# G-weighted flow equivalence
joint with Mike Boyle

Warwick
July 8, 2003
1 hour

## Outline

Part I : Twistwise Flow Equivalence (TWFE) up to 1997

Part II : New Framework; ~~new results~~
Positive equivalence and $E(\mathbb{Z}G)$ equivalence.

Part III Progress on TWFE, at last!

Through-out this talk $G$ is a finite group.
In Parts I & III $G = \mathbb{Z}/2 = \langle t \mid t^2 = 1 \rangle$.

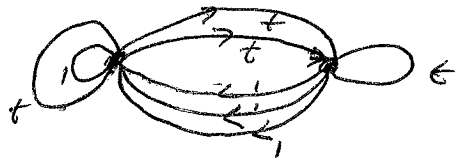
Recall $\mathbb{Z}G$ = integral group ring of $G$
= formal sums: $5g_1 - 6g_2 + g_3 + 7\underset{1}{\overset{}{e}}$

Stopping time = 5:45

## Part I

Let $G = \mathbb{Z}/2 = \langle t \mid t^2 = 1 \rangle$.

Given a square matrix $A(t)$ over $\mathbb{Z}_+ G = \{a + bt \mid a \geq 0, b \geq 0\}$ we get a $G$ labeled graph.

Ex $\begin{bmatrix} 1+t & 2t \\ 3 & t \end{bmatrix} \longleftrightarrow$ 

(not sofic)

And a SFT $(\sigma, X_{A(1)})$ and a label function $f : X_{A(1)} \to G$, $f(x)$ = label on $x_0$.

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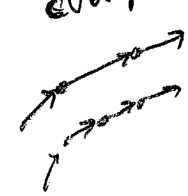
Def: Call these matrices "twist matrices."

Def: SSE move: $A = RS$, $B = RS$

Def PS move: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ a_{11} & 0 & a_{12} & \cdots & & a_{1n} \\ \vdots & \vdots & & & & \\ a_{n1} & 0 & a_{n2} & \cdots & & a_{nn} \end{bmatrix}$

$\downarrow$ time change



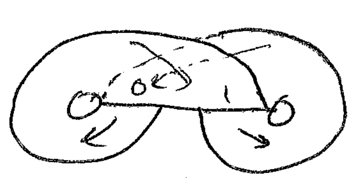
Def (really a Thm of S following PS)

$A(t)$ is twistwise flow eq to $B(t)$ iff $\exists$ a seq of SSE and PS moves from $A$ to $B$. This is an eq. relation.

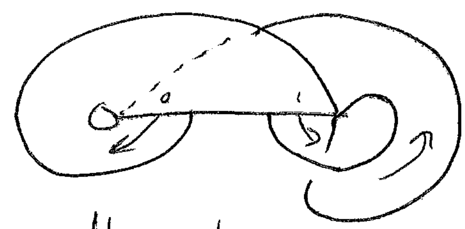
__Motivation__ (Very intuitive due to time $< \infty$)

__Template__: ← Williams
branched 2-manifold with semi-flow used to model ~~certain~~ invariant sets of certain flows on 3-manifolds (e.g. Smale flows)

__Ex__




Lorenz




Horseshoe

- Invariant set has

* Cantor set as cross-section

- 1-1 correspondence with bi-$\infty$ seq of 0's 1's (full 2-shift)
- has incidence matrix $\begin{bmatrix}1&1\\1&1\end{bmatrix}$

Invariant set is top. eq. Matrix is same

In "ordinary" symbolic dynamics ($G$ trivial, $t=1$) $A$ and $B$ gor $X_A$, $X_B$ are flow eq iff their suspension flows, $F_A$, $F_B$ are top eq.

**Thm** (Parry-Sullivan) SSE and PS generate flow eq.

**Def** $PS(A) = \det(I-A)$ $\quad BF(A) = \dfrac{\mathbb{Z}^n}{(I-A)\mathbb{Z}^n}$

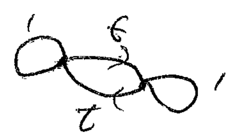
**Thm** [Franks 84] For nontrivial, irr, matrices over $\mathbb{Z}_+$ PS + BF determine flow eq.

Yet, Lorenz and H.S. "look different".
Idea (S) encode twist $\begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix}$ with $t^2=1$.

Use unit normal bundle to get "Ribbon sets"
Define twistwise flow eq to be homeo of Ribbon sets. Prove earlier def is a thm.

