Flows

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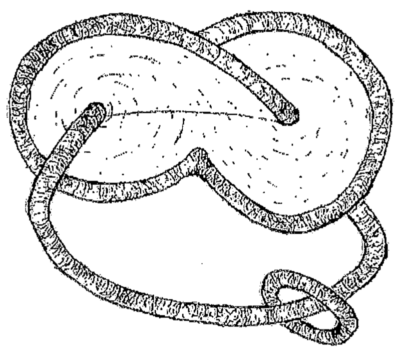

Show Me Undergraduate Math Conference

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Manifolds

ROUGH DEFINITION: An n-dimensional **manifold** is a set M where for each point $p \in M$ there is an open neighborhood U_p that is homeomorphic to $\mathbb{R}^n$ – that is there is a continuous bijection between U_p and $\mathbb{R}^n$.

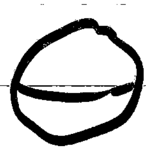
Example: The open unit disk in $\mathbb{R}^2$, $D = \{(r, \theta) | r < 1\}$ is homeomorphic to all of $\mathbb{R}^2$ via

$$\Theta = \theta \qquad\qquad R = \tan\left(\frac{\pi}{2} r\right)$$

Examples of 1-dimensional manifolds:

the real number line.

Examples of 2-dimensional manifolds:



Sphere

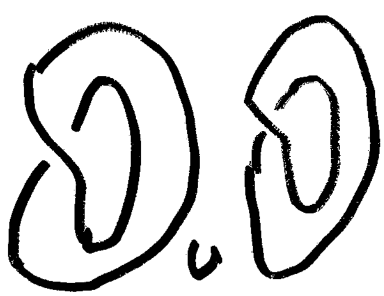
Torus

double torus

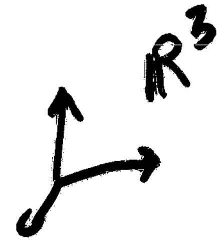
Sphere ↳ = union of two closed disks.

Torus "

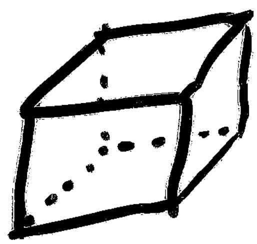
identify opposite edges



Klien Bottle = union two Möbius bands along boundary

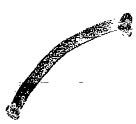


Examples of 3-dimensional manifolds:



Identify opposite faces = 3-torus

The Definition can be modified to include manifolds with boundary.



Solid torus

One can glue two solid tori to make the 3-sphere.

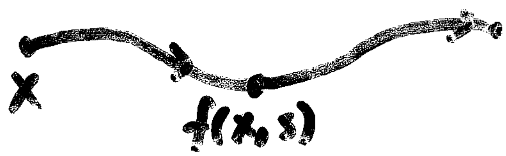
Solid ball with solid torus deleted

Flows

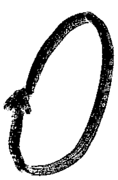
DEFINITIONS.

(1) A **flow** on a manifold M is a continuous function $f : M \times \mathbb{R} \to M$ such that for all $x \in M$ and all $s, t \in \mathbb{R}$

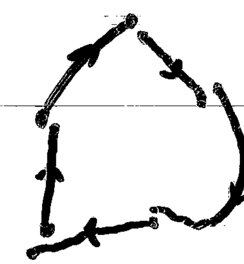
$$f(x, 0) = x$$
$$f(f(x, s), t) = f(x, s + t)$$



(2) For each $x \in M$ let $\mathcal{O}(x) = \{f(x, t) \in M \mid t \in R\}$. It is called the **orbit** of x – also called the **trajectory** of x.



(3) If $\mathcal{O}(x) = \{x\}$ then x is a **fixed point**. A flow is **nonsingular** if it has no fixed points. If $\mathcal{O}(x)$ is a closed loop then the orbit is **periodic**.



