_1(S^1) \cong Z_g$. Let $x$ and $x'$ be points on the edges of $\bar{P}$ which are identified in $B$, and let $x$ be a path formed by the line segments $x_0x$, $x'x_0$. $x$ is a loop in $B$ based at $x_0$. Choose a base point preserving homeomorphism $x \times S^1 \to x' \times S^1$ which induces $\chi([x]) : \pi_1(S^1) \to \pi_1(S^1)$.
Identifying pairs of fibers over the edges of \( P \) by this homeomorphism gives a manifold \( \mathcal{M}_0^3 \) with boundary \( \partial D_0 \times S^1 \). Now suppose \( \partial D_0 \times S^1 \) is trivially fibered by circles \( \omega \) such that \( [\omega] = Q_0 + bF \in \pi_1(\partial D_0 \times S^1) \) where \( Q_0 \) generates \( \pi_1(\partial D_0) \) and \( F \) generates \( \pi_1(S^1) \). We close \( \mathcal{M}_0^3 \) with a solid torus \( \mathcal{N}(F) \) by a homeomorphism \( h: \partial \mathcal{N}(F) \to \partial \mathcal{M}_0^3 \) such that for \( M \) a meridian of \( \mathcal{N}(F) \), \( M \sim Q_0 + bF \), to obtain \( \mathcal{M}_0^3 = \mathcal{M}_0^3 \cup h\mathcal{N}(F) \). \( \chi \) is called the characteristic and \( b \) the obstruction term. By removing the fibers over open disks \( D_i, i = 1, \ldots, n \) in \( B \) we obtain \( \mathcal{M}_i^3 \) with \( n \) boundary components \( \partial D_i \times S^1 \). Suppose \( \partial D_i \times S^1 \) is trivially fibered by circles \( \omega_i \) such that \( [\omega_i] = \alpha_i Q_i + \beta_i F_i \), where \( Q_i \) generates \( \pi_1(\partial D_i) \), \( F_i \) generates \( \pi_1(S^1) \), \( (\alpha_i, \beta_i) = 1 \), and \( 0 < \alpha_i < \beta_i \). By replacing the solid tori removed by \( \mathcal{N}(F_i) \) such that for \( M_i \) a meridian of \( \mathcal{N}(F_i) \), \( M_i \sim \alpha_i Q_i + \beta_i F_i \), we obtain a closed manifold fibered by \( S^1 \) over \( B \). \( F_i \) is a singular fiber of type \( \alpha_i \) and has a trivial product neighborhood if and only if \( \alpha_i = \pm 1 \).

The fundamental group of \( \mathcal{M}^3 \) is given in terms of the \( (\alpha_i, \beta_i) \), \( b \), and \( \chi \) by Van Kampen’s theorem.

\[
\pi_1(\mathcal{M}^3) = (A_i, B_i, (C_i), Q_0, Q_1, \ldots, Q_n, F) \prod_{i=1}^{n} [A_i, B_i]Q_1 \cdots Q_nQ_0 = 1
\]

\[
\left( \prod_{i=1}^{n} C_iQ_1 \cdots Q_nQ_0 = 1 \right)\]

\[
A_i^{-1}FA_i = F^{\alpha_i(A_i)}, B_i^{-1}FB_i = F^{\beta_i(B_i)}, (C_i^{-1}FC_i = F^{\gamma_i(C_i)}),
\]

\[
[F, Q_i] = 1, Q_0F^\alpha = 1, Q_i^\beta F_i = 1.
\]

2. Fibering the complement of a knot.

**Theorem 2.** The complement of a knot \( K \) can be singularly fibered in the sense of Seifert if and only if \( K \) is a torus knot.

**Proof.** Let \( K(r, s) \) be a torus knot lying on a standardly embedded torus in \( S^3 \). The diagram illustrates the case \( r = 3, s = 2 \).

We have a fibering of \( S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1 \} \) given by \( (z_1, z_2) = (z_\lambda, z_\bar{\lambda}) \) for \( \lambda \in S^1 \) (that is, a partition of \( S^1 \) into orbits \( S^1 \)) over \( B = S^2 \) with the unit circle as a singular fiber of type \( \alpha_i = s \) and the \( z \)-axis as a singular fiber of type \( \alpha_i = r \). Each nonsingular fiber is an \((r, s)\) torus knot. If we remove a regular neighborhood of the torus knot, we have \( S^3 - \mathcal{N}(K) \) singularly fibered.

