Transverse Foliations to nonsingular Morse-Smale flows and Bott-integrable Hamiltonian systems

Michael C. Sullivan

Department of Mathematics
Southern Illinois University
Carbondale, IL 62901-4408
mikesulliva (at) math (dot) siu (dot) edu
http://www.math.siu.edu/sullivan

Flows on Manifolds

Let M be a closed Riemannian 3-manifold.

A flow ϕ_t on M is a group action of $\mathbb{R}$, the real numbers. That is for all x in M and any two real numbers s, t we have

$$\phi_0(x) = x$$
$$\phi_{s+t}(x) = \phi_s(\phi_t(x))$$

A point $x \in M$ is a **fixed point** for a flow if $\phi_t(x) = x$ for all $t \in \mathbb{R}$.

A flow without fixed points is called a **non-singular** flow.

The set $\mathcal{O}(x) = \{\phi_t(x) \,|\, t \in \mathbb{R}\}$ is called the orbit of x.

A point x is said to be **periodic** if it is not a fixed point and for some $\tau \neq 0$ we have $\phi_\tau(x) = x$. In this case $\mathcal{O}(x)$ is an embedding of the 1-sphere (a circle) in M.

An embedded 1-sphere in a 3-manifold is called a **knot**.

We will be working with knotted periodic orbits.

Let d be a metric on M.

The **chain recurrent set**, $\mathcal{R}$, of a flow ϕ_t on M is the set of points $x \in M$ such that for any $\epsilon > 0$ there exits points $x = x_1, x_2, \ldots, x_n$ and positive real numbers $t_1 < t_2 < \cdots t_n$, with $d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$ for $i = 1, \ldots, n-1$ and $d(\phi_{t_n}(x_n), x_1) < \epsilon$.

A chain recurrent set has a **hyperbolic structure** provided the tangent bundle equals $E^u \oplus E^c \oplus E^s$, each invariant under $D\phi_t$ where E^c is spanned by the vector generating ϕ_t and there are positive constant C and a such that

$$||D\phi_t(\mathbf{v})|| \leq Ce^{-at}||\mathbf{v}|| \text{ for } \mathbf{v} \in E^s, t \geq 0$$

and

$$||D\phi_t(\mathbf{v})|| \leq Ce^{at}||\mathbf{v}|| \text{ for } \mathbf{v} \in E^u, t \geq 0.$$

Theorem [**Smale**]. Suppose the chain recurrent set $\mathcal{R}$ of a flow has a hyperbolic structure. Then the connected components of $\mathcal{R}$ are a finite disjoint union of compact invariants set $\mathcal{R}_i$, $i = 1, \ldots, k$ and each $\mathcal{R}_i$ contains a orbit that is dense in $\mathcal{R}_i$. The $\mathcal{R}_i$ are called the **basic sets** of the flow.

The set of points moving toward an orbit as $t \to \infty$ forms is **stable manifold**.

The set of points moving toward an orbit as $t \to -\infty$ forms is **unstable manifold**.

If the topology of a flow is preserved under small perturbations the flow is **structural stable**.

Nonsingular Morse-Smale (NMS) Flows

Definition 1 A flow ϕ on a manifold M is a **Nonsingular Morse-Smale flow** if the following hold.

- The chain recurrent set is hyperbolic.
- The stable and unstable manifolds of basic sets meet transversely.
- Each basic set consists of a single periodic orbit.

For M a compact manifold, it follows that NMS flows have a finite number of periodic orbits.

Notation: The periodic orbits are indexed by: 0 for an attractor, 1 for a saddle, and 2 for a repeller.

Not all 3-manifolds can support NMS flows. John Morgan (1979) has given a theorem that characterizes just which 3-manifolds do. In higher dimensions Asimov (1975) has shown that all manifolds with Euler characteristic 0 support NMS flows.

Wada's Theorem

Which indexed links can be realized as invariant sets of NMS flows of S^3?

Theorem 1 (Wada) Let $\mathcal{F}$ be the set of indexed links which can be realized as the collection of periodic orbits of a Nonsingular Morse-Smale flow on S^3, respecting index. Then $\mathcal{F} = \mathcal{W}$, where $\mathcal{W}$ is defined on the next few transparencies.

Wada's Moves

Definition 2 Let $\mathcal{W}$ be the collection of indexed links determined by the following axioms:

W0: The Hopf link indexed by 0 and 2 is in $\mathcal{W}$.

W1: If $L_1, L_2 \in \mathcal{W}$ then $L_1 \circ L_2 \circ u \in \mathcal{W}$, where u (here and below) is an unknot in S^3 indexed by 1.

W2: If $L_1, L_2 \in \mathcal{W}$ and K_2 is a component of L_2 indexed by 0 or 2, then $L_1 \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W3: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 with indices 0 and 2 (resp.), then $(L_1 - K_1) \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W4: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 (resp.) each with index 0 or 2, then

$$((L_1, K_1)\#(L_2, K_2)) \cup m \in \mathcal{W},$$

where $K_1 \# K_2$ shares the index of either K_1 or K_2 and m is a meridian of $K_1 \# K_2$ indexed by 1.

W5: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L replacing a tubular neighborhood of K with a solid torus with three closed orbits, K_1, K_2, and K_3. K_1 is the core and so has the same knot type as K. K_2 and K_3 are parallel (p, q) cables of K_1. The index of K_2 is 1. The indices of K_1 and K_3 may be either 0 or 2, but at least one of them must be equal to the index of K.

W6: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L by changing the index of K to 1 and placing a $(2, q)$-cable of K in a tubular neighborhood of K, indexed by i.

W7: $\mathcal{W}$ is minimal. That is, $\mathcal{W} \subset \mathcal{W}'$ for any collection, $\mathcal{W}'$, satisfying W0-W6.

Remark: The last condition, W7, means that $\mathcal{W}$ is generated from the indexed Hopf link in S^3 by applying operations W1-W6.

Transverse Foliations to Flows

Definition 3 A **2-dimensional foliation** $\mathcal{F} = \{L_\alpha\}$ of a 3-manifold M is a partition of M such that $\forall x \in M\ \exists$ a chart $(U_x, \phi : U_x \to \mathbb{R}^3)$ such each connected component of $\phi(U \cap L_\alpha)$ is of the form $\{(x, y, z) \in \phi(U_x) \,|\, z = \text{ a constant}\}$. The L_α's are called the leaves of the foliation.

Definition: An indexed link on a 3-manifold has the **Linking Property** if for every closed orbit that bounds a disk, there is an attracting or repelling closed orbit that has nonzero algebraic linking number with that disk.

Theorem [Goodman; Yano] A nonsingular Morse-Smale flow on a 3-manifold has a transverse 2-dimensional foliation (each flow line meets any leaf transversely) if and only if its periodic orbits satisfy the linking property.

Wada Moves and Transverse Foliations

Theorem [S-2008]: The set of indexed links that can be realized as the set of periodic orbits of nonsingular Morse-Smale flows on S^3 that have transverse foliations is the subset of $\mathcal{W}$ generated by W0, W4, W5, & W6.

Bolt-Integrable Hamiltonian Systems

Let M^4 be a compact, smooth, 4-manifold with a symplectic structure. Denote local coordinates by (p_1, p_2, q_1, q_2). Let $h : M^4 \to \mathbb{R}$ be a smooth function. The **skew-symmetric gradient** of h yields the **Hamiltonian vector field**

$$\text{sgrad } h = \left\langle -\frac{\partial h}{\partial q_1}, -\frac{\partial h}{\partial q_2}, \frac{\partial h}{\partial p_1}, \frac{\partial h}{\partial p_2} \right\rangle.$$

Let $r \in \mathbb{R}$ be a regular value of h. Then $Q = h^{-1}(r)$ is a 3-manifold and sgrad h induces a flow on Q.

This flow is **Bolt-integrable** if there exists a smooth real valued function f on some neighborhood U of Q in M^4 such that ...

(a) The functions f and h are *independent* – meaning their gradients are linearly independent at each point.

(b) There exists a function $\lambda : \mathbb{R} \to \mathbb{R}$ such that the Poisson bracket $\{f, h\} \equiv \sum_{i=1,2} f_{p_i} h_{q_i} - h_{p_i} f_{q_i}$ can be written as $\{f, h\} = \lambda \circ h$. (The Poisson bracket depends only on the "energy level".)

(c) At $r = h(Q)$ we have $\lambda(r) = \lambda'(r) = 0$.

(d) The set of critical points of f on Q is the union of disjoint non-degenerate submanifolds. In fact it is made up of circles and tori if we assume f is orientable.

The Link between NMS Flows & Hamiltonian Systems

The non-degeneracy requirement allows us to index the loops of critical points as attracting, repelling or saddle-like.

In 1998 Casasayas, Alfaro, & Nunes studied such indexed links of fixed points of Hamiltonian systems.

They showed that these links were a subset of the NMS links.

On S^3 they showed that this subset of $\mathcal{W}$ is generated by W0, W4, W5, & W6. This is the same subset in Theorem [S-2008].

Contact Flows: Ghrist & Etnyre

In 1999 Ghrist & Etnyre studied gradient flows of 3-manifolds tangent to plane fields associated to a contact structure. In these flows there are indexed links of fixed points.

They showed that these links were a subset of the NMS links.

On S^3 they showed that this subset of $\mathcal{W}$ is generated by W0, W4, W5, & W6. This is the same subset in Theorem [S-2008].

$\mathbf{S}^2 \times \mathbf{S}^1$

A natural question is whether this result holds for all 3-manifolds that can be realized by $Q = h^{-1}(r)$ for some Hamiltonian h.

The problem in approaching this is that we do not have "Wada moves" to generate the NMS flows on other 3-manifolds. However, one can establish a partial result on $S^2 \times S^1$.

Cordero, Martinez and Vindel have defined a series of Wada-like moves that generate NMS flows of $S^2 \times S^1$, but they have not shown completeness. However, the ones they generate that correspond to Hamiltonian systems do satisfy the linking condition of Goodman and Yano.