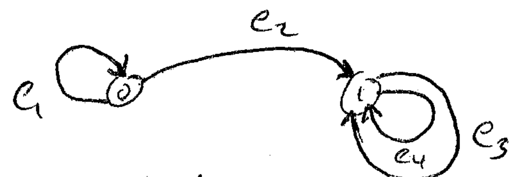
Symbolic Dynamics: SFTs.

Ex 1



$$\longleftrightarrow \quad M = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}$$

$\mathcal{G}_M$

Let Σ_M be the set of all bi-infinite sequences of edges allowed in the graph $\mathcal{G}_M$. This example is <u>reducible</u>.

$$\ldots e_1 e_1 e_1 . e_1 e_1 \ldots \in \Sigma_M$$
$$\ldots e_1 e_1 e_2 . e_3 e_3 e_4 e_3 \ldots \in \Sigma_M$$

Ex 2

$\longleftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. <u>Irreducible</u>.

Def A square matrix is <u>irreducible</u> if $\forall\, i,j\ \exists\, n$ s.t. $M^n(i,j) \neq 0$. ($\mathcal{G}_A$ is irr. iff M is.)

Def [SFT] For each square matrix over $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ we give Σ_A a metric $d(\bar{x}, \bar{y}) = \sum_{i \in \mathbb{Z}} \frac{\delta(x_i, y_i)}{2^{|i|}}$, where $\delta(x_i, y_i) = \begin{cases} 1 & x_i = y_i \\ 0 & x_i \neq y_i \end{cases}$. Then it can be shown that

Σ_A is a Cantor set.

Let $\sigma : \Sigma \to \Sigma$ be the shift map: $\sigma(\bar{x}) = \bar{y} \Leftrightarrow y_i = x_{i+1}$. It is a homeo. The pair (Σ, σ) is called a <u>shift of finite type</u>, or an SFT.

* Draw picture 

Def [Top. Conj.] Given (Σ, σ) and (Σ', σ'), SFTs, we say they are <u>topologically</u> <u>conjugate</u> if $\exists$ homeo $h : \Sigma \to \Sigma'$ s.t.

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "h"'] & \Sigma \arrow[d, "h"] \\ \Sigma' \arrow[r, "\sigma'"'] & \Sigma' \end{tikzcd}$$

commutes,

i.e. $\sigma' \circ h = h \circ \sigma$.

Def Let A and B be square matrices over $\mathbb{Z}_+$. A <u>shift move</u> (also called an SSE-move) from A to B is a dual decomposition $A = RS$, $B = SR$, where R and S are over $\mathbb{Z}_+$, but need not be square.

Ex: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = [2]$. Then $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 1]$ and $B = [1 \ 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Def A is Strong Shift Equivalent (SSE) to B if $\exists$ a chain of shift moves from A to B. (SSE is an equivalence relation.)

Thm [Williams] Σ_A is top conj. to Σ_B iff A is SSE to B.

Rmk: There is no known algorithm to check this. It may be undecidible.

Twist-wise Flow Equivalence

UMD
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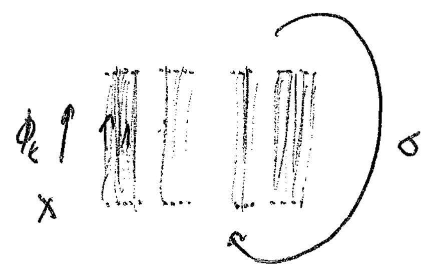
Def Let A be a square nonneg. matrix.
Let (X, σ) be the SFT for A.
Let (F, ϕ_t) be defined as follows

$$F = X \times \mathbb{R} \Big/ \{x, t+1\} \sim \{\sigma(x), t\}$$

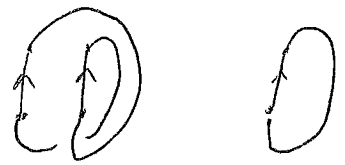
$$\phi_t([x, s]) = [x, s+t].$$

(F, ϕ_t) is called the <u>mapping torus</u> or the <u>suspension flow</u> of (X, σ). See LM § 13.6



Def (F_A, ϕ_t) and (F_B, ϕ_t) are <u>top eq</u> if $\exists$ homeo from F_A to F_B taking flow lines to flow lines and preserving the flow direction. We say matrices A and B are <u>flow eq</u> (FE) if their suspensions are top. eq.

