Twistwise Flow Equivalence

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I. Symbolic Dynamics

SFT, shift maps, top. conj., SSE, computable?

II. Flow Eq.

SSE + P.S. $\Leftrightarrow$ FE. Complete computable invariants known.

III Smale Flows on 3-manifolds

Chain recurrent set, hyperbolicity, basic sets, saddle sets, templates, stable manifold twisting

IV. TWFE.

Matrices over $\mathbb{Z}_+$ $\mathbb{Z}/2$. Invariants. Complete?

V. K-Thy approach.

$SL(\mathbb{N}, \mathbb{Z}G)$, elementary matrices, PE, BFE, complete alg. invariant, but is it computable?

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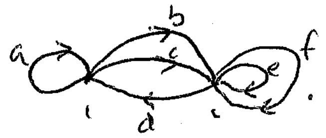
I. Symbolic Dynamics

Given an $n \times n$ matrix over the nonneg. integers $\mathbb{Z}_+$ we determine a shift of finite type (SFT) as in this example.

Let $M = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Then G_M is . $E_M = \{a, \dots, f\}$.

X_M = all bi-infinite sequences of letters (edges) from E_M that are allowed as paths in G_M.

Ex: $\dots aa.bfedaac \dots \in X_M$ but $\dots ad \dots \notin X_M$.

Fact $(M^p)_{ij}$ = # paths of length p from vertex i to vertex j.
Hence $\operatorname{tr} M^p$ = # of loops of length p.

Metric $d(\bar{x}, \bar{y}) = \sum_{i=-\infty}^{\infty} \frac{\delta(x_i, y_i)}{2^{|i|}}$, $\delta(x_i, y_i) = \begin{cases} 0 & x_i = y_i \\ 1 & x_i \neq y_i \end{cases}$.

Def M is irreducible if $\forall ij \; \exists p$ s.t. $(M^p)_{ij} \neq 0$.

Fact If M is irreducible and not a permutation matrix then X_M is a Cantor space.

The shift map $\sigma: X_M \to X_M$ is $(\sigma(\bar{x}))_i = x_{i+1}$. It is a "step along a path". It is a homeo.

Ex $\sigma(\dots aa.bfdaac \dots) = \dots aab.fdaac \dots$

X_M and σ form a dynamical system.

Two SFTs, X_A and X_B are <u>topologically conjugate</u> if $\exists$ homeo $h: X_A \to X_B$ s.t.

$$\begin{tikzcd} X_A \arrow[r, "\sigma"] \arrow[d, "h"'] & X_A \arrow[d, "h"] \\ X_B \arrow[r, "\sigma"] & X_B \end{tikzcd}$$

commutes.

This implies periodic seq's are taken to per. seq's of the same least period. As dynamical systems they are equivalent.

<u>Q</u>: When are two SFT's top. conj.?

Let A and B be sq. matrices over $\mathbb{Z}_+$. They need not be the same size. An <u>SSE-move</u> from A to B is a dual factoring

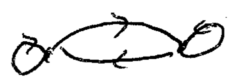
$$A = RS \qquad B = SR$$

for R and S over $\mathbb{Z}_+$ and not necessarily square.

If $\exists$ a finite seq. of SSE-moves from A to B we say A and B are <u>strong shift equivalent (SSE)</u>.
(It is an eq. relation.)

<u>Thm</u> (Williams '73*) X_A is top. conj to X_B iff $A \overset{SSE}{\sim} B$.

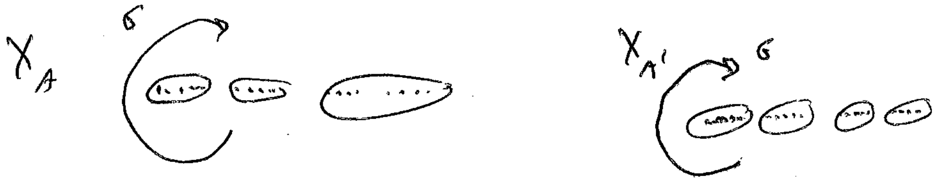
<u>Ex</u> $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $[2] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.




Notice $\mathrm{tr} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = \mathrm{tr}\, [2]^k$.

* I said 79 in the talks, but it was 1973.

The basic idea behind the proof is that:
~~$X_A \overset{\text{top. conj}}{\sim} X_B$~~ Given X_A we can recover A by putting a Markov partition on X_A and defining A_{ij} = # of "pieces" of $\sigma(M_i) \cap M_j$. We can change the partition by "splitting" or "amalgimations" and get a new matrix, say A', with $X_A \sim X_{A'}$.

