Summer 2002
SFU

A new approach to symbolic dynamics

Based on Mike Boyle's paper Positive K-Theory and Symbolic Dynamics.

3 ideas:

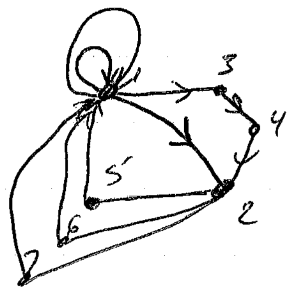
① Replace the shift relation $(A \sim B \Leftrightarrow A = RS, B = SR)$ with something like row operations, realized by ~~elementary~~ multiplications by elementary matrices, sort of. This will mean working with $\mathbb{N} \times \mathbb{N}$ matrices since A and B can have different sizes.

② Work with polynomial matrices for greater compactness and algebriac richness.

③ Make CW complexes to model SSE. Relate K-theory to homotopy groups to get invariants.

Poly matrices:

Ex $B = \begin{pmatrix} 2t & t^3+t \\ 3t^2 & 0 \end{pmatrix} \in M\left(t\mathbb{Z}_+[t]\right)$



$$B^{\#} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Associate to B the SFT given by $B^{\#}$.
The Rome of B is $\{1,2\}$.
A route is a path starting and ending in the Rome that only meets the Rome twice. R_A = all routes.

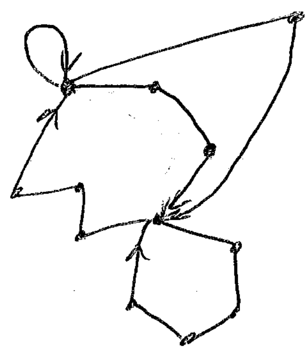
Elementary Move Let $i,j,k>0$, $i \neq j$. Let $E = E(i,j,k)$ = the identity except that $E_{ij} = t^k$.
Let $A \xrightarrow{ijk} B$ be determined by

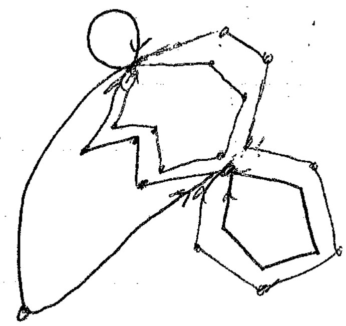
$$E(I-A) = I-B \qquad \left(B = I - E(I-A)\right)$$

Note: we need ijk to be s.t. $B \in M(t\mathbb{Z}_+[t])$

Ex $A = \begin{pmatrix} t & t^2+t^3 \\ t^4 & t^5 \end{pmatrix}$ $E = \begin{pmatrix} 1 & t^3 \\ 0 & 1 \end{pmatrix}$

Then $B = \begin{pmatrix} t+t^7 & t^2+t^8 \\ t^4 & t^5 \end{pmatrix}$





Clearly the SFT for $A^\#$ and $B^\#$ are conjugate. Can all conjugacies be realized by chains of elementary moves (left and right) and there inverses?

No. But yes if we enlarge $M(t\,\mathbb{Z}_+[t])$ a bit.

Ex: $A = [2t]$ $B = \begin{bmatrix} t & t \\ t & t \end{bmatrix}$. Conjugate but not related by ele. moves since (?) some sizes are different. NO

Ex This example shows an elementary equivalence chain need not preserve the size of the rome set.

$$A = \begin{bmatrix} t & t \\ t & 0 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

$$E_1 (I - A) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-t & -t \\ -t & 1 \end{bmatrix} = \begin{bmatrix} 1-t-t^2 & 0 \\ -t & 1 \end{bmatrix}$$

$$\text{so} \quad B_1 = \begin{bmatrix} t+t^2 & 0 \\ t & 0 \end{bmatrix}$$

$$(I - B_1) E_2 = \begin{bmatrix} 1-t-t^2 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1-t-t^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{So} \quad B_2 = \begin{bmatrix} t+t^2 & 0 \\ 0 & 0 \end{bmatrix} = [t+t^2]$$

~~NZC~~:

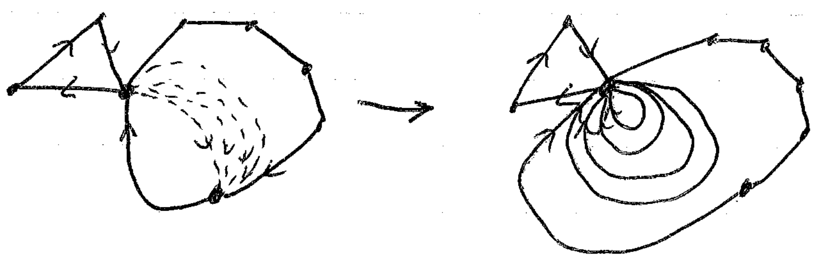
We allow A to be in $M(\mathbb{Z}[t])$ and think of ~~...~~ edges with constant labels as having "zero length." Then we restrict A so that its has no cycles of zero length. This is the NZC condition:

$$A(t) \text{ is NZC iff } \operatorname{tr}(A^n(0)) = 0 \quad \forall n > 0.$$

Now to get an SFT from such a matrix we ~~...~~ squeeze out the slack (zero length paths).