Suppose \( \mathcal{M}^3 = S^3 - \mathcal{N}(K) \) is singularly fibered. Let \( F \sim mL + nM \) where \( F \) is a fiber on \( \delta \mathcal{M}^3 \) and \( (M, L) \) is a meridian-longitude pair for \( \mathcal{N}(K) \). If \( m \neq 0 \), then \( M \not\sim F \) on \( \delta \mathcal{M}^3 \). Hence, there exists a singularly fibered solid torus \( sT^3(v/u) \) and a fiber preserving homeomorphism \( h: \delta sT^3 \to \delta \mathcal{M}^3 \) which takes a meridian of \( sT^3 \) to \( M \) by Lemma 6 of Seifert [4]. Hence, \( \mathcal{M}^3 \cup h sT^3 = S^3 \) and \( S^3 \) is singularly fibered with \( K \) as a fiber of multiplicity \( m \).
If $m \neq \pm 1$, then $K$ is a singular fiber and hence unknotted. If $m = \pm 1$, then $K$ is an ordinary fiber and hence a torus knot. If $m = 0$, $F \sim nM$ where $M$ generates $H_1(S^3 - N^-(K)) \cong \mathbb{Z}$. But if $\mathcal{M} = S^3 - N^-(K)$ is singularly fibered, then

$$\pi_1(\mathcal{M}) = (A_i, B_i, (C_i), Q_0, Q_1, \ldots, Q_n, F| \prod_{i=1}^n [A_i, B_i]Q_i \cdots Q_nQ_0 = 1$$

$$\prod_{i=1}^n C_iQ_i \cdots Q_nQ_0 = 1$$

$$A_i^{-1}FA_i = F^{\alpha_i(A_i)}, B_i^{-1}FB_i = F^{\alpha_i(B_i)}, (C_i^{-1}FC_i = F^{\alpha_i(C_i)})$$

$$[F, Q_i] = 1, Q_iF^{\beta_i} = 1, Q_i^xF^{\beta_i} = 1, 1 \leq i \leq n - 1$$

$$\simeq (A_i, B_i, (C_i), Q_i, \ldots, Q_{n-1}, F | A_i^{-1}FA_i = F^{\alpha_i(A_i)}, B_i^{-1}FB_i = F^{\alpha_i(B_i)},$$

$$(C_i^{-1}FC_i = F^{\alpha_i(C_i)})$$

$$[F, Q_i] = 1, Q_i^xF^{\beta_i} = 1, 1 \leq i \leq n - 1).$$

Abelianizing, we see that $g = 0$ ($k = 0$). Setting $F = 1$, we see that $i = 1$ unless $n = \pm 1$ in which case $\alpha_i = \pm 1$, a contradiction. Hence $\pi_1(\mathcal{M}) = (Q_i, F|Q_i^xF^{\beta_i} = 1)$ and $K$ is a torus knot of type $(\alpha_i, \beta_i)$.

**NOTE:** Theorem 2 can also be proved with results from [1] and [5].

3. The fibered manifolds obtained by elementary surgery along a torus knot.

**PROPOSITION 3.1.** If an elementary surgery of type $(p, q)$ is per-
formed along $K(r,s)$ and $|\sigma| = |rsp + q| \neq 0$, then the manifold obtained is singularly fibered with fibers of multiplicities $\alpha_i = s, \alpha_2 = r$, and $\alpha_3 = |\sigma| = |rsp + q|$.

**Proof.** In performing the surgery, we remove a fiber neighborhood of a nonsingular fiber $K$ to obtain $S^3 - \mathcal{N}(K)$ and then close $S^3 - \mathcal{N}(K)$ with $sT^3$ such that $M' \sim pL - qM$ where $M'$ is a meridian of $sT^3$, $L$ is a longitude of $\mathcal{N}(K)$, and $M$ is a meridian of $\mathcal{N}(K)$. If $F$ is a fiber on $\partial \mathcal{N}(K)$, then $sT^3$ contains a singular fiber of multiplicity $|\sigma|$. If $|\sigma| \neq 0$ or $\sigma = 0$, the 3-manifold obtained is a Seifert fiber space with three singular fibers of multiplicities $\alpha_i = s, \alpha_2 = r$, and $\alpha_3 = |\sigma|$. The space is topologically a product of a disk with 3 holes and $S^3$ if we remove regular neighborhoods of the $z$-axis, unit circle, $K(r,s)$, and an additional nonsingular fiber. If $\alpha_3 = |\sigma| = 1$, $u = 1$ and $v = 0$. The $sT^3$ added is nonsingularly fibered, so the resultant manifold has only two nonsingular fibers of types $\alpha_1 = s$ and $\alpha_2 = r$.