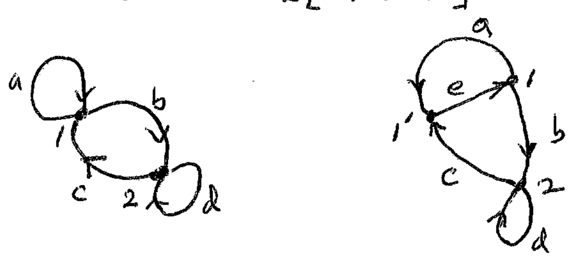
Ex 1 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $[1]$ are FE.



For permutation matrices just count closed orbits. This is called the <u>trivial case</u>. Notice these are not SSE.

Ex 2 $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $[2]$ are FE since they are SSE. Think about this. FE is coarser than SSE.

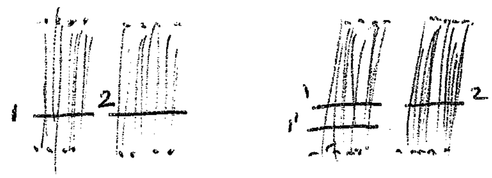
Ex 3 $\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}$ and $\begin{array}{c|ccc} & 1' & 1 & 2 \\ \hline 1' & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{array}$ are FE.



Every path that went through 1 now goes through 1' first.

.... aa.aa.... $\longleftrightarrow$ ...eaea.eaea... so not SSE.

... aac.dbacddd $\longleftrightarrow$ ... eaeaec.dbeaecddd... FE since we have an "orbit" map.



$\leftarrow$ new Markov partion element, parallel to old new. Flow is unchanged, but sybolic dynamics is not preserved.

Def PS move: $\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \ddots \\ \vdots & & \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 1 & 0 & \cdots \\ a_{11} & 0 & a_{12} & \cdots \\ a_{21} & 0 & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Thm [PS] Together SSE and PS generate FE.

<u>Idea of Proof</u>: Refine and slide Markov partition.

Zeta at $t=1$.

<u>Invariants</u>: $PS(A) = \det(I - A)$ (Parry–Sullivan number) [PS]

$BF(A) = \mathbb{Z}^n/(I-A)\mathbb{Z}^n$ (Bowen–Franks group) [BF].

<u>Thm</u> [F] These are complete for nontrivial irreducible square matrices over $\mathbb{Z}_+$.

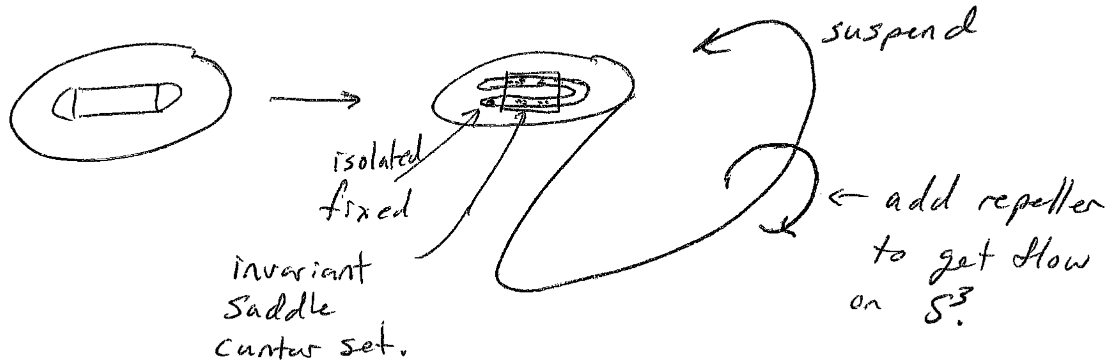
<u>Idea of Proof</u>: Find normal forms.

<u>Rmk</u> $|BF| = |PS|$ unless $|PS| = 0$, then $|BF| = \infty$.

