Invariants of Twist-Wise Flow Equivalence

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DEFINITION: Two nonnegative square matrices are FLOW EQUIVALENT if their induced SFTs have topologically equivalent suspension flows.

THEOREM: [John Franks, 1984] Two nonnegative irreducible integer matrices, A and B, neither of which is in the trivial flow equivalence class are flow equivalent if and only if

$$\det(\mathbf{I} - \mathbf{A}) = \det(\mathbf{I} - \mathbf{B}) \qquad \text{(Parry-Sullivan number)}$$

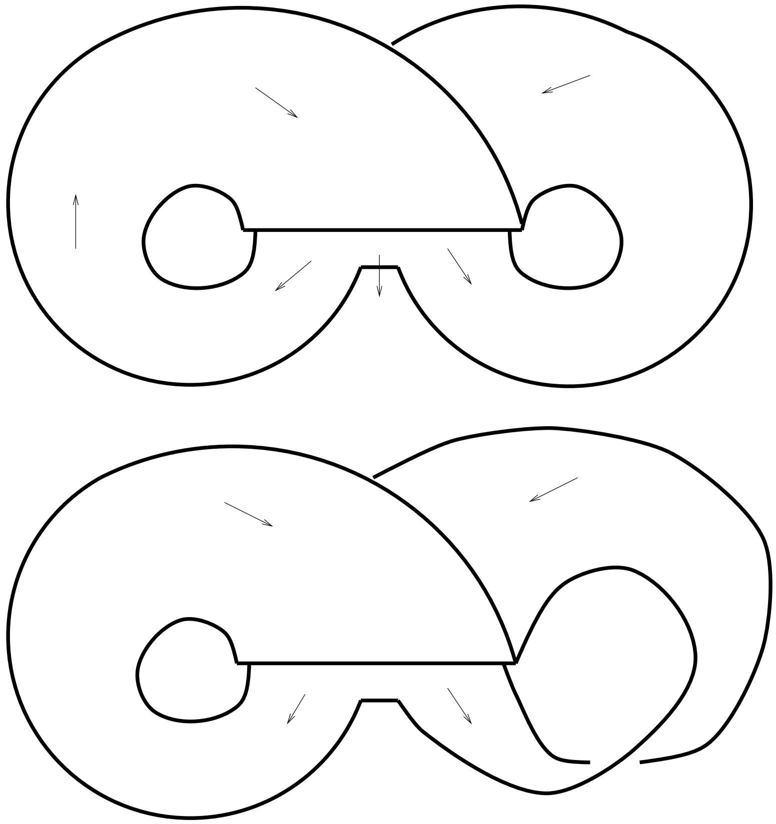
and

$$\frac{\mathbf{Z}^{\mathrm{n}}}{(\mathbf{I} - \mathbf{A})\mathbf{Z}^{\mathrm{n}}} \cong \frac{\mathbf{Z}^{\mathrm{m}}}{(\mathbf{I} - \mathbf{B})\mathbf{Z}^{\mathrm{m}}} \qquad \text{(Bowen-Franks group)}$$

where n and m are the sizes of A and B respectively.

NOTE: Extended by D. Huang to the reducible case.

MOTIVATING PROBLEM:



Templates are used to model 1-dimensional basic sets of saddle type in flows on 3-manifolds. The inverse limit of the invariant orbits of a template reproduce the basic set. These two templates have the same incidence matrix, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$**, and hence are not distinguished by Franks' theorem.**

DEFINITION: Let $\{1, t\}$ represent the group Z_2. A TWIST MATRIX is a square matrix over the semi-group-ring Z^+Z_2, i.e. entries are of the form $a + bt$ where a and b are nonnegative integers.

DEFINITION: The RIBBON SET of a 1-dimensional basic set is the basic set's local stable manifold. For a saddle set in a 3-manifold's flow the ribbon set looks like a swirling mass of ribbons.

Twist matrices and ribbons sets are connected in a manner analogous to the way incidence matrices are connected to 1-dimensional basic sets For the second template shown before an associated twist matrix is $\begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix}$.

DEFINITION: Two twist matrices are TWIST-WISE FLOW EQUIVALENT if their associated ribbon sets are topologically equivalent.

DEFINITIONS: Let $\mathbf{A(t)}$ be a twist matrix.