(4) The **chain recurrent** set of a flow f on M is

$\mathcal{R} = \{x \in M \mid$ For every $\epsilon > 0$, there exists points

$x_0 = x, x_1, x_2, ..., x_k$ in M and $t_1, t_2, ..., t_k$ in $\mathbb{R}^+$

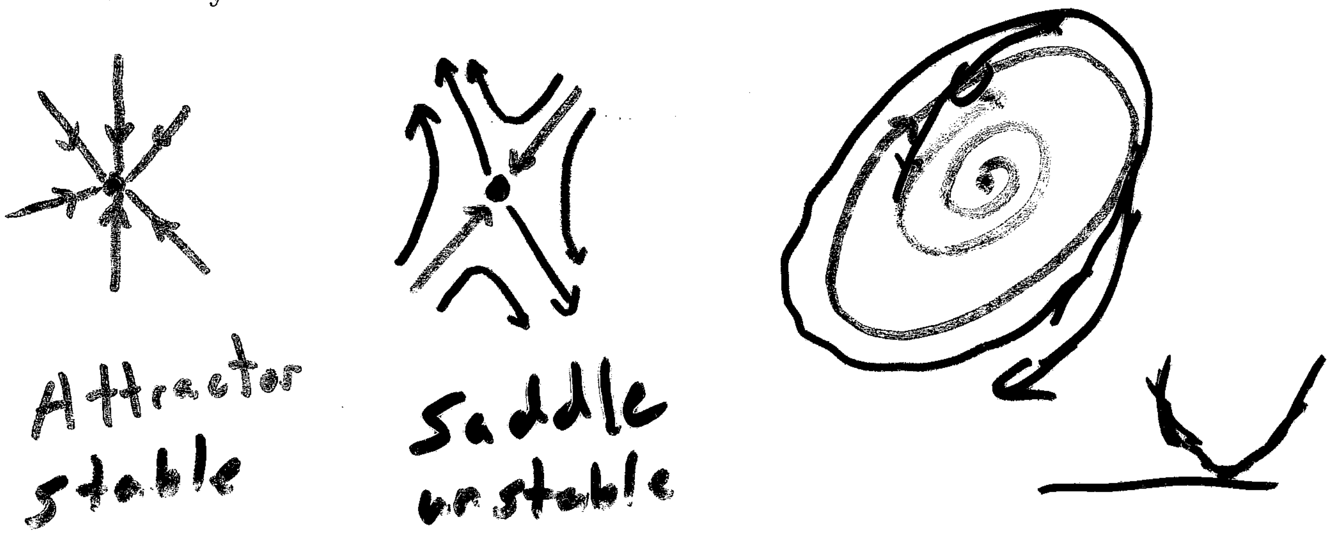
such that distance$(f(x_i, t_i), x_{i+1}) < \epsilon$ for $i = 1, ..., k - 1$

and distance$(f(x_k, t_k), x_0) < \epsilon\}$

Examples: Fixed points and closed loops. Chaotic examples later.

Local Stability and Structural Stability

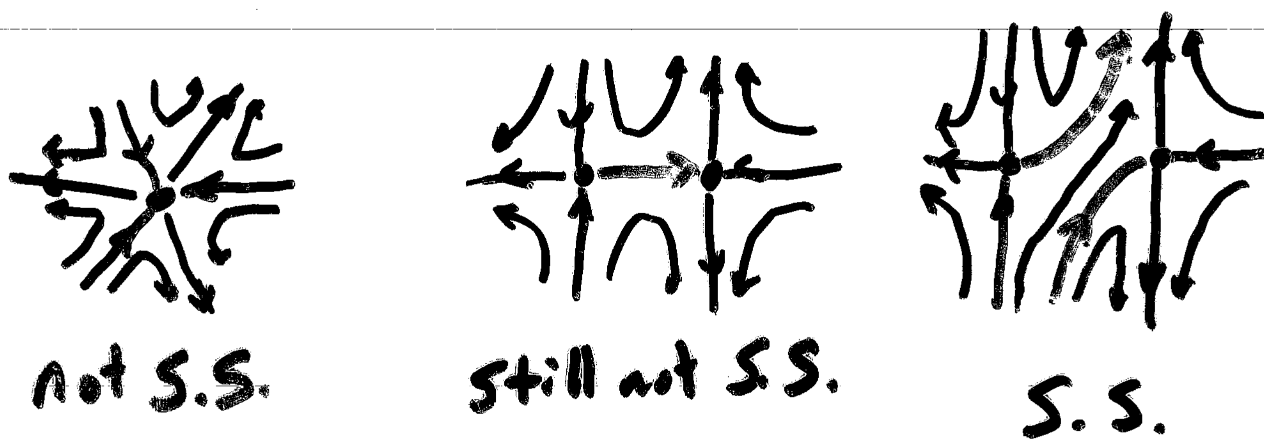
Local Stability means orbits near an invariant set stay near by.