**Def** $PS^{\pm}(A(t)) = PS(A(\pm 1)) = \det(I - A(\pm 1))$
$BF^{\pm}(A(t)) = BF(A(\pm 1))$
$BF^{\partial}(A(t)) = BF((A+B)\otimes I_2 + (A-B)\otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ "double cover"
$O(A) = \begin{cases} \text{yes} & \text{if no loop in graph has weight } t \\ \text{no} & \text{otherwise} \end{cases}$

**Thm** [S] These are invariants of TWFE.




$O$ = yes

Are they complete???

**Ex** $\begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ $\quad PS^{\pm} = \pm 1 \Rightarrow$ group $= 0$

$O$ = no

Yet I could not find a seq of SSE + PS moves connecting them — 1997

Part II  G is any finite group

Def  Two G-labeled SFT's $\left(\begin{array}{ll} \sigma: X \to X & \text{is shift.} \\ f: X \to G & \text{label map} \end{array}\right)$ are G-cong.

if  $\exists\ h: X_A \to X_B$ s.t. $h \circ \sigma_A = \sigma_B \circ h$

and  $\exists\ c: X_A \to G$ s.t. $f_A = f_B \circ h \cdot c \cdot (c \circ \sigma_A)^{-1}$

Thm [Parry via Boyle]  G-cong $\Leftrightarrow$ SSE over $\mathbb{Z}_+G$.

Obviously, this is harder than SSE over $\mathbb{Z}_+$!

Def (Really Thm)  The eq. rel. given by SSE and PS over $\mathbb{Z}_+G$ we call G-flow eq.

Mike Boyle and I have a thm that simplifies the problem of G-flow eq, for irr matrices. Boyle proved this thm in the case G trivial, but allowing for reducible matrices.

**Def** [Positive Equivalence]

Notation: $(U,V): A \to UAV$.

Let $E = E_{ij}(g)$ be the elementary matrix with off diagonal matrix entry $g$ at $(i,j)$.

Assume $g$ is a summand of $A_{ij}$, where $A$ is an ~~N×N~~ square matrix of finite support over $\mathbb{Z}_+G$.

Note: We allow $\infty$ $N \times N$ matrices

Then each of the matrix equivalences

$(E,I): (I-A) = E(I-A) = I-B$, $(E^{-1},I)$:

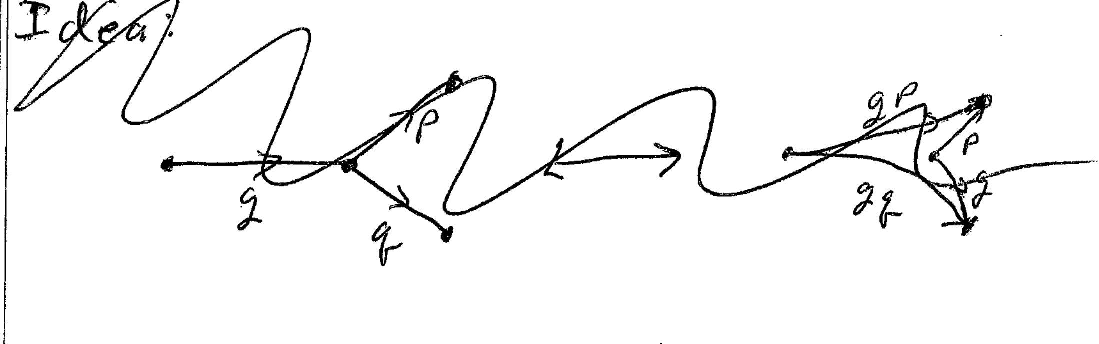
$(I,E)$: $(I,E^{-1})$: check these

is called a basic pos. eq. (BPE) over $\mathbb{Z}_+G$. ~~Provided we stay nonneg.~~

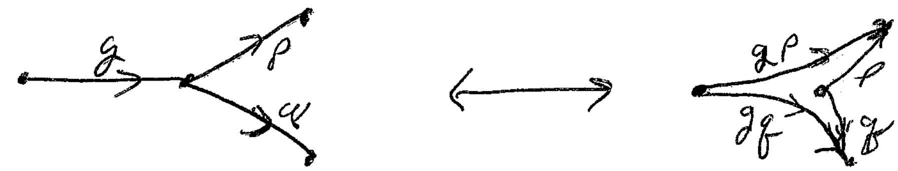
A chain of BPE's is a pos eq (PE).