Let $\mathbf{PS^+(A) = \det(I - A(1))}$.

Let $\mathbf{PS^-(A) = \det(I - A(-1))}$.

Let $$\mathbf{BF^+(A) = \frac{Z^n}{(I - A(1))Z^n}}.$$

Let $$\mathbf{BF^-(A) = \frac{Z^n}{(I - A(-1))Z^n}}.$$

Let $$\mathbf{BF^{\partial}(A) = \frac{Z^{2n}}{(I - A(T))Z^{2n}}}, \text{ where } \mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}.$$
Notice $\mathbf{T^2 = I}$. ($\mathbf{A(T)}$ means replace each $\mathbf{a + bt}$ with the $\mathbf{2 \times 2}$ block $\mathbf{aI + bT}$.)

THEOREM: [M.S.] These are invariants of twistwise flow equivalence.

Two twist matrices are twist equivalent if and only if there is a finite sequence of the three matrix moves shown below taking one to the other.

The shift move: $\mathbf{A} \overset{s}{\sim} \mathbf{B}$ if there exists rectangular matrices R and S, over Z^+Z_2, such that $\mathbf{A} = \mathbf{RS}$ and $\mathbf{B} = \mathbf{SR}$.

The expansion move: $A \overset{e}{\sim} B$ **if** $A = [A_{ij}]$ **and**

$$B = \begin{bmatrix} 0 & A_{11} & \cdots & A_{1n} \\ 1 & 0 & \cdots & 0 \\ 0 & A_{21} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1} & \cdots & A_{nn} \end{bmatrix},$$

or vice versa.

The twist move: $A \overset{t}{\sim} B$ **if** $A = [A_{ij}]$ **and**

$$B(t) = \begin{bmatrix} A_{11} & tA_{12} & \cdots & tA_{1n} \\ tA_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ tA_{n1} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

The first two matrix moves generate flow equivalence [W. Parry and D. Sullivan, 1975]. The first generates STRONG SHIFT EQUIVALENCE [Williams, 1973]. Our theorem is proved by showing invariance under the three matrix moves.

GEOMETRIC MOTIVATIONS:

Shift: Splitting, amalgamating or relabeling Markov partition elements.

Expansion: Inserting or deleting "parallel" Markov partition element.

Twist: Switching orientation of a Markov partition element.

EXAMPLE:

For $\begin{bmatrix} t & 1 \\ 1 & 1 \end{bmatrix}$, $PS^+ = PS^- = -1$.

Since, $|PS^{\pm}|$ **is the order of** $BF^{\pm}$ **(respectively) we get that both** $BF^{\pm}$ **groups are trivial groups.**

For $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$, $PS^+ = -1$ **while** $PS^- = 1$.

Thus these two matrices are in different twist-wise flow equivalence classes.

EXAMPLE:

For $\begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix}$, $PS^+ = -1$ **and** $PS^- = 1$ **just as with** $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$.

The double cover invariant yields no additional information since BF^∂ **is the trivial group in both cases.**

But, we have not been able to find a sequence of moves $(\overset{s}{\sim}, \overset{e}{\sim}, \overset{t}{\sim})$ **that would show these two matrices to be twist equivalent.**

Since $\begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix}$ **has a "period one" Möbius band in its ribbon set, and** $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ **does not, such a sequence would have to include an expansion move, perhaps many.**

EXAMPLE:

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **gives** $PS^+ = PS^- = -1$**, as did** $\begin{bmatrix} t & 1 \\ 1 & 1 \end{bmatrix}$**.**

Passing to the double cover gains us nothing. Yet, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **has an orientable ribbon set while** $\begin{bmatrix} t & 1 \\ 1 & 1 \end{bmatrix}$ **does not.**

So, they cannot be twist equivalent. Thus, our invariants cannot be complete. Of course one can always take orientability itself as an invariant. Still this is frustrating as the motivating force has been to find a systematic means to capture orientation data.