We will be more concerned with Structural Stability.

This means small enough "perturbations" can create only "topologically equivalent" flows.

Two flows, say f on M and g on N, are **topologically equivalent** if there is a homoeomorphism from M to N that takes orbits of f to orbits of g respecting the flow direction.



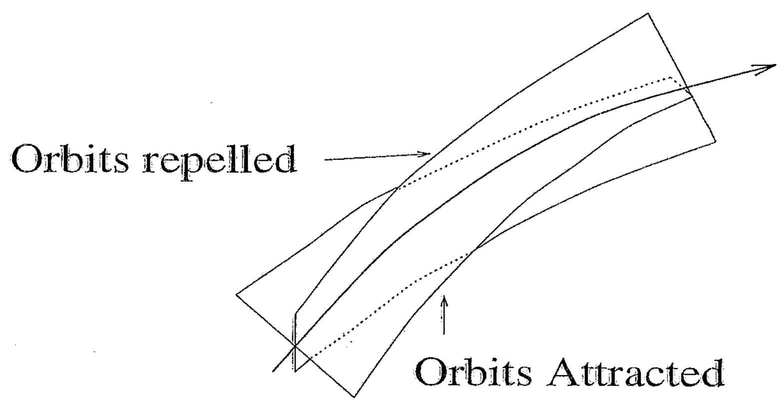


Figure 1.1: Stable and Unstable manifolds for an orbit.

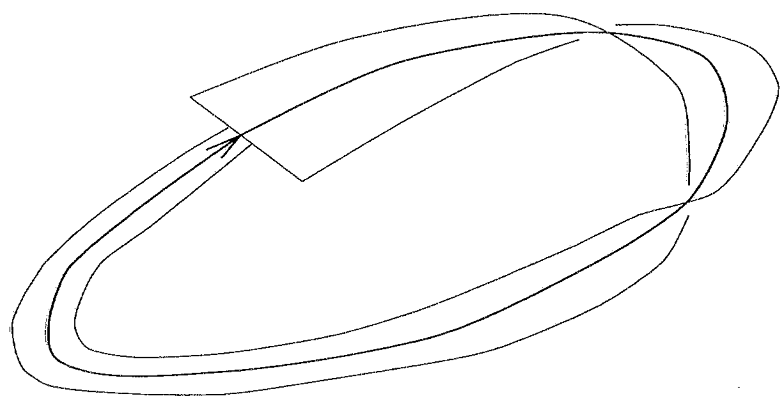
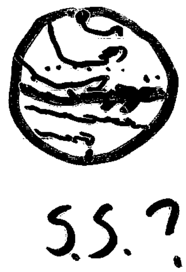
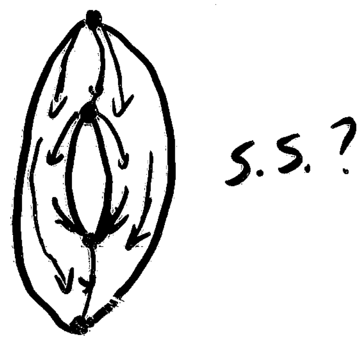
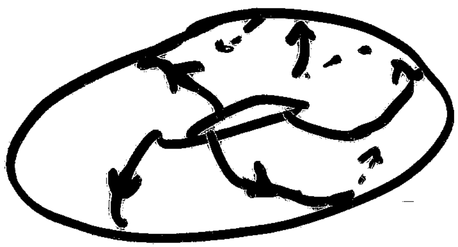


Figure 1.2: Stable manifold for a closed orbit with a full twist.