Notation: $I-A \overset{+}{\sim} I-B$

Idea:



Idea: ~~Example~~




(mention ess. irr.)

**Prop** ~~[illegible]~~ Let $A$ and $B$ be square ← could be different sizes matrices over $\mathbb{Z}_+ G$. Then $(I-A)_\infty \overset{SE}{\sim} (I-B)_\infty$ iff $S_A = (\sigma_A, X_A)$ f.r. $S_B = (\sigma_B, X_B)$ are $G$-flow equivalent.

**Rmk** This is nice because SSE and PS are less natural than BPE. But, ~~one has~~ we have to be careful to stay positive along each step. Or do we?

**Def** [Weights Groups or Class]

Suppose $A$ is a finite square irreducible matrix over $\mathbb{Z}_+G$. Let $i$ be a vertex in the graph for $A$. Define $W_i(A)$ to be the subgroup of $G$ generated by weights of loops from $i$ to $i$.

**Thm:** If $G$ is abelian all the $W_i(A)$ are equal and we call $W(A) = W_i(A)$, the <u>weights</u> ~~group~~ of $A$.

**Thm:** If $G$ is nonabelian all the $W_i(A)$ are conjugate and we define $W(A)$ to be the conjugacy class in $G$ of $W_i(A)$.

For $G = \mathbb{Z}/2$ $W(A) = \mathbb{Z}/2$ or is trivial group.

So, $W(A)$ generalizes orientation. It was motivated by the "$\Gamma$" group defined in Parry-Schmidt, 1984.

**Thm** Let $A$ be irr matrix over $\mathbb{Z}_+G$, with $H = W(A)$. Then, $(I - A) \overset{+}{\sim} (I - B)$ where $B$ is over $\mathbb{Z}_+H$.

$\exists B$ over $\mathbb{Z}_+H$ s.t.

# MAIN THEOREM

**Thm** ~~Let G be a finite group and let A and B~~
Let A and B be essentially irreducible, nontrivial matrices over $\mathbb{Z}_+ G$.
Assume $W(A) = W(B) = G$.
Then: ~~$\mathcal{O}_A$ and $\mathcal{O}_B$ are G-flow equivalent~~

① If $(U,V) : (I-A)_\infty \to (I-B)_\infty$ is an $E(\infty, \mathbb{Z}G)$ eq, it ~~(can be factored)~~ ~~into~~ is a positive equivalence.

② $S_A$ and $S_B$ are G-flow eq iff $\exists$ a $E(\infty, \mathbb{Z}G)$ eq from $(I-A)_\infty$ to $(I-B)_\infty$.

Rmks on Proof: ① ⇒ ② is immediate.

Proof of ① is messy, but elementary.
Talk about very positive matrices.

Part III $\quad G = \mathbb{Z}/2$.

Ex: Let $A = \begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$$(E, I)(I - A) = I - B \Rightarrow A \overset{twfe}{\sim} B.$$

~~Rmk: $SL(\mathbb{Z}G) = E(\mathbb{Z}G)$ for $G = \mathbb{Z}/2$~~

However, $\mathbb{Z}G$ is still not a PID and $\rightarrow (1+t)(1-t) = 0$
there is not a normal form for SL-eq.
But,

Thm (Normal Form). Let $M$ be an ~~square~~ $n \times n$ matrix over $\mathbb{Z}G$. Write $M = A + Bt$ with $A$ and $B$ $n \times n$ matrices over $\mathbb{Z}$. If $\det(A+B)$ is odd, then $M$ is $E(\mathbb{Z}G)$-eq to a Smith normal form. This form corresponds to $(C, D)$ where $C$ and $D$ are Smith normal forms of $A+B$ and $A-B$, resp.

Cor: hold if $A+B$ has only one even elementary divisor.

Cor: In this case $PS(\pm 1)$, $BF(\pm 1)$, $O$ are complete

Cor: $\exists$ a matrix $M$ over $\mathbb{Z}G$ that is not $E(\mathbb{Z}G)$-eq to a triangular matrix; so no Smith normal form in general.

~~Prop~~

Cor If $\det A \oplus B = $ odd (or ...), then $M$ is $E(\mathbb{Z}G)$-eq to its transpose.

Rmk Time reversal. Always true for $G$ = trivial [Franks].

Cor But, $\exists$ $M$ over $\mathbb{Z}G$ that is not $E(\mathbb{Z}G)$-eq to its transpose.

GL (?)

## Conclusion

TWFE reduced to alg.

Computations are easier.

New fact about time reversal of flows.