Knotted Periodic Orbits in Flows

Michael Sullivan

Southern Illinois University

March 2018

Harris-Stowe University

mikesullivan (at) math (dot) siu (dot) edu
http://www.math.siu.edu/sullivan

Knots and Links

Knots and Links

NOTATION.

- $\mathbb{R}^n$ is the Euclidean space of dimension n.
- $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
- $S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1\} \approx \mathbb{R}^3 \cup \{\infty\}$.

Knots and Links

NOTATION.

- $\mathbb{R}^n$ is the Euclidean space of dimension n.
- $S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$.
- $S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \,|\, w^2 + x^2 + y^2 + z^2 = 1\} \approx \mathbb{R}^3 \cup \{\infty\}$.

DEFINITION. A **knot** is a smooth one-to-one map K of S^1 into $\mathbb{R}^3$ or S^3.

$$K : S^1 \to \mathbb{R}^3 \text{ or } S^3$$

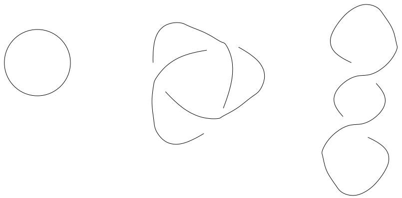
Knots and Links

NOTATION.

- $\mathbb{R}^n$ is the Euclidean space of dimension n.
- $S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$.
- $S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \,|\, w^2 + x^2 + y^2 + z^2 = 1\} \approx \mathbb{R}^3 \cup \{\infty\}$.

DEFINITION. A **knot** is a smooth one-to-one map K of S^1 into $\mathbb{R}^3$ or S^3.

$$K : S^1 \to \mathbb{R}^3 \text{ or } S^3$$



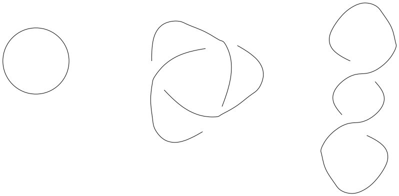
Knots and Links

NOTATION.

- $\mathbb{R}^n$ is the Euclidean space of dimension n.
- $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
- $S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1\} \approx \mathbb{R}^3 \cup \{\infty\}$.

DEFINITION. A **knot** is a smooth one-to-one map K of S^1 into $\mathbb{R}^3$ or S^3.

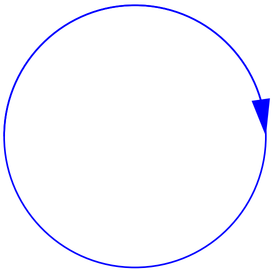
$$K : S^1 \to \mathbb{R}^3 \text{ or } S^3$$

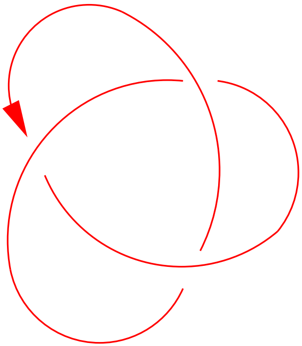


Two knots are **equivalent** (have the same **knot-type**) if one can be smoothly deformed into the other.

Knots can be Given an Orientation

Knots can be Given an Orientation





Knots and Links

Knots and Links

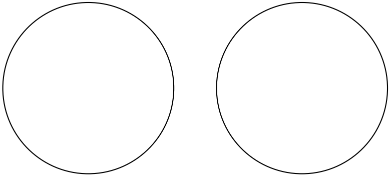
DEFINITION. A **link** is a smooth one-to-one map L of a disjoint union of a finite number of copies of S^1 into $\mathbb{R}^3$ or S^3.

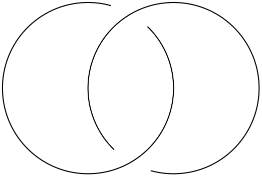
$$L : S^1 \times \{1\} \cup \cdots \cup S^1 \times \{n\} \to \mathbb{R}^3 \text{ or } S^3$$

Knots and Links

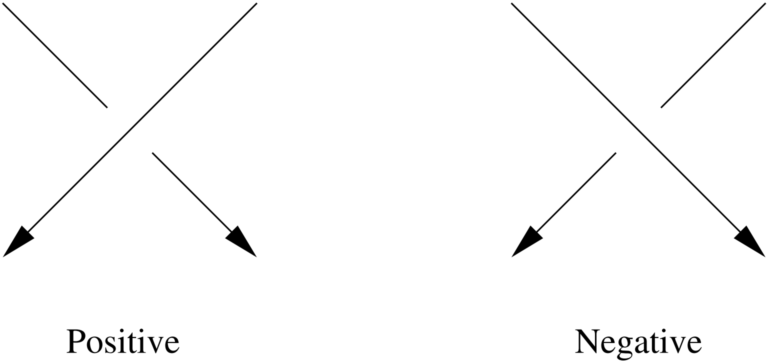
Definition. A **link** is a smooth one-to-one map L of a disjoint union of a finite number of copies of S^1 into $\mathbb{R}^3$ or S^3.

$$L : S^1 \times \{1\} \cup \cdots \cup S^1 \times \{n\} \to \mathbb{R}^3 \text{ or } S^3$$





There are Two Crossing Types



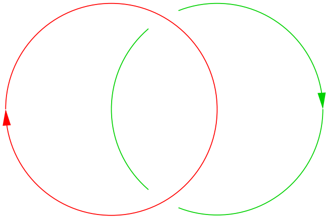
Linking Number

Linking Number

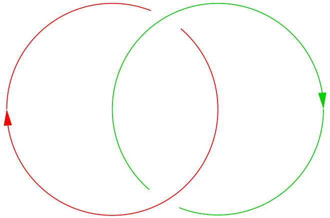
Given an oriented link of two components the **linking number** is one half the sum of the crossing signs for each crossing of the two component knots.

Linking Number

Given an oriented link of two components the **linking number** is one half the sum of the crossing signs for each crossing of the two component knots.



Linking Number = 0.



Linking Number = -1.

Linking

Linking

Two knots are said to be **unlinked** is they can be smoothly deformed so that one component is inside a ball while the other is outside. Otherwise they are **linked**.

Linking

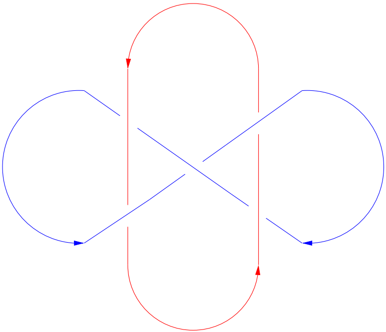
Two knots are said to be **unlinked** is they can be smoothly deformed so that one component is inside a ball while the other is outside. Otherwise they are **linked**.

If two knots are unlinked, their linking number is zero. The converse is false.

Linking

Two knots are said to be **unlinked** is they can be smoothly deformed so that one component is inside a ball while the other is outside. Otherwise they are **linked**.

If two knots are unlinked, their linking number is zero. The converse is false.



Whitehead link.

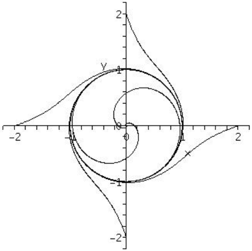
A Knot in a Flow

A Knot in a Flow

$$\begin{aligned}\dot{x} &= x + y - x(x^2 + y^2)\\ \dot{y} &= -x + y - y(x^2 + y^2)\end{aligned}$$

A Knot in a Flow

$$
\begin{aligned}
\dot{x} &= x + y - x(x^2 + y^2) \\
\dot{y} &= -x + y - y(x^2 + y^2)
\end{aligned}
$$



Source: Boyce & DiPrima, 10^{th} edition, page 566.

A Flow with Lots of Knots

A Flow with Lots of Knots

LORENZ EQUATIONS.

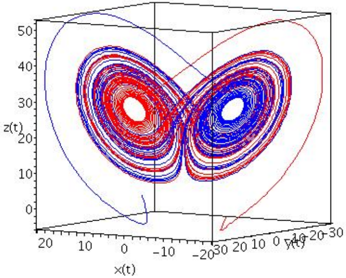
$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned}$$

$$\sigma = 10 \qquad r = 28 \qquad b = 8/3$$

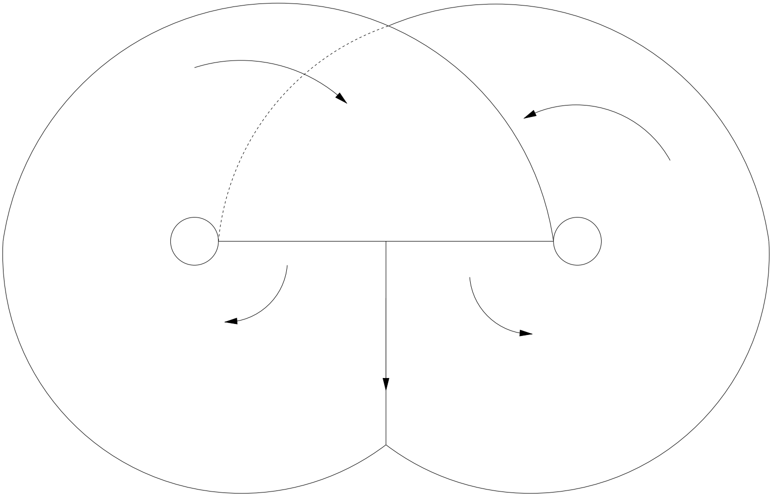
LORENZ EQUATIONS.

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= -bz + xy\end{aligned}$$

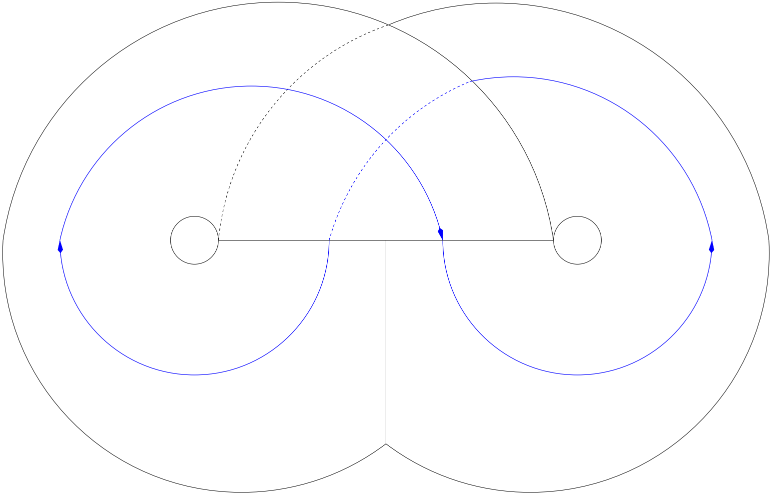
$$\sigma = 10 \qquad r = 28 \qquad b = 8/3$$



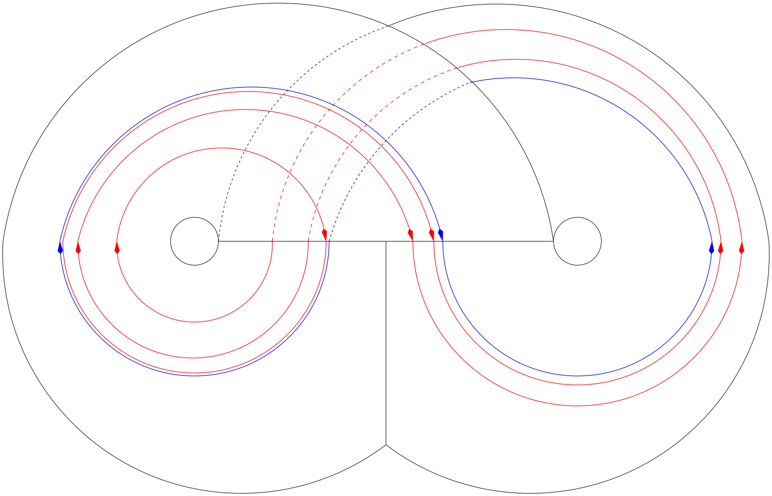
A Template for Lorenz Orbits



A Template for Lorenz Knots



A Template for Lorenz Knots



How many different Lorenz knots are there?

How many different Lorenz knots are there?

Theorem. There are infinitely many distinct Lorenz knots. [Franks & Williams, 1983]

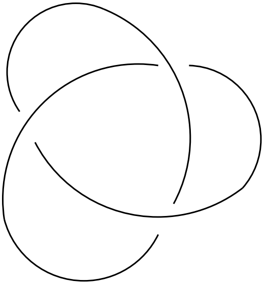
Lorenz Knots are Prime

Lorenz Knots are Prime

This was proven by Williams in 1984. What does it mean?

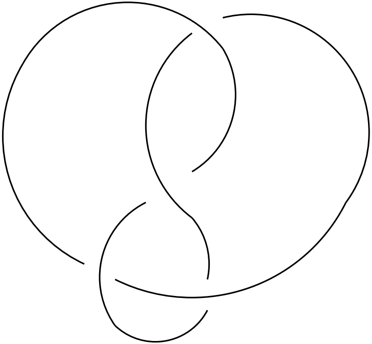
Lorenz Knots are Prime

This was proven by Williams in 1984. What does it mean?



Trefoil

Prime

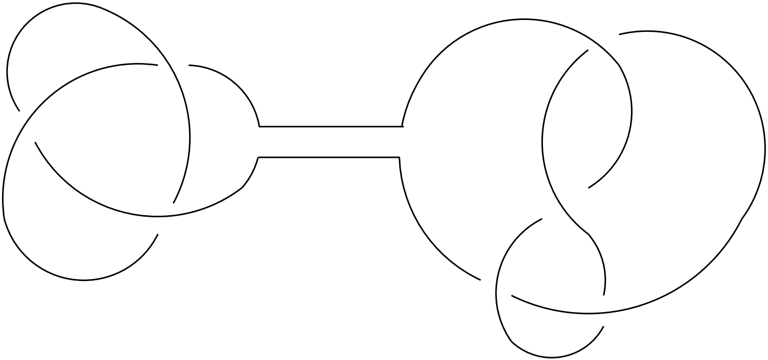


Figure–8

Prime

Lorenz Knots are Prime

This was proven by Williams in 1984. What does it mean?



Trefoil # Figure–8

Composite

Any two interior Lorenz knots are Linked

Any two interior Lorenz knots are Linked

Interior means we exclude the two unknots in the boundary of the Lorenz template.

Any two interior Lorenz knots are Linked

Interior means we exclude the two unknots in the boundary of the Lorenz template.

Proof: Any pair of knots in the interior of the Lorenz template must cross and all the crossings in the Lorenz template are of the same sign. Thus, the linking number will be nonzero.

Torus Knots

Torus Knots

A **torus** is the surface of a donut shape. More technically

$$T^2 = S^1 \times S^1.$$

Torus Knots

A **torus** is the surface of a donut shape. More technically

$$T^2 = S^1 \times S^1.$$

A **torus knot** is a knot that can be embedded into a torus.

Torus Knots

A **torus** is the surface of a donut shape. More technically

$$T^2 = S^1 \times S^1.$$

A **torus knot** is a knot that can be embedded into a torus.

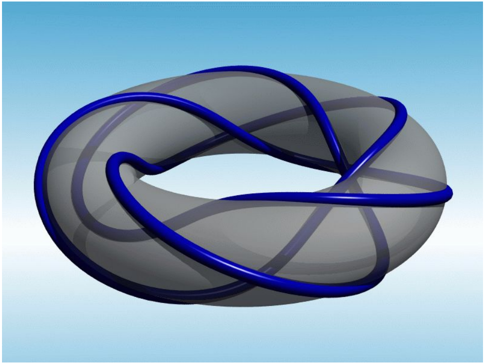


Image source: http://makerhome.blogspot.com/2014/01/day-150-trefoil-torus-knots.html

Torus Knots are Lorenz Knots

Torus Knots are Lorenz Knots

Proof!

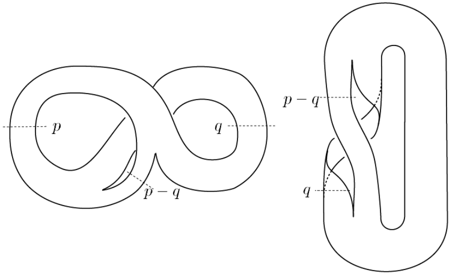


Image source: Knots and Links in Three-Dimensional Flows, by Ghrist, Holmes, Sullivan.

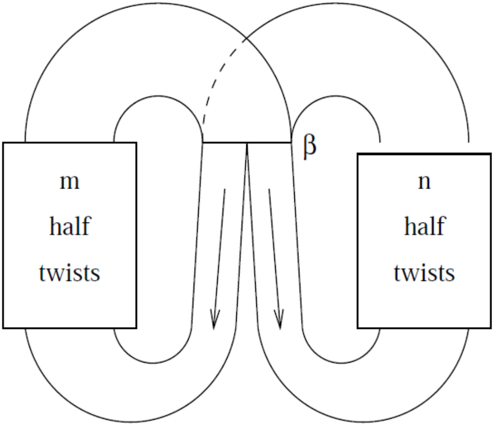
Lorenz-like Templates

Lorenz-like Templates

If we put some number of twists in each band we get a **Lorenz-like template**. They are denoted by $L(m, n)$ where m and n give the number of half twists in the respective bands as shown.

Lorenz-like Templates

If we put some number of twists in each band we get a **Lorenz-like template**. They are denoted by $L(m, n)$ where m and n give the number of half twists in the respective bands as shown.



Template Relationships

Template Relationships

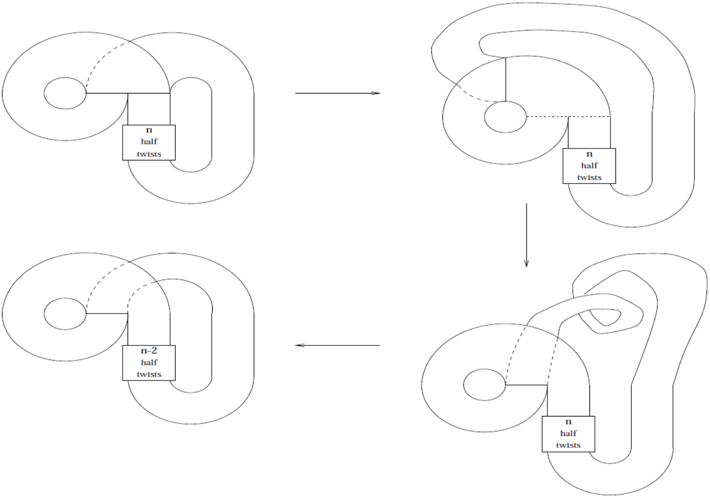
Theorem. Regarding $L(m,n)$ as the set of all knots in $L(m,n)$ we have

$$\cdots \supset L(0,-4) \supset L(0,-2) \supset L(0,0) \supset L(0,2) \supset \cdots$$

and

$$\cdots \supset L(0,-3) \supset L(0,-1) \supset L(0,1) \supset L(0,3) \supset \cdots$$

Proof!



Template Relationships

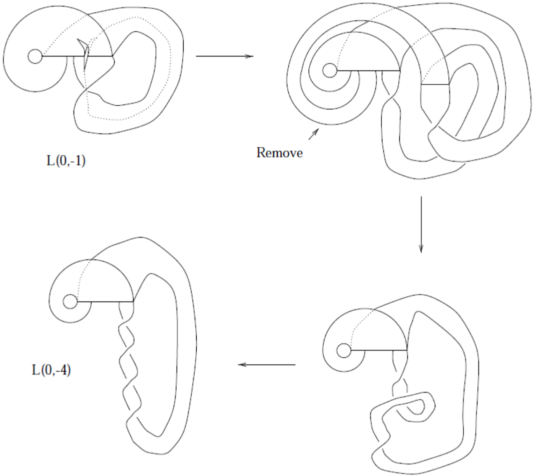
Template Relationships

Theorem. $L(0,-4) \subset L(0,-1)$.

Template Relationships

Theorem. $L(0, -4) \subset L(0, -1)$.

Proof!



Universal Templates

Universal Templates

A template is said to be **universal** if it contains all knots.

Universal Templates

A template is said to be **universal** if it contains all knots.

Birman and Williams had conjectured these did not exist.

Universal Templates

A template is said to be **universal** if it contains all knots.

Birman and Williams had conjectured these did not exist.

Theorem. $L(0, -1)$ contains all knots. [Ghrist, 1996]

Universal Templates

A template is said to be **universal** if it contains all knots.

Birman and Williams had conjectured these did not exist.

Theorem. $L(0, -1)$ contains all knots. [Ghrist, 1996]

The proof is hard.

An ODE with all knots

An ODE with all knots

Theorem. There exists an open set of parameters $\beta \in [6.5, 10.5]$ for which periodic solutions to the differential equation

$$\begin{aligned}\dot{x} &= 7|y - \phi(x)| \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y\end{aligned}$$

where

$$\phi(x) = 2x/7 - 3\left[|x+1| - |x-1|\right]/14,$$

are modeled by $L(0, -1)$.
[Ghrist & Holmes, 1996]

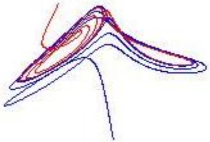
Robinson's Attractor and Templates

Robinson's Attractor and Templates

$$\begin{aligned}
\dot{x}(t) &= y \\
\dot{y}(t) &= x - 2x^3 - 0.71y + 1.644x^2 y + yz \\
\dot{z}(t) &= -0.7061z + 0.1x^2
\end{aligned}$$

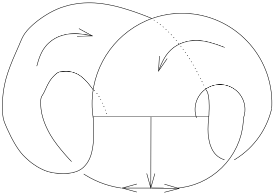
Robinson's Attractor and Templates

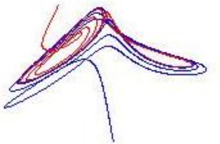
$$\begin{aligned}
\dot{x}(t) &= y \\
\dot{y}(t) &= x - 2x^3 - 0.71y + 1.644x^2y + yz \\
\dot{z}(t) &= -0.7061z + 0.1x^2
\end{aligned}$$



Robinson's Attractor and Templates

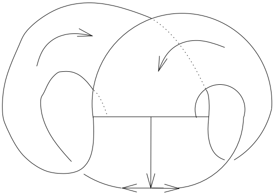
$$
\begin{aligned}
\dot{x}(t) &= y \\
\dot{y}(t) &= x - 2x^3 - 0.71y + 1.644x^2y + yz \\
\dot{z}(t) &= -0.7061z + 0.1x^2
\end{aligned}
$$

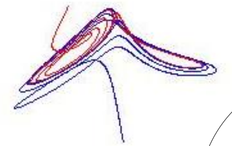


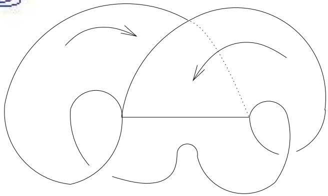


Robinson's Attractor and Templates

$$
\begin{aligned}
\dot{x}(t) &= y \\
\dot{y}(t) &= x - 2x^3 - 0.71y + 1.644x^2 y + yz \\
\dot{z}(t) &= -0.7061z + 0.1x^2
\end{aligned}
$$

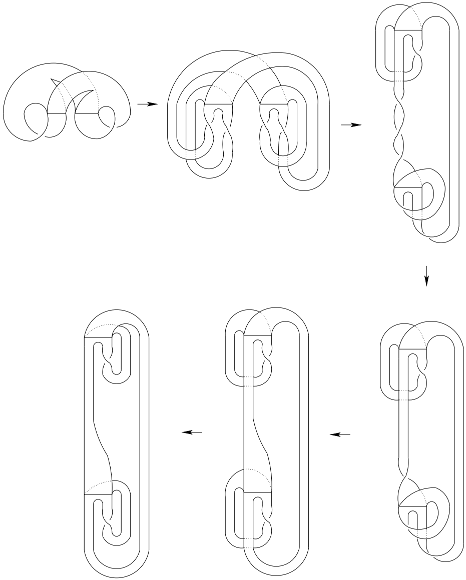






An Alternative form of $L(-1,-1)$

An Alternative form of $L(-1, -1)$



Current work

Current work

It is not universal because ...

Current work

It is not universal because ...

$L(-1,-1)$ knots can be presented with all crossing the same sign.

Current work

It is not universal because ...

$L(-1,-1)$ knots can be presented with all crossing the same sign.

My former PhD student Ghazwan Alhashimi has shown that all $L(-1,-1)$ knots are fibered, but this is rather technical.

Current work

It is not universal because ...

$L(-1,-1)$ knots can be presented with all crossing the same sign.

My former PhD student Ghazwan Alhashimi has shown that all $L(-1,-1)$ knots are fibered, but this is rather technical.

$L(-1,-1)$ contains composite knots. It is not known what the maximum number of prime factors is. My current PhD student Mansour Faraj is going to try to find out.