<u>Ex</u>

$$C = \begin{pmatrix} t^3 & 4+t^5 \\ t & 0 \end{pmatrix} \in M(\mathbb{Z}_+(t))_{NZC}$$



$$D = \begin{pmatrix} t^3+4t & t^5 \\ t & 0 \end{pmatrix} \in M(t\mathbb{Z}_+(t))$$

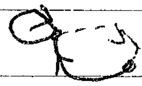
and clearly C sse D. This can be generalized:

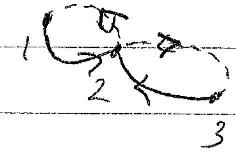
$$\text{So} \quad M(\mathbb{Z}_+[t])_{NZC}/_{sse} = M(t\mathbb{Z}_+[t])/_{sse}$$

~~Thm [BW] Any matrix in $M(\mathbb{Z}_+[t])_{NZC}$ is sse to one in $M(t\mathbb{Z}_+[t])$.~~

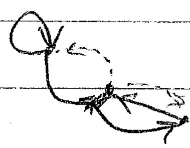
$\underline{Ex}$ The equivalence of $[2t]$ and $\begin{bmatrix} t & t \\ t & t \end{bmatrix}$.

$$[2t] \to \begin{bmatrix} t & 1 \\ t & 0 \end{bmatrix} \to \begin{bmatrix} 0 & t & 0 \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix}$$



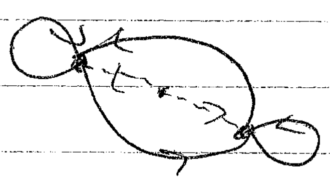


$$\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -t & 0 \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix} = \begin{bmatrix} 1-t & 0 & -t \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix}$$



$$\begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix} \begin{bmatrix} 1-t & 0 & -t \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix} = \begin{bmatrix} 1-t & 0 & -t \\ -1 & 0 & -1 \\ -t & 0 & 1-t \end{bmatrix}$$



$$\begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ t & 0 & t \end{bmatrix}$$

relabel vertices

$$\begin{bmatrix} t & t & 0 \\ t & t & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-t & -t & 0 \\ t & 1-t & 0 \\ -1 & -t & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \to \begin{bmatrix} t & t \\ t & t \end{bmatrix}$$

The main theorem, due to BW, and unpublished, is that chains of elementary moves in $M(\mathbb{Z}_+[t])_{NZC}$ give

SSE of the the SFTs and vice versa.

A technical tool used in this proof, which we will not do, is .

<u>Path Shifts</u> Let $A \in M(\mathbb{Z}_+[t])_{NZC}$, be given

Let $A_{ij} = t^{l_1} + t^{l_2} + \cdots + t^{l_n}$. Then $A_{ij}(1) = n$.

Let $R_{ij} = \{ (i,j,k,l) \mid 1 \le k \le n,\ l = l_k \}$

<u>Ex:</u> For $A_{12} = t + t + t^3 + t^7$, $n = 4$ and $R_{12} =$

$\{(1,2,1,1), (1,2,2,1), (1,2,3,3), (1,2,4,7)\}$

= routes from 1 to 2.

Let $R = \bigcup_{ij} R_{ij}$. Let $\mathcal{A} = \{ (i,j,k,l,t) \mid (i,j,k,l) \in R, t \in \mathbb{Z} \}$

Let $\Sigma = \{ (s_n)_{-\infty}^{\infty} \in \mathcal{A}^{\mathbb{Z}} \mid \forall n$ if $s_n = (i,j,k,l,t)$ and

$s_{n+1} = (i', j', k', l', t')$ then $t' = t + l$, $j = i' \}$.

But Σ is too big. Let $s \sim s'$ if $\exists\, m \in \mathbb{Z}$ s.t. $s_{n+m} = s'_n$ $\forall n$. Let $P_A = \Sigma/\sim$.

What does it mean to "shift" in P_A?

Let $\delta : P_A \to P_A$ be, $\delta(s) = s'$, where if $s_n(i,j,k,l,t)$ then $s'_n = (i,j,k,t-1)$.

It is not obvious that (P_A, δ) really is a SFT. But,

<u>Thm:</u> (P_A, δ) is top. conj to $(\Sigma_{A^\#}, \sigma)$.

<u>Thm</u> Every top. conj. of <u>path spaces</u> is realized by a chain of ele. moves.

Where is the K-theory?

~~Let M = N×N matrices over~~

Let R be a ring containing a semiring S_+ that contains "0" and "1."

Let $E(R_+)$ be $N \times N$ matrices $= I$ in all except one entry E_{ij}, $i \neq j$ with $E_{ij} \in R_+$.