Flows on Surfaces



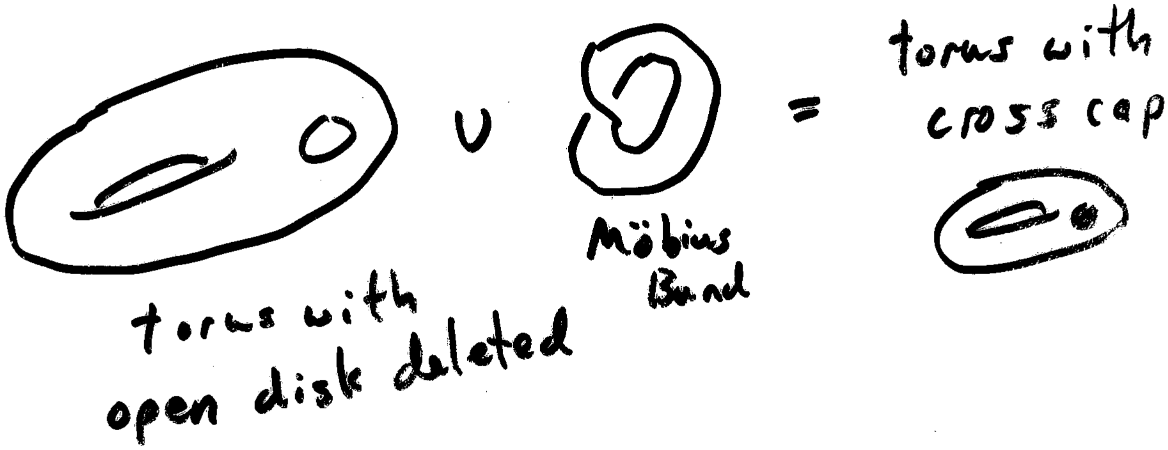




On any ~~compact~~ closed orientable surface a structurally stable flow will have chain recurrent $\mathcal{R}$ that is the union of a finite number of fixed points (attractors, repellers, or saddles) and periodic orbits (attractors or repellers). [See Palis & de Melo.]

This is also true for flows on the Klein Bottle and the ~~torus with a cross cap.~~

It is unknown for other orientable surfaces.

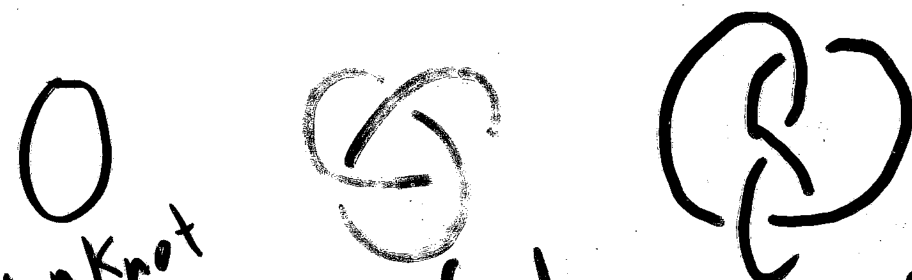


Knots

A **knot** is just a closed curve in 3-manifold. In flows on 3-manifolds periodic orbits form knots.

Two knots, K_1 and K_2, in M are of the same **type** if M can be continuously deformed so that K_1 now coincides with K_2. (The technical term for the deformation is **ambient isotopy**.)