Assuming a given fixed orientation on $\mathcal{H}(p,q)$, we can determine the $\beta_i$ and the obstruction term $b$ in terms of $p$. $H_1(\mathcal{H}(p,q))$ is cyclic of order $\beta_1 \alpha_1 \alpha_2 \alpha_3 + \beta_2 \alpha_1 \alpha_3 + \alpha_1 \alpha_2 \beta_2 > 0$ ($\beta_1 \alpha_1 + \beta_2 \alpha_2 + \alpha_1 \beta_2$ for $|\sigma| = 1$); on the other hand $H_1(\mathcal{H}(p,q))$ is cyclic of order $|q| = rsp \mp \sigma$. Equating $\beta_1 \alpha_1 \alpha_3 + \beta_2 \alpha_1 \alpha_3 + \alpha_1 \beta_2 \alpha_3 + \alpha_1 \beta_2 \alpha_3$ ($\beta_1 \alpha_1 + \beta_2 \alpha_2 + \alpha_1 \beta_2$ for $|\sigma| = 1$) and $q = rsp \mp \sigma$, we can solve for the $\beta_i$ and $b$. For example, if $(r,s) = (3,2)$ and $\sigma = 5$, then the Seifert manifolds obtained are given by the following symbols:

- $(\mathcal{O}, \alpha, 0 \mid p-6/5; 2,1; 3,1; 5,1)$ if $p = 1 \pmod{5}$
- $(\mathcal{O}, \alpha, 0 \mid p-7/5; 2,1; 3,1; 5,2)$ if $p = 2 \pmod{5}$
- $(\mathcal{O}, \alpha, 0 \mid p-8/5; 2,1; 3,1; 5,3)$ if $p = 3 \pmod{5}$
- $(\mathcal{O}, \alpha, 0 \mid p-9/5; 2,1; 3,1; 5,4)$ if $p = 4 \pmod{5}$.

If $|\sigma| = 1$, then the manifold is a lens space $L(|q|, x)$. The Seifert invariants do not determine $x$; we determine $x$ in the next proposition.

**Proposition 3.2.** If an elementary surgery of type $(p,q)$ is performed along $K(r,s)$ and $|\sigma| = |rsp + q| = 1$, then the manifold is a lens space $L(|q|, ps)$.

**Proof.** Let $T_i$ be a standardly embedded torus in $S^3$ as shown below and let $T_i$ be $S^3 - T_i$. Let $(M_i, L_i)$ be a standard meridian-
longitude pair for $T_1$, $(M_2, L_2) = (L_1, M_1)$ for $T_2$. $K \sim F \sim rM_1 + sL_1$.

We remove $\mathcal{N}(K)$ so that $T_3$ is still a solid torus and replace it with $sT^3$ such that $M' \sim pL - qM \sim pF \mp M (\sigma = \pm 1)$ and so $L' \sim F$. $sT^3UT_1$ is a solid torus $T_3$ ($sT^3 \cap T_1 \simeq S^1 \times I$) since a longitude of $sT^3$, $L' \sim F$. Let $M_3$ be a meridian of $T_3$. We want to determine $x$ such that $M_3 \sim |q|L_2 + xM_2$.

Now $M' \sim pF \mp M \sim p(rM_i + sL_i) \mp M = prM_i + psL_i \mp M$
also $M_3 \sim L_1 - rM$, $L_3 \sim M_i + sM$
and $M_3 \sim M_i \mp sM' \sim M_i \mp s(prM_i + psL_i \mp M) = (1 \mp rsp)M_i$
$\mp ps^2L_i + sM \sim (1 \mp rsp) (L_2 - sM) \mp ps^2(M_i + rM) + sM$
$= (1 \mp rsp)L_2 - sM \pm rs^2pM \mp ps^2M_2 \mp rs^2pM + sM$
$= |q|L_2 \mp ps^2M_2$
so we have $L(|q|, ps^2)$. The diagrams illustrate the case $r = 3$, $s = 2$, $\sigma = 1$, $q = -(2)(3) + 1 = -5$, and $x = -2(2)$.

