Transverse Foliations to nonsingular Morse-Smale flows on the 3-sphere and Bott-integrable Hamiltonian systems & Contact Structures

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Nonsingular Morse-Smale Flows

Definition 1 A flow ϕ on a manifold M is a **Morse-Smale flow** if the following hold.

- The chain recurrent set is hyperbolic.
- The stable and unstable manifolds of basic sets meet transversely.
- Each basic set consists of a single closed orbit or fixed point.

For M a compact manifold, it follows that Morse-Smale flows have a finite number of periodic orbits and fixed points. A **nonsingular flow** is a flow without fixed points.

Notation: Invariant sets are indexed by: 0 for an attractor, 1 for a saddle, and 2 for a repeller.

Not all 3-manifolds can support NMS flows. John Morgan has given a theorem that characterizes just which 3-manifolds do. (In higher dimensions Asimov has shown that all manifolds with Euler characteristic 0 support NMS flows.)

Wada's Theorem

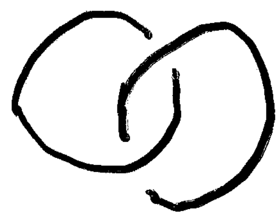
Which indexed links can be realized as invariant sets of NMS flows of S^3, you ask?

Theorem 1 (Wada) Let $\mathcal{F}$ be the set of indexed links which can be realized as the collection of periodic orbits of a nonsingular Morse-Smale flow on S^3, respecting index. Then $\mathcal{F} = \mathcal{W}$, where $\mathcal{W}$ is defined on the next few transparencies.

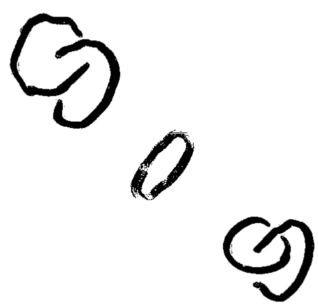
Wada's Links

Definition 2 Let $\mathcal{W}$ be the collection of indexed links determined by the following axioms:

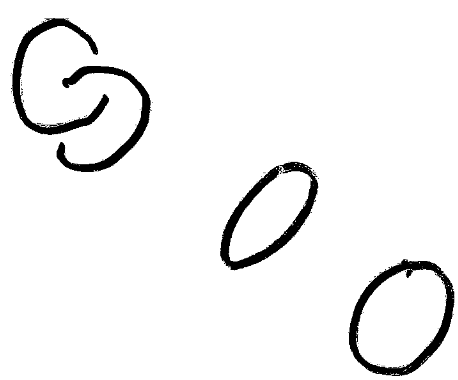
W0: The Hopf link indexed by 0 and 2 is in $\mathcal{W}$.



W1: If $L_1, L_2 \in \mathcal{W}$ then $L_1 \circ L_2 \circ u \in \mathcal{W}$, where u (here and below) is an unknot in S^3 indexed by 1.



W2: If $L_1, L_2 \in \mathcal{W}$ and K_2 is a component of L_2 indexed by 0 or 2, then $L_1 \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

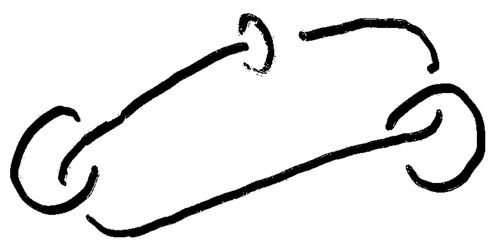


W3: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 with indices 0 and 2 (resp.), then $(L_1 - K_1) \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W4: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 (resp.) each with index 0 or 2, then

$$((L_1, K_1) \# (L_2, K_2)) \cup m \in \mathcal{W},$$

where $K_1 \# K_2$ shares the index of either K_1 or K_2 and m is a meridian of $K_1 \# K_2$ indexed by 1.



W5: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L replacing a tubular neighborhood of K with a solid torus with three closed orbits, K_1, K_2, and K_3. K_1 is the core and so has the same knot type as K. K_2 and K_3 are parallel (p, q) cables of K_1. The index of K_2 is 1. The indices of K_1 and K_3 may be either 0 or 2, but at least one of them must be equal to the index of K.

W6: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L by changing the index of K to 1 and placing a $(2, q)$-cable of K in a tubular neighborhood of K, indexed by i.