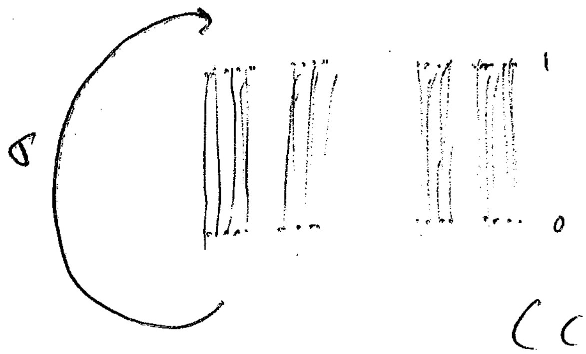


~~Any~~ X_A is top. conj to X_B iff $\exists$ seq of these moves. These moves are SSE-moves and any SSE-move can be factored into "splittings" and "amalgimations".

Many invariants of SSE are known. But it is not known if $\exists$ an algorithm to decide SSE. This is a big ~~open~~ and long standing <u>open question</u>!

II. Flow Equivalence (FE)

Let X_A be a SFT. Let $F_A = X_A \times [0,1] / (x,0) \sim (\sigma(x),1)$.



$\downarrow \phi_t$ Place a flow on F_A as indicated.

(Cantor spaghetti ??)

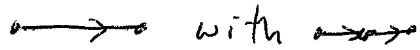
F_A, ϕ_t is the <u>suspension flow</u> for X_A, σ.

Two susp. flow (F_A, ϕ_t) and (F_B, ψ_t) are <u>top. conj.</u> if $\exists$ homeo : $F_A \to F_B$ that takes flows lines to flow lines preserving the flow direction.

We say A is <u>flow eq. (FE)</u> to B if F_A is top. conj. to F_B.

<u>Ex</u> $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $[1]$ are FE but not SSE.

<u>Easy Fact</u> SSE does imply FE.

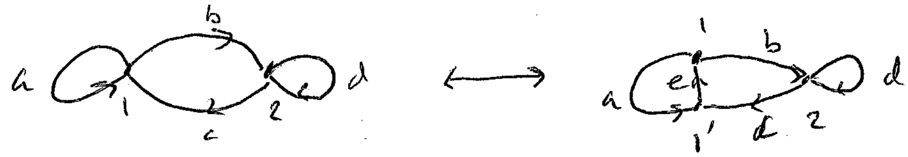
Given two graphs placing an edge  will give ~~FE systems.~~ top. conj. SFT.

Q: How can we decide if two sq matrices over $\mathbb{Z}_+$ are FE?

Def A **PS-move** is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ a_{11} & 0 & a_{12} & \cdots & & \\ a_{21} & 0 & a_{22} & \cdots & & \\ & \vdots & & & & \\ & 0 & & & & \end{bmatrix}$$

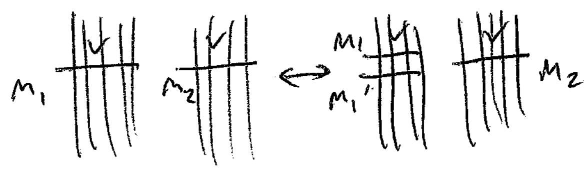
Ex $\begin{bmatrix}1&1\\1&1\end{bmatrix} \xleftrightarrow{PS} \begin{bmatrix}0&1&0\\1&0&1\\1&0&1\end{bmatrix}$.



FE but not SSE.

Looking flows

Markov Partition of cross section



flow did not really change!

Thm's

1. SSE + PS generate FE (Parry and D. Sullivan '75)
2. Let $PS(A) = \det(I-A)$. It is an FE invariant. (P&S '75)
3. Let $BF(A) = \mathbb{Z}^n / (I-A)\mathbb{Z}^n$ (a finitely generated abelian gp.)

   It is an FE invariant (Bowen and Franks '77)

   Note: $|BF(A)| = |PS(A)|$ if $PS(A) \neq 0$ but the sign of $PS(A)$ is an independent invariant.

4. PS + BF form a complete set of invariants for FE. (Franks '84)

These are easy to compute and distinguish using Smith normal form.

III Smale Flows

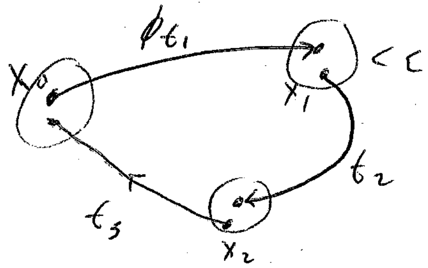
Assume M is a compact 3-manifold. The Smale flows on M are the structurally stable flows with one-dimension chain recurrent sets.

<u>Def</u> For a flow ϕ_t on M, $x \in M$ is <u>chain recurrent</u> if

$\forall\, \varepsilon > 0, T > 0$ $\exists\, x_i \in M$ $(i = 0, \dots, n)$, $x_0 = x$ and t_i, $i = 1, \dots, n$, s.t.