Let M be a collection of $N \times N$ matrices over R that differ from the identity in at most finitely many entries.

Let $\mathcal{C}(M, E(S_+))$ be the category

with objects M and hom's given by:

$C \xrightarrow[\text{fwd}]{E} D$

① $EC = D$ or $CE = D$, $C, D \in M$, $E \in E(R_+)$ is a forward elementary positive move

② or backwards.

③ chains of these

$$K_1^+(M, E(S_+)) = \mathcal{C}(M, E(S_+)) / \sim$$

Def Let A and B be matrices in $M(\mathbb{Z}_+[t])_{NZC}$. Let $\mathcal{E}$ be the elementary matrices, i.e. $E \in \mathcal{E}$ if $E = Id$ except for one off diagonal entry that is of the form t^k, $k \geq 0$. Then $A \sim B$ if A can be taken to B be a chain of moves of these forms:

$$E(I-A) = (I-B) \quad \text{left forward elementary move}$$
$$(I-A)E = (I-B) \quad \text{right forward elementary move}$$
$$E^{-1}(I-A) = (I-B) \quad \text{left backward ele. move}$$
$$(I-A)E^{-1} = (I-B) \quad \text{right backward ele. move.}$$
$$PAP = B \quad \text{vertex relabelling move}$$

where $E \in \mathcal{E}$ and P is a permutation matrix. Remember, all matrices are really $\mathbb{N} \times \mathbb{N}$.

Ex $[2t] \sim \begin{bmatrix} t & t \\ t & t \end{bmatrix}$.

a. $[2t] \to \begin{bmatrix} 2t & 0 \\ t & 0 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ right backward

b. $\begin{bmatrix} 2t & 0 \\ t & 0 \end{bmatrix} \to \begin{bmatrix} t & 1 \\ t & 0 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ left backward

c. $\begin{bmatrix} t & 1 \\ t & 0 \end{bmatrix} \to \begin{bmatrix} 0 & t \\ 1 & t \end{bmatrix}$ relabelling.

d. $\begin{bmatrix} 0 & t \\ 1 & t \end{bmatrix} \to \begin{bmatrix} 0 & t & 0 \\ 1 & t & 0 \\ 0 & t & 0 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$ right backward

e. $$\begin{bmatrix} 0 & t & 0 \\ 1 & t & 0 \\ 0 & t & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & t & 0 \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ left backwards

f. $$\begin{bmatrix} 0 & t & 0 \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ left forward.

g. $$\begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ t & 0 & t \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$$ left forward

h. $$\begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ t & 0 & t \end{bmatrix} \longrightarrow \begin{bmatrix} t & t & 0 \\ t & t & 0 \\ 1 & 1 & 0 \end{bmatrix}$$ relabelling

i. $$\begin{bmatrix} t & t & 0 \\ t & t & 0 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} t & t & 0 \\ t & t & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ right forward

j. $$\begin{bmatrix} t & t & 0 \\ t & t & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} t & t & 0 \\ t & t & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} t & t \\ t & t \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ right forward.

Calculations

a. $\begin{bmatrix} 1-2t & 0 \\ -t & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1-2t & 0 \\ 0 & 0 \end{bmatrix} \to [2t]$ ✓

b. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-t & -1 \\ -t & 0 \end{bmatrix} = \begin{bmatrix} 2t & 0 \\ -t & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2t & 0 \\ t & 0 \end{bmatrix}$ ✓

c. clear

d. $\begin{bmatrix} 1 & -t & 0 \\ -1 & 1-t & 0 \\ 0 & -t & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix} = \begin{bmatrix} 1 & t & 0 \\ -1 & 1-t & 0 \\ 0 & -t & 0 \end{bmatrix} \to \begin{bmatrix} 0 & t & 0 \\ 1 & t & 0 \\ 0 & t & 0 \end{bmatrix}$ ✓

e. left $\begin{bmatrix} 1 & -t & 0 \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -t & 0 \\ -1 & 1-t & 0 \\ 0 & -t & 1 \end{bmatrix}$

$\to \begin{bmatrix} 0 & t & 0 \\ 1 & t & 0 \\ 0 & t & 0 \end{bmatrix}$ ✓

f. $$\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -t & 0 \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix} = \begin{bmatrix} 1-t & 0 & -t \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix} \to \begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ 0 & t & 0 \end{bmatrix}$$

g. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix} \begin{bmatrix} 1-t & 0 & -t \\ -1 & 1 & -1 \\ 0 & -t & 1 \end{bmatrix} = \begin{bmatrix} 1-t & 0 & -t \\ -1 & 1 & -1 \\ -t & 0 & 1-t \end{bmatrix}$$

$$\to \begin{bmatrix} t & 0 & t \\ 1 & 0 & 1 \\ t & 0 & t \end{bmatrix}$$

h. clear

i. easy

j. easy.

<u>Problem</u>: Draw all the graphs.