W7: $\mathcal{W}$ is minimal. That is, $\mathcal{W} \subset \mathcal{W}'$ for any collection, $\mathcal{W}'$, satisfying W0-W6.

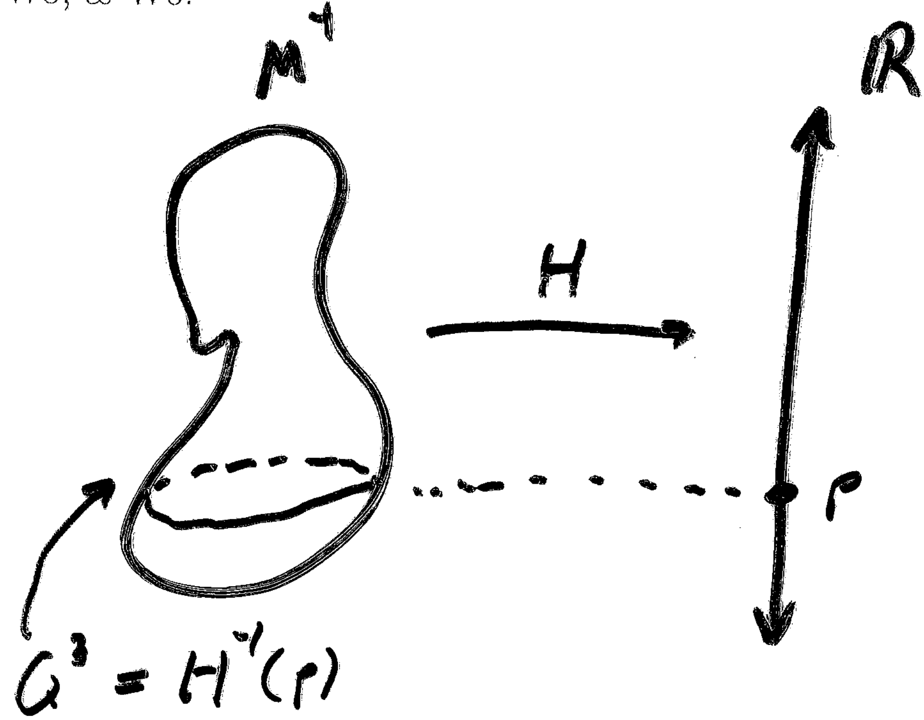
Remark 1 The last condition, W7, means that $\mathcal{W}$ is generated from the indexed Hopf link in S^3 by applying operations W1-W6.

Bolt-Integrable Hamiltonian Systems

In 1998 Casasayas, Alfaro, & Nunes studied indexed links of fixed points of flows induced by certain Hamiltonian systems.

They showed that these links were a subset of the NMS links.

On S^3 they showed that this subset of $\mathcal{W}$ is generated by W0, W4, W5, & W6.



Contact Flows: Ghrist & Etnyre

In 1999 Ghrist & Etnyre studied gradient flows of 3-manifolds tangent to plane fields associated to a contact structure. In these flows there are indexed links of fixed points.

They showed that these links were a subset of the NMS links.

On S^3 they showed that this subset of $\mathcal{W}$ is generated by W0, W4, W5, & W6.

Transverse Foliations to Flows

Definition 3 A **2-dimensional foliation** $\mathcal{F} = \{L_\alpha\}$ of a 3-manifold M is a partition of M such that $\forall x \in M\ \exists$ a chart $(U_x, \phi : U_x \to \mathbb{R}^3)$ such each connected component of $\phi(U \cap L_\alpha)$ is of the form $\{(x,y,z) \in \phi(U_x) \,|\, z =$ a constant$\}$. The L_α's are called the leaves of the foliation.

Definition 4 An indexed link on a 3-manifold has the **Linking Property** if for every closed orbit that bounds a disk, there is an attracting or repelling closed orbit that has nonzero algebraic linking number with that disk.

Theorem 2 (Goodman; see also Yano) A nonsingular Morse-Smale flow on a 3-manifold has a transverse 2-dimensional foliation (each flow line meets any leaf transversely) if and only if its periodic orbits satisfy the linking property.

Wada moves and Transverse Foliations

Theorem 3 (S) The set of indexed links that can be realized as the set of periodic orbits of nonsingular Morse-Smale flows on S^3 that have transverse foliations is the subset of $\mathcal{W}$ generated by W0, W4, W5, & W6.

THE BIG QUESTION