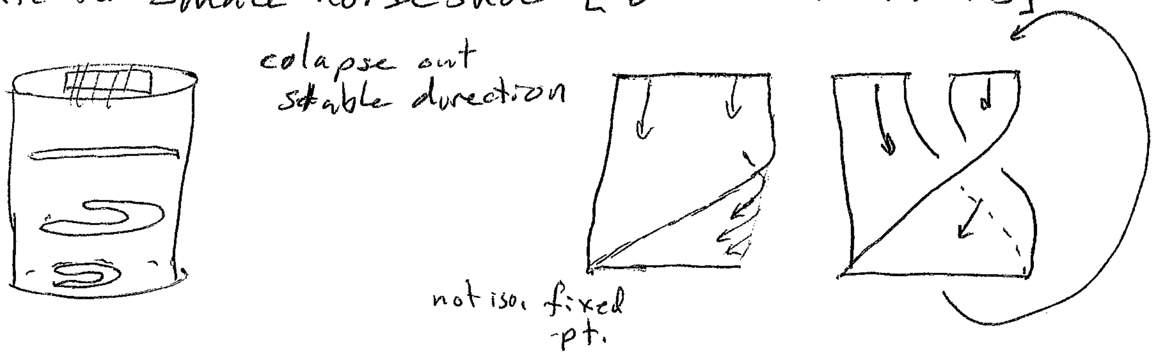
<u>Rmk</u> D. Huang has found a complete computible set of invariants for the reducible case.

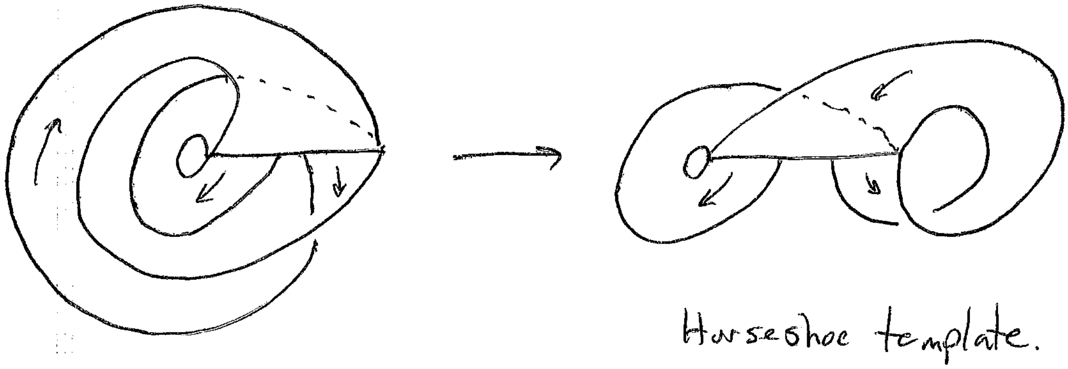
<u>Application</u>: Smale flows and templates.

Smale horseshoe:



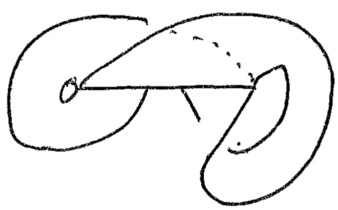
Template for Smale horseshoe [Birman & Williams]



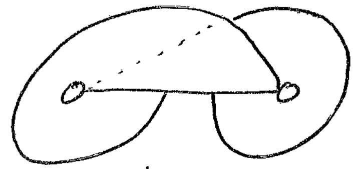


Horseshoe template.

But there are many templates:



$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$



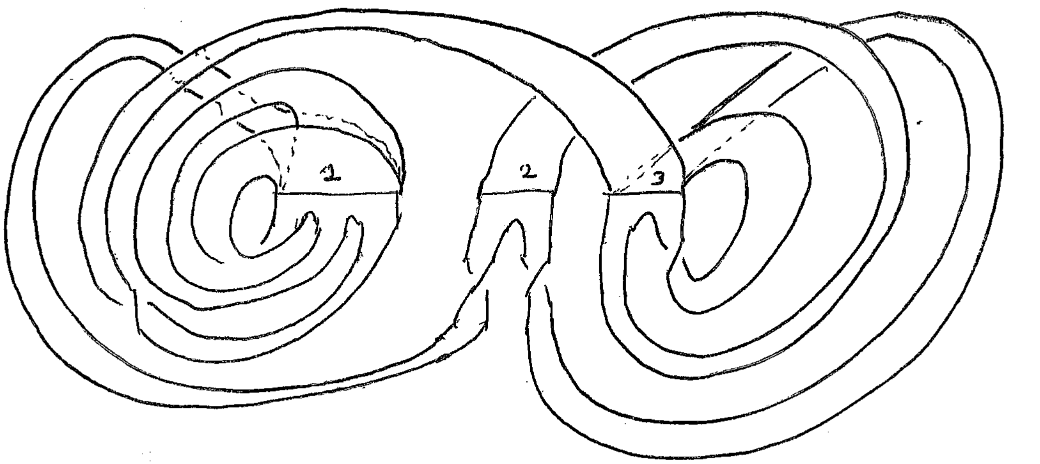
Lorenz

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Sure look different, but have same symbolic dymacis.
How to distinguish them? Idea: use t for "twist."

$\begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Let $G = \mathbb{Z}/2 = \{1, t\}$ with $t^2 = 1$. Consider square matrices over $\mathbb{Z}_+ G$. Example:



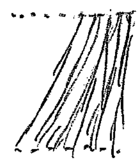
, gives $\begin{bmatrix} t+1 & 0 & t \\ t & 0 & t \\ 0 & 1 & t \end{bmatrix}$ as its "twist matrix."

For more on templates see [GHS].

Def Given a matrix $A(t)$ over $\mathbb{Z}_+[\mathbb{Z}/2]$ (a twist matrix) we define the <u>ribbon set</u> R of $A(t)$ to be a certain fiber bundle over the suspension flow of $A(1)$. Let (F, ϕ) be the susp. flow. The fiber will be $I = (-1, 1)$.

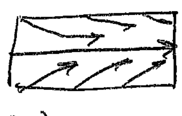
First suppose $A(t)$ has only 1's t's and 0's as entries. Then place an oriented Markov partition on F which induces $A(1)$. $\to \{d_1, \dots d_k\}$.

Let $F_{ij} = \{x \in F \mid x \in \phi_t(y) \quad y \in d_i \quad \underbrace{\phi_T(y) \in d_j}_{T \text{ first time}}, \; 0 \le t \le T\}$



Let $R_{ij} = F_{ij} \times I$ (some $R_{ij} = \emptyset$)

Attach these so that the core is F and the gluing of the fibers are ~~the~~ the id if $A_{ij} = 1$ and mult. by -1 if $A_{ij} = t$. Call this set R. We can place a flow on R that agrees with ϕ on F and each R_{ij} looks like:



Claim: For more general $A(t)$ one can still find a "good" Markov partition to use [Franks] (elementary)

Claim: The R is independent of the choice of Markov partition. <u>Pf:</u> Given M_1, M_2, M_1 can be transformed to M_2 by splittings, amalgamations and isotopies. Sp. and am. do not effect R; iso. induce iso.

Def Two twist matrices are <u>TWFE</u> if they have top. eq. Ribbon sets.

Rmk For a basic saddle set F of a Smale flow on M $\bigcup_{x\in F} E^s(x) \subset TM$ is a ribbon set for a twist matrix induced by a fine enough Markov partition. Think of it as the <u>infinitesimal</u> <u>stable</u> <u>manifold</u> of F.

Def* Given A over $\mathbb{Z}_+G$ get a G-labeled graph $\mathcal{G}_A$. Let $\tilde{f}_A$: Edges $\to G$ be determined by the labeling. Let $f_A(\bar{x}) = \tilde{f}_A(x_0)$ define $f_A : X_A \to G$.

Def* Two matrices A and B over $\mathbb{Z}_+G$ are G-conjugate if

$$\exists\, h : X_A \to X_B \quad \text{s.t.} \quad h \circ \sigma_A = \sigma_B \circ h \quad \text{and}$$
$$\exists\, c : X_A \to G \quad \text{s.t.} \quad f_A = f_B \circ h + c - c \circ \sigma_A .$$

Thm* [Parry via Boyle] G-conj. $\Leftrightarrow$ SSE over $\mathbb{Z}_+G$.