Examples of Knots:

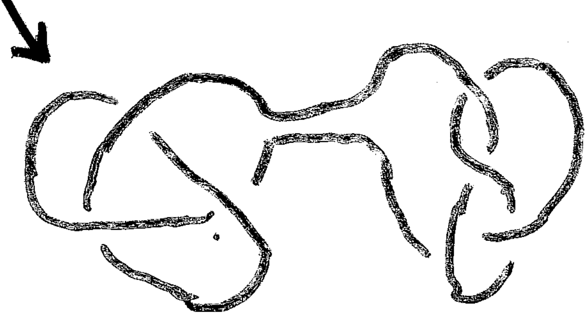


unknot

trefoil

figure-8 knot

Prime verses composite Knots:



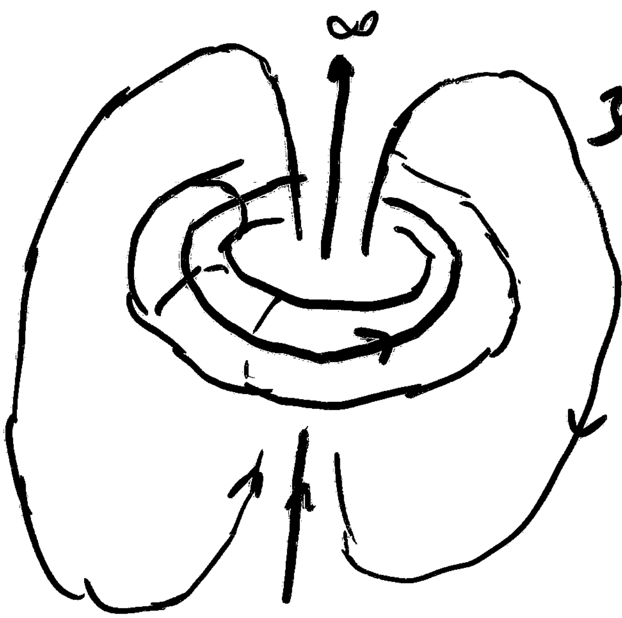
Every knot can be factored uniquely into prime knots!

Connected sum of trefoil and figure-8 knot.

Flows on 3-manifolds

Hopf Flow:

Is S.S.



There is a smooth flow on the 3-sphere with no fixed points or periodic orbits. [Kupperberg]

not SS

There is a ~~sm~~ smooth flow on the 3-sphere with all knots [Ghrist]

Is S.S.

Morse-Smale Flows

Rough Definition: A flow is a Morse-Smale flow Chain Recurrent set is a finite collection of points and periodic orbits.

- S. S.
- Are only S.S. on closed orientable surfaces.
- But this is not true on 3-manifolds.
- Only some 3-manifolds support nonsingular Morse-Smale Flows.

Smale Flows

Rough Definition: A flow is a Smale flow if each component of the Chain Recurrent is 1-dimensional or a fixed point. We will look mainly at nonsingular Smale flows on S^3.

Smale showed that the chain recurrent set of a Smale flow can have infinitely many periodic orbits. These are organized into chaotic saddle sets.

All other components of the chain recurrent set are isolated fixed points or isolated periodic orbits.