Remark. Distinct surgeries along a given torus knot yield distinct lens spaces; however, the same lens space may be obtained by surgering different torus knots. For example, a $(2,11)$ surgery on $K(3, 2)$ gives $L(11, 8)$, a $(1,11)$ surgery on $K(5, 2)$ gives $L(11, 4)$ which is homeomorphic to $L(11, 8)$, but a $(1,11)$ surgery on $K(4, 3)$ gives $L(11, 9)$ which is not homeomorphic to $L(11, 8)$.

4. The nonfibered, nonprime manifolds.

**Proposition 4.** If an elementary surgery of type $(p, q)$ is performed along $K(r, s)$ and $|\sigma| = |rsp + q| = 0$, then the manifold obtained is the connected sum of two lens spaces $L(r, s)\#L(s, r)$ and is not singularly fibered.

**Proof.** If $|\sigma| = |rsp + q| = 0$, then $p = 1$, since $p$ and $q$ are relatively prime, $p > 0$, and $r > s > 0$. By Kneser's conjecture the manifold obtained is a connected sum since the fundamental group is a free product $\pi_1(\mathcal{N}(p, q)) \cong (a, b|a^r = b^s, a^r = 1)$.

Let $S^3$ be the union of two solid tori $T_1$ and $T_2$, $(M_1, L_1)$ a standard meridian-longitude pair for $T_1$, $(M_2, L_2) = (L_1, M_1)$ for $T_2$, $K$ an $(r, s)$ curve on $T_1$. Let $\mathcal{N}_K$ be a regular neighborhood of the knot with meridian-longitude pair $(M, L)$. We remove $\mathcal{N}_K$ from $S^3$ forming a depression along $K$ in each of $T_1$ and $T_2$ but leaving each a solid torus.

We sew back a solid torus $sT^e$ with meridian $M'$ so that $M' \sim L-qM \sim K$. A meridian goes to one edge of the depression; another meridian goes to the other edge since they are parallel. Thus we may assume that the $\partial sT^e$ between two meridians is sewn to each half of the picture. Each half would be a lens space except that a 3-cell is
missing—the 3-cell which is the other half of $sT^n$.

We now consider how the two halves of the picture are identified. The boundaries of $T_1$ and $T_2$ outside of the depression are identified, as are the meridional disks of $sT^n$. The boundaries are annuli and the disks are sewn to them so as to make 3-spheres. Filling in these 3-spheres would give $L(r, s)$ and $L(s, r)$ since $M' \sim F \sim rM_1 + sL_1 \sim sM_2 + rL_2$. Hence the manifold obtained is $L(r, s) \# L(s, r)$.

5. Conjectures. A natural question to ask is whether Seifert manifolds can be obtained by elementary surgery along a knot other than a torus knot. We conjecture that the answer to this question is "no" in light of the following information:

1. If the fundamental group of a Seifert manifold is infinite, then the subgroup generated by the fiber is an infinite cyclic normal subgroup, the center of the group in case the characteristic is trivial [4].

2. All the known finite fundamental groups of closed 3-manifolds are groups of Seifert manifolds. All the possible finite fundamental groups have a nontrivial center. In case the order of the group is even, the unique element of order 2 lies in the center. In case the order of the group is odd, the group is cyclic and the center is the whole group [3].

3. Waldhausen has a partial converse to 1. If $\mathcal{M}^3$ is an irreducible 3-manifold such that $\pi_1(\mathcal{M}^3)$ has a nontrivial center and either $H_1(\mathcal{M}^3)$ is infinite or $\pi_1(\mathcal{M}^3)$ is a nontrivial free product with amalgamation, then $\mathcal{M}^3$ is a Seifert manifold [5].

4. Burde and Zieschang have shown that if the fundamental group of the complement of a knot has a nontrivial center, then the knot is a torus knot and the center is infinite cyclic [1].

Conjecture 1. If $\mathcal{M}^3$ is a Seifert manifold and $\mathcal{M}^3$ is obtained
by elementary surgery along a knot $K$, then $K$ is a torus knot.

Conjecture 2. If $\mathcal{M}^3$ is a lens space obtained by elementary surgery along a knot $K$, then $K$ is a torus knot.

Conjecture 3. If $\mathcal{M}^3$ is obtained by elementary surgery along a knot $K$ and $\pi_1(\mathcal{M}^3)$ is finite, then $K$ is a torus knot.

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