Thm [S] TWFE is generated by SSE over $\mathbb{Z}_+[\mathbb{Z}/2]$ and PS.

Rmk For more general G, G-weight FE is defined by SSE over $\mathbb{Z}_+G$ and PS.

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* G is any group.

Invariants of TWFE

$$\det(I - A(t)) = a + bt$$
$$PS^{+}(A) = PS(A(1))$$
$$PS^{-}(A) = PS(A(-1))$$

$BF^{+} = BF(A(1))$

$BF^{-} = BF(A(-1))$

$BF^{\partial} = BF(A(T))$ $\quad T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \overset{A(t)}{\begin{bmatrix} 3+t & 2+2t \\ 2 & 4t \end{bmatrix}} \to \overset{A(T)}{\begin{bmatrix} 3 & 1 & 2 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 0 & 0 & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix}}$

$\hookrightarrow$ BF of boundary flow.

Are they complete?

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} t & 1 \\ 1 & 1 \end{bmatrix}$$

$I - [\ ] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ $\qquad$ $I - [\ ] = \begin{bmatrix} 1-t & -1 \\ -1 & 0 \end{bmatrix}$

$PS^{\pm} = -1$ $\qquad$ $PS^{\pm} = -1$

$\Rightarrow$ all BF groups trivial $\qquad$ $\Rightarrow$ BF's trivial.

Invariants not complete.

But orientibility is a computable invariant.

Are $PS^{\pm}$, $BF^{\pm,\partial}$ and Or complete? UNKNOWN.

Question: Are $\begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ TWFE? [1997]

Lind

Boyle & Wagoner

<u>Enter k-theory</u>: See Boyle, BW (?) and books by Silvester and Rosenberg.

(Back to $G = \{1\}$). Embed $A \to \begin{bmatrix} A & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \\ 0 & & & \\ \vdots & & & \end{bmatrix} = A_\infty$ $\mathbb{N} \times \mathbb{N}$.

Let U and V be in $SL(\underset{\mathbb{N}\times\mathbb{N}}{\mathbb{N}}, \mathbb{Z})$ ← integer entries.

<u>Define</u> $(U,V)(A) = I - U(I-A)V = B$ (so, $U(I-A)V = I-B$; sometimes we write $(U,V)(I-A) = I-B$.) These are just row and column operations.

<u>Define</u>, for $i \neq j$, $E_{ij} = I + 1_{ij} \in SL$. (For $G = \{1\}$ E_{ij} and inverses generate SL.)

<u>Def</u> If $A_{ij} > 0$ then (E_{ij}, I), (I, E_{ij}) and their inverses are <u>basic</u> <u>pos</u> <u>eqs</u>. A finite chain of bpe's is a <u>pos</u> <u>eq</u>.

<u>Lemma</u>: [Boyle/Franks] If $A \xrightarrow{+} B$ then A is FE to B.

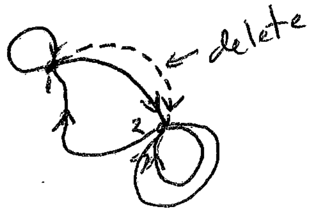
<u>"Proof"</u>:

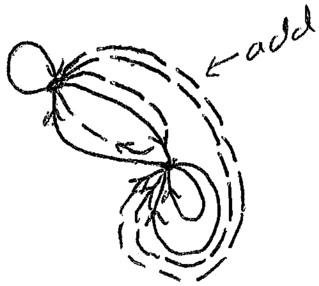
Examples

1) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad (E,I)(A) = B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1-1 & -2 \\ -1 & 1-2 \end{bmatrix}$ key

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix}$$

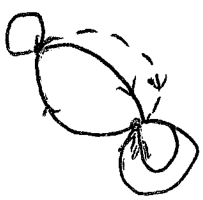




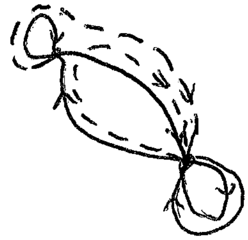
One edge from 1 to 2 is deleted. But we add an edge for each 2 edge path, that started with the removed edge.

