Weighted Flow Equivalence of Shifts of Finite Type

Michael Sullivan
Southern Illinois University
USA
msulliva@math.siu.edu
http://www.math.siu.edu/sullivan

(Based on joint work with Mike Boyle)

Pan-American Advanced Studies Institute on Differential Equations and Nonlinear Analysis
Universidad de Chile, Santiago, Chile
January 2005

Classification of SFTs up to FE

DEFINITION: PS(A) = det(I-A) is the Parry-Sullivan number of A.

DEFINITION: BF(A) $= \dfrac{\mathbb{Z}^n}{(I-A)\mathbb{Z}^n}$ is the Bowen-Franks group of A.

THEOREM (Franks): Let A and B be square matrices of nonnegative integers that are non-trivial and irreducible. Then they are flow equivalent if and only if

(1) PS(A) = PS(B)

(2) BF(A) $\cong$ BF(B)

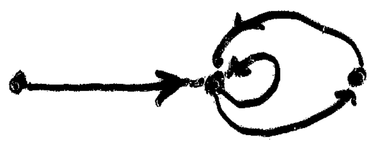
Types of SFTs

Trivial: Finite number of orbits. Permutation matrix.

Irreducible: Path from any vertex to any vertex. For any (i,j) there is an n with $M^n(i,j)>0$.

Essentially Irreducible Matrix: Matrix has a unique principal submatrix that is irreducible. Passing to the IRREDUCIBLE CORE does not change the shift space.

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad core(M) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



$$X_M = X_{cor(M)}$$

Strong Shift Equivalence (SSE)

Let A & B be square matrices over $\mathbb{Z}_+$. We say there is an SSE-move from A to B if there are matrices R & S over $\mathbb{Z}_+$ with

$$A = RS \text{ and } B = SR.$$

Two square matrices, A & B, over $\mathbb{Z}_+$ are Strong Shift Equivalent (SSE) if there is a sequence of SSE-moves from A to B.

FUNDAMENTAL THEOREM of SFTs (Williams): Two SFTs are topologically conjugate if and only if their matrices are SSE.

It is unknown if SSE is decidable, but many computable invariants exist.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

So $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is SSE to $\begin{bmatrix} 2 \end{bmatrix}$

Shifts of Finite Type (SFT)

A SHIFT SPACE is a subset of the set of bi-infinite sequences of symbols from a finite alphabet that is invariant under the SHIFT MAP. The topology is the subset topology of the product topology. (compact)

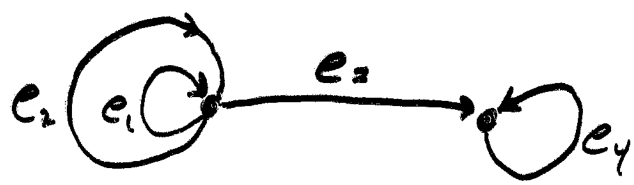
$$\sigma(\ldots 001.1001\ldots) = \ldots 0011.001\ldots$$

Shifts spaces are TOPOLOGICALLY CONJUGATE if there is a homeomorphism between them that commutes with the shift maps.

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "\sigma_Y"] & Y \end{tikzcd} \qquad h \circ \sigma_X = \sigma_Y \circ h$$

A directed graph or its adjacency matrix determines a special type of shift space called a SFT, up to topological conjugacy. We illustrate with an example.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

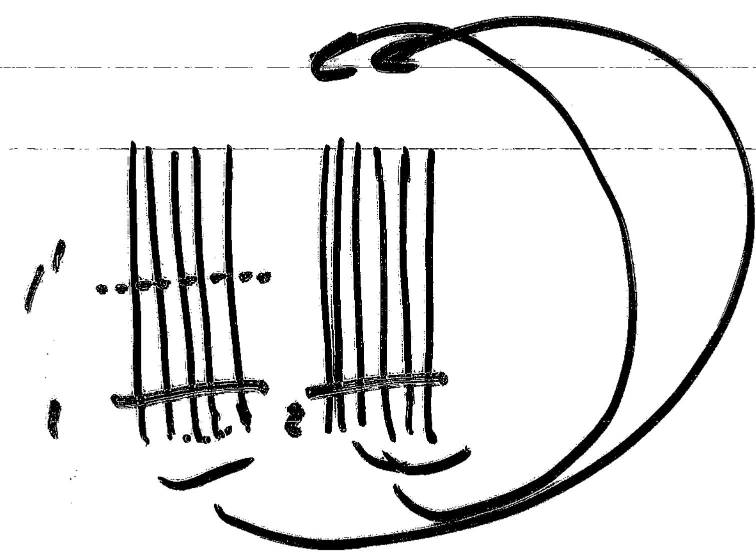


$$X_A = \left\{ \begin{array}{c} \text{All sequences} \\ \text{of } e_1 \text{ and } e_2 \end{array} \right\} \cup \{ e_4^\infty \} \cup \left\{ \overset{\text{any } e_1 \,\&\, e_2}{\ldots\ldots} e_3\, e_4\, e_4 \ldots \right\}$$

The PS-move

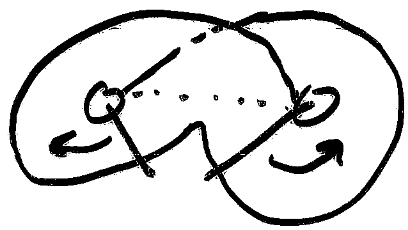
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & a_{11} & \cdots & a_{1n} \\ 1 & 0 & \cdots & 0 \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

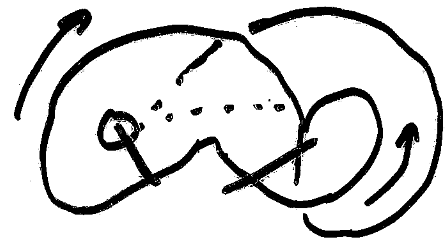
THEOREM (Parry & Sullivan): FE of matrices is generated by SSE and the PS-move.



Twistwise Flow Equivalence

Let $G = \mathbb{Z}/2$.





The indicated cross sections are both full 2-shifts. Thus, they are FE.

But they are not TWFE.

Create TWIST MATRICES:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$$

Invariants of TWFE

- $\mathrm{PS}^+(\mathrm{A}) = \mathrm{PS}(\mathrm{A}(1))$.
- $\mathrm{PS}^-(\mathrm{A}) = \mathrm{PS}(\mathrm{A}(-1))$.
- $\mathrm{BF}^+(\mathrm{A}) = \mathrm{BF}(\mathrm{A}(1))$.
- $\mathrm{BF}^-(\mathrm{A}) = \mathrm{BF}(\mathrm{A}(-1))$.
- $\mathrm{BF}^\partial(\mathrm{A}) = \mathrm{BF}(\mathrm{A}(\mathrm{T}))$, where $\mathrm{T}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\mathrm{Or}(\mathrm{A}) =$ yes if no Möbius , no otherwise.

EXAMPLE:

$$\mathrm{A}=\begin{pmatrix} 0 & t \\ 1 & 1 \end{pmatrix} \qquad \mathrm{B}=\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$$

All invariants are the same. But are they TWFE???