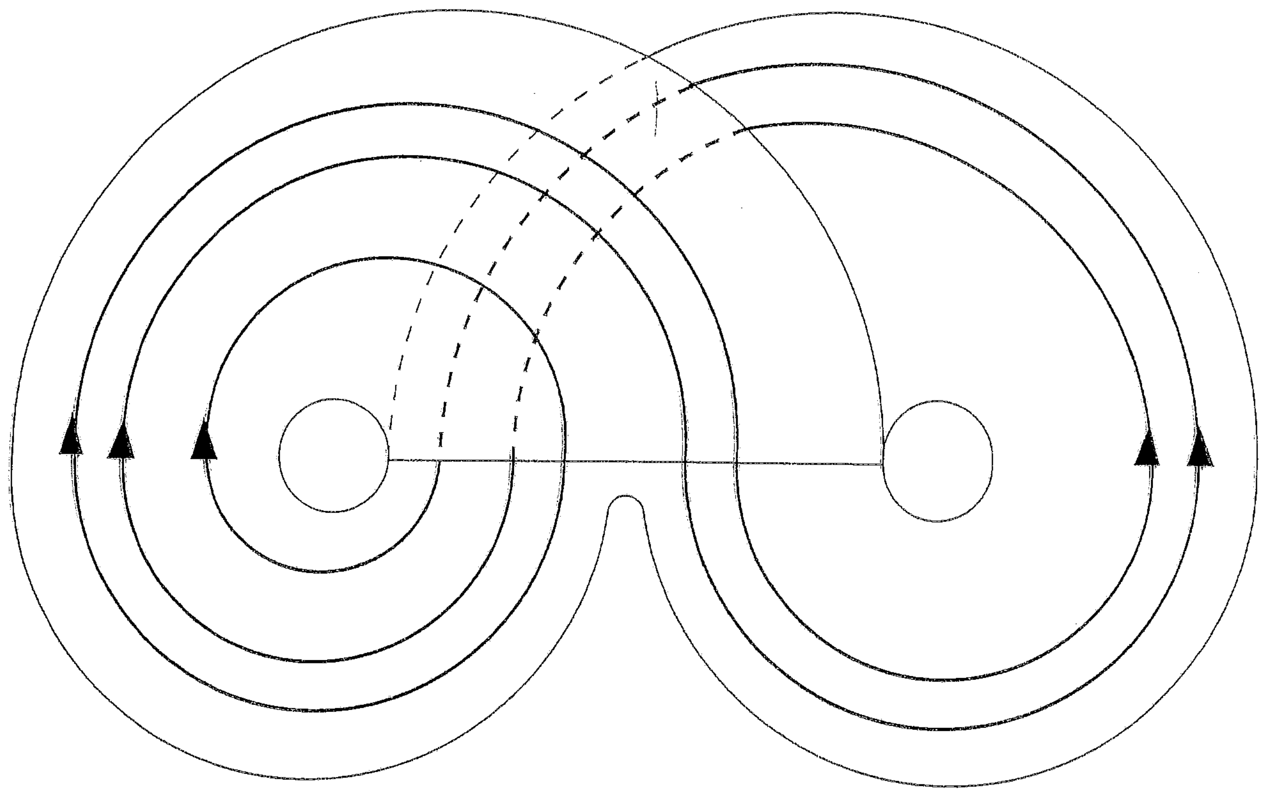
Smale flows are structurally sable.

But how can one study a chaotic saddle set? What would one look like?

Templates

ROUGH DEFINITION: A **template** is a branched 2-manifold with a semi-flow.

Examples:



Lorenz template
with trefoil orbit.

⟶ Knots on Templates ⟵

- Templates can be braided and have infinitely many knot types (Franks & Williams).
- Lorenz knots are prime positive braids (Williams, 1983).
- Some templates contain all knots and links (Ghrist, 1997).
- A positive braided template has a bound on the number of prime factors of its knots (S., 2005).

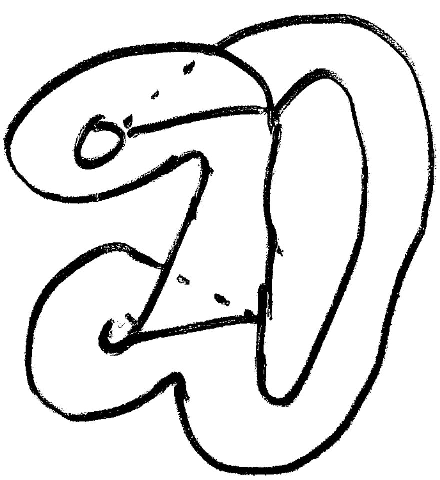
Examples:

$L(0,n)$ $n \geq 0$: Prime

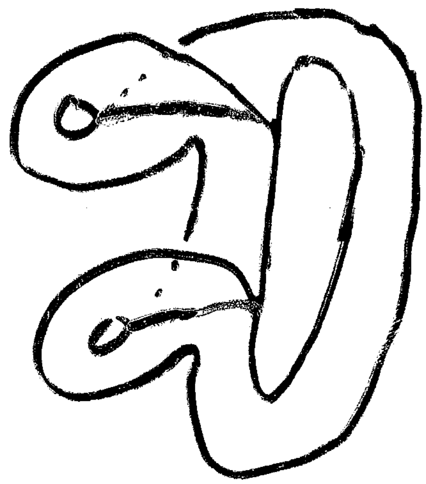
$L(0,n)$ $n < 0$: all knots!

$L(n,m)$ $n,m > 0$: max 2 prime factors.

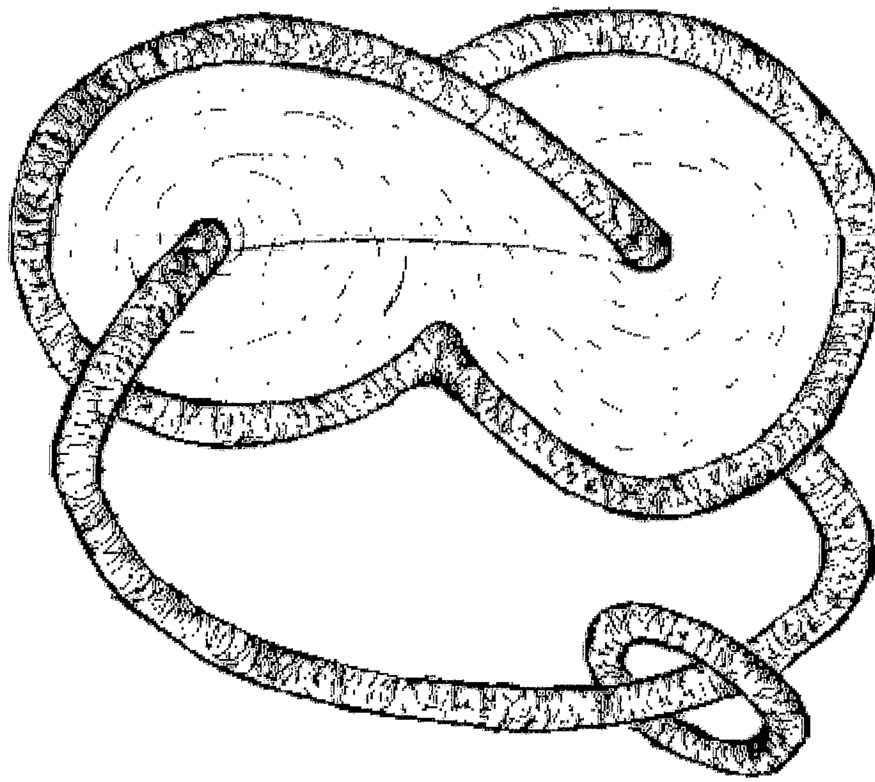
$L(-1,-1)$: max 3 prime factors.



All knots



Prime only.



Open Questions

- Very little is known about Smale flows on other 3-manifolds.

- How can you tell if two templates are the "same"?

- Suppose you have two knots, perhaps linked together. Can you build a Smale flow with one an attractor and the other a repeller? [with no fixed points.]

- When does a template have all knots? Is there a test?

Further Reading

Knots and Links in Three-Dimensional Flows – Ghrist, Holmes & Sullivan

Geometric Theory of Dynamical Systems – Palis & de Melo

Differential Equations and Dynamical Systems – L. Perko

Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields – Guckenheimer & Holmes

Knots and Link – Dale Rolfsen

Knots – Burde & Zieschang

Formal Knot Theory – Kauffman

An introduction to knot theory – W. Lickorish

Knots and Surfaces – Gilbert & Porter

Introduction to Topological Manifolds – John M. Lee

A Combinatorial Introduction to Topology – Michael Henle

The Topology of Chaos – Gilmore & Lefranc