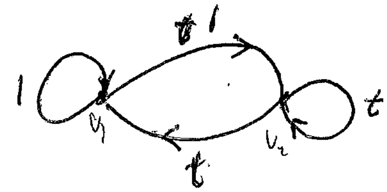
GMU
Nov
2002

# Twistwise Flow Eq

We study graphs with edge labels from $\mathbb{Z}/2 = \langle t \mid t^2=1\rangle$

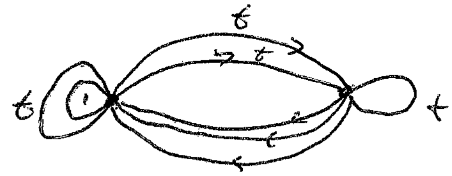
Ex




This graph corresponds to the matrix $\begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix}$.

Given a square martix with entries in $\mathbb{Z}_+[\mathbb{Z}/2]$ $\rightarrow \{a+bt \mid a,b \geq 0\}$
we get a graph.

Ex $\begin{bmatrix} 1+t & 2t \\ 3 & t \end{bmatrix}$



Call these matrices "twist martices."

**Def** SSE move : $A = RS \quad B = SR$

Ex: $[1+t] = [1 \; 1]\begin{bmatrix} 1 \\ t \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}[1 \; 1]$

PS move: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \longleftrightarrow \begin{bmatrix} a_{11} & 1 & 0 & 0 & \cdots & 0 \\ a_{11} & 0 & a_{12} & & \cdots & a_{1n} \\ \vdots & \vdots & & & & \vdots \\ a_{n1} & 0 & a_{n2} & & \cdots & a_{nn} \end{bmatrix}$

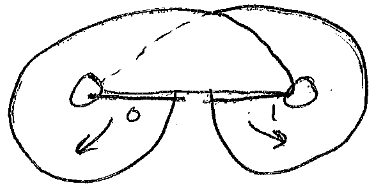
[Def/Thm] A is twistwise flow equivalent to B if $\exists$ a sequence of SSE and PS moves from A to B. This is an equivalence relation.

# Why do this?

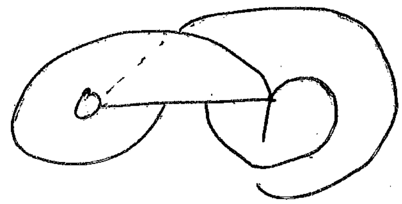
Motivation: Sketch only because of time.

Template: branched 2-manifold with semi-flow used to model certain invariant sets of flows on 3-manifolds.

Ex




Lorenz



Horseshoe

- Invariant set is Cantor set. cross section
- 1-1 correspondence with bi-infinite sequences of 0's and 1's.
- Has incidence matrix $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

Invariant set is top eq. Matrix is the same.

In "ordinary symbolic dynamics" $(t=1)$ (Martix $\leftrightarrow$ graph $\leftrightarrow$ set of bi-infinite seq's) ~~temp~~ template
A and B are flow eq if their "suspension flows" (think templates invariant flows) are top. eq.

Thm [Parry-Sullivan] SSE and PS generate flow eq.

Def $PS(A) = \det(I-A)$ $\quad BF(A) = \dfrac{\mathbb{Z}^n}{(I-A)\mathbb{Z}^n}$

Thm [Franks] For nonpermutation, irreducible square matrices over $\mathbb{Z}_+$, PS and BF determine flow eq.

Yet, Lorenz and H.S. "look different." Idea: encode twist $\begin{bmatrix}1 & 1\\ t & t\end{bmatrix}$

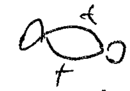
**Def** $PS^{\pm}(A(t)) = PS(\cancel{A}(\pm 1)) = \det(I - A(\pm 1))$

$BF^{\pm}(A(t)) = BF(A(\pm 1)) = \mathbb{Z}^n / (I - A(\pm 1))(\mathbb{Z}^n)$

$BF^{\partial}(A(t))$ = something a little fancier.

$O(A)$ = yes if no loop in $\mathcal{G}_A$ has weight $t$.

= no otherwise.

$\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}$ 

$O(A) = $ yes

**Thm** [S] These are all invariants of TWFE.

Are they complete? Unknown.

**Ex** $\begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{bmatrix}$ and $\begin{bmatrix} t & t & 1 \\ t & t & t \\ 1 & t & t \end{bmatrix}$ yield

$PS^{+} = -2 \quad BF^{+} = \mathbb{Z}/2$

$PS^{-} = 0 \quad BF^{-} = \mathbb{Z} \oplus \mathbb{Z}/3$

$BF^{\partial} = \mathbb{Z} \oplus \mathbb{Z}/3 \qquad O =$ ~~yes~~ no

$\begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$

$PS^{\pm} = \pm 1 \Rightarrow$ groups $= 0$.

$O(\cdot) =$ NO.

But I could not find seq of SSE and PS moves connecting them.

---

$\boxed{\frac{1997}{2002}}$

**Def** $(I - A)_{\infty} = \left[\begin{array}{c|c} I - A & 0 \\ \hline 0 & I \end{array}\right]$ infinite matrix.

**Thm** [Boyle - S.] If $A$ and $B$ are irreducible, non ar. ($O$ = no) then $A$ is TWFE to $B$ iff $\exists$ an $SL_{\infty}(\mathbb{Z}_+[\mathbb{Z}/2])$ equivalence between $(I-A)_{\infty}$ and $(I-B)_{\infty}$.

Cor All previously known invariants follow.

Ex $\quad A = \begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$E(I-A) = I-B.$$

Rmk True for $\mathbb{Z}_+[G]$, any finite group $G$, with an additional condition that replaces orientability.

Def Let $A$ be such that $A_{ij} = a_{ij} + b_{ij}t$ has $a_{ij} > 0$ (resp $b_{ij} > 0$) and let $E$ be the elementary matrix with $E_{ij} = 1$ (resp $t$) then

$$E(I-A) \mapsto I-B$$
$$(I-A)E \to I-B$$

are basic positive equivalences. A pos eq is a chain of ba p. eq (including inverses)

Lemma [Boyle S.] $\quad PE \Leftrightarrow FE$.

Thm [Boyle S.] (hard) $\quad SLE$ can be decomposed into $PE$.

<u>Open Problem</u>: We need a normal form for martices over $\mathbb{Z}[\mathbb{Z}/2]$, but this ring is not a PID. ~~This is~~