2) $(I,E)(A) = B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix}$$



delete same edge



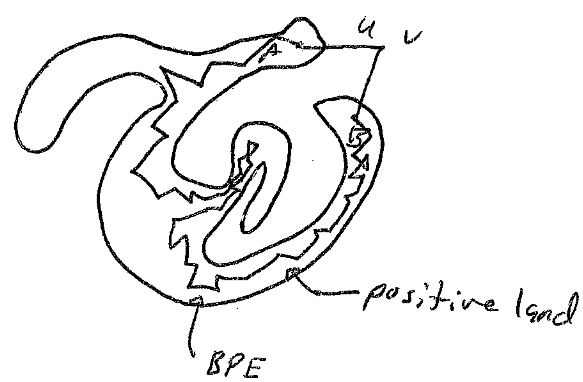
add an edge for each 2edge path that ended with the deleted edge.

3) $A = \begin{bmatrix} 5 & 0 \\ 2 & 7 \end{bmatrix}$. $(E,I)(A)$ and $(I,E)(A)$ are undefined.

Thm [Boyle] BPE generate FE.

Outline of Proof SSE can be broken down into basic splittings and their inverses (amalgamations). One factors basic splittings into BPE. PS can also be so factored.

Thm [Boyle] Let $U, V \in SL$ $(U,V)(A) = B$ with both A and B nonneg. Then (U,V) can be factored into B.P.E.s.

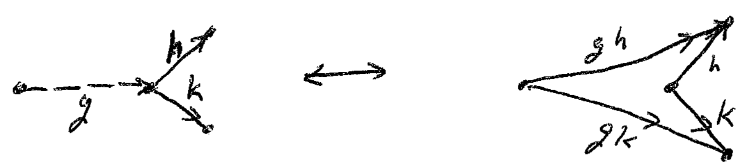


Pf: Hard. Messy. First make A and B "very positive."

Cor: Completeness of PS + BF follows.

Now back to $\mathbb{Z}G$.

The idea of BPE still works:



labels multiple

And [B+S] Boyles theorems generalize.
I should point out, for $G = \{1\}$ Boyle does not assume matrices are irreducible. For G finite group we do assume and use irreducibility.

I should point something else out. In order to prove a G-weighted version of the "positive factorization theorem," we need to assume that the subset of G realized as labels of closed orbits is the same for both matrices. (The definition is a little more complicated than this when G is nonabelian.)

<u>Def</u> Let A be a square matrix over $\mathbb{Z}G$, G a finite abelian group. Let $L(A) = \{g \in G \mid g \text{ appears as a summand in trace } A^n \text{ for some } n\}$. The definition of $L(A)$ for G finite but not abelian is messier.

<u>Thm</u> [Boyle–S.] Let G be a finite group and let A and B be nontrivial irreducible square matrices over $\mathbb{Z}_+G$. Then A and B are G-flow eq. iff $L(A) = L(B)$ and $\exists$ an $El(\mathbb{Z}G)$ equivalence from $(I-A)_\infty$ to $(I-B)_\infty$.

<u>Rmk</u> For $G = \mathbb{Z}/2$, $L(A)$ is the orientation invariant.

$G = \mathbb{Z}/2$

<u>Ex:</u> $A = \begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Then $(E, I)(A) = E \begin{bmatrix} 1 & -t \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -t \\ -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix} = B.$

(abuse of notation)

Thus A is TWFE to B.

Here is a basic positive factorization. Let $Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
$Q_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Then

$$(I, Q_1), (I, Q_2) \; (E, I) \; (I, Q_2^{-1}) \; (I, Q_1^{-1})$$

is a seq of BPE's from A to B.

$$\underset{A}{\begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}} \to \begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} t & 1 \\ 1 & 1+t \end{bmatrix} \to \begin{bmatrix} 1+t & 1+t \\ 1 & 1+t \end{bmatrix} \to \begin{bmatrix} 1+t & t \\ 1 & 1 \end{bmatrix} \to \underset{B}{\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}}$$

<u>Final Rmk</u> We still need to find a normal form for matrices over $\mathbb{Z}[\mathbb{Z}/2]$, but this ring is not a PID.

?

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