EXAMPLE:

Now consider $A = \begin{bmatrix} 3 & 1+t \\ 1+t & 3 \end{bmatrix}$**, and** $B = \begin{bmatrix} 3 & 1+t \\ 2 & 3 \end{bmatrix}$**.**

We get $PS^+ = 0$**,** $BF^+ = Z \oplus Z_2$**,** $PS^- = 4$**, and** $BF^- = Z_2^2$ **for both matrices.**

But $BF^\partial(A) = Z \oplus Z_4$ **while** $BF^\partial(B) = Z \oplus Z_2^2$**.**

Thus, A **and** B **are in distinct twist-wise flow equivalence classes.**

Matrix	PS^+	BF^+	PS^-	BF^-	BF^D
$t11\,111\,111$	-2	Z_2	-4	Z_4	Z_8
$1t1\,111\,111$	-2	Z_2	0	Z	Z
$tt1\,111\,111$	-2	Z_2	-2	Z_2	Z_2^2
$t11\,1t1\,11t$	-2	Z_2	0	$Z \oplus Z_3$	$Z \oplus Z_3$
$tt1\,1t1\,111$	-2	Z_2	-4	Z_4	Z_8
$1t1\,t11\,111$	-2	Z_2	2	Z_2	Z_2^2
$ttt\,111\,111$	-2	Z_2	0	Z	Z
$t11\,t11\,111$	-2	Z_2	-2	Z_2	Z_2^2
$tt1\,tt1\,111$	-2	Z_2	-2	Z_2	Z_2^2
$t11\,1t1\,111$	-2	Z_2	-6	Z_6	$Z_2 \oplus Z_6$
$ttt\,ttt\,ttt$	-2	Z_2	4	Z_4	Z_8
$ttt\,ttt\,1tt$	-2	Z_2	6	Z_6	$Z_2 \oplus Z_6$
$tt1\,ttt\,1tt$	-2	Z_2	0	$Z \oplus Z_3$	$Z \oplus Z_3$
$0t1\,111\,111$	-3	Z_3	-1	0	Z_3
$011\,1t1\,111$	-3	Z_3	-5	Z_5	Z_{15}
$0t1\,1t1\,111$	-3	Z_3	-3	Z_3	Z_3^2
$01t\,101\,1t1$	-4	Z_4	0	Z	$Z \oplus Z_2$
$01t\,101\,11t$	-4	Z_4	0	Z	$Z \oplus Z_2$
$011\,t01\,1t1$	-4	Z_4	-2	Z_2	Z_8
$011\,t01\,11t$	-4	Z_4	2	Z_2	Z_8
$01t\,ttt\,110$	-4	Z_4	6	Z_6	Z_{24}
$011\,t01\,110$	-4	Z_2^2	0	$Z \oplus Z_2$	$Z \oplus Z_2^2$

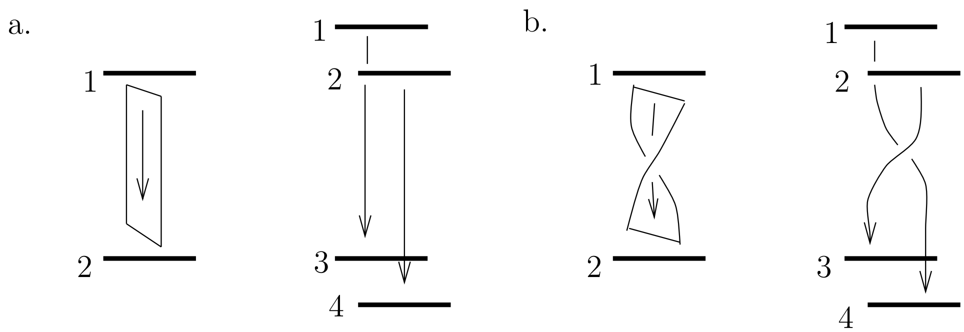
Other representations of Z_2 give nothing new.

THEOREM: [Hua and Reiner, 1951] Let M be an $n \times n$ matrix with $M^2 = I$. Then M is similar over the integers to the direct sum of matrices of the form [1], [-1] and T.

PROPOSITION: [M.S.] Other matrix representations of Z_2 yield only redundant invariants.

PROOF: Follows from Theorem of Hua and Reiner.

Double covers



$$\mathbf{A}(\mathbf{t}) \longrightarrow \mathbf{A}(\mathbf{T})$$

$$\mathbf{a} + \mathbf{b}\mathbf{t} \longrightarrow \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{a} \end{bmatrix}$$