$T < t_1 < t_2 < \cdots < t_n$ and $\| \phi_{t_i}(x_{i-1}) - x_i \| < \varepsilon$ $i = 1, \dots n$

and $\| \phi_{t_n}(x_{n-1}) - x \| < \varepsilon$.



The <u>chain recurrent set $\mathcal{R}(\phi_t)$</u> = all ch. rec pts.
Obviously all periodic orbits are in $\mathcal{R}(\phi_t)$. (If $\mathcal{R}(\phi_t)$ contains only a finite number of periodic orbits and fixed pts the flow is a <u>Morse-Smale flow</u>.)

<u>Thm</u> (Smale '67) $\mathcal{R} = \bigcup_{\text{finite}} \mathcal{B}_i$ s.t. each $\mathcal{B}_i$ is compact, contains a dense orbit and is invariant under the flow. They are called the <u>basic sets</u>

In Smale flow (on 3-manifolds) there are three types of basic sets: attractors, repellers and saddle sets. (I.e., they are <u>hyperbolic</u>.) The first two must be isolated closed orbits. Saddle sets can also be susp. of nontrivial SFT. We call these basic sets <u>chaotic saddle sets</u>.

Each orbit γ of a saddle set has a 2-dim (un) stable mfld consisting of orbits that converge to γ as $t \to \infty$ ($t \to -\infty$).

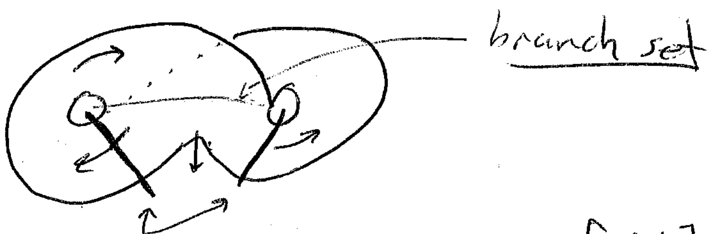
The ε-local (un) stable mfld is the 2ε width "ribbon" in the (un) st. mfld that contains γ as its core. It is either an annulus, a Möbius band or homeo to $\mathbb{R} \times [-\varepsilon, \varepsilon]$. There is a corresponding ribbon in the normal bundle.

(The Cantor spaghetti becomes Cantor fettuccine!)

Templates are branched 2-manifolds with semi-flows used to model chaotic saddle sets.

A template for a saddle set is constructed by taking a "tight" enough nbhd and collapsing out the local stable manifolds.

Ex Lorenz template, $L(0,0)$.

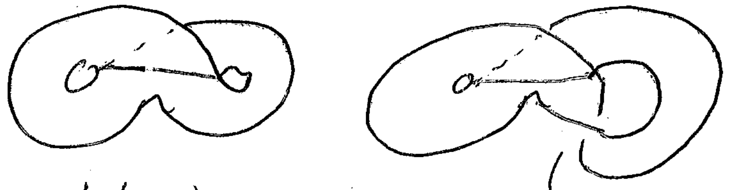


$\rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Different cross sections (or "tighter" nbhds) give different, but FE, incidence matrices.

IV Twistwise Flow Eq (TWFE)

Now consider



$L(0,0)$

half twist

$L(0,1)$ = "Horse shoe template"

They both have $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ as incident matrices, so the corresponding saddle sets are top. conj. But what about the twisting? How to capture this? Idea: for $L(0,1)$ use $\begin{bmatrix}1&1\\t&t\end{bmatrix}$ and set $t^2 = 1$.

More formally let m_i be disks for a "fine enough" markov partition. Then let ρ (rho) be the "first return map". Define the <u>twist matrix</u> to be

$$A(t)_{ij} = \begin{cases} 1 & \rho(m_i) \cap m_j \neq \phi \quad \rho|m_i \text{ is orientation preserving} \\ t & \rho(m_i) \cap m_j \neq \phi \quad \rho|m_i \text{ is orientation reversing} \\ 0 & \rho(m_i) \cap m_j = \phi . \end{cases}$$

Since one works with power of $A(t)$ think of it as being over $\{a+bt \mid a,b \in \mathbb{Z}_+\} \cong \mathbb{Z}_+(\mathbb{Z}/2)$.

↳ semi-group ring.

Given a twist matrix one can construct a template and hence a flow on a susp. SFT, together with the twisting data. The latter can be used to determine a "ribbon set" (which lives in the tangent bundle of M for techincal reasons). Two ribbon sets, $R_{A(t)}$ and $R_{B(t)}$, are <u>top. eq.</u> if $\exists$ homeo : $R_A \to R_B$ that takes annuli to annuli, Mobius bands to Mobius band, and infinite strips to infinite strips preserving to flow direction. If $R_{A(t)} \sim R_{B(t)}$ we say $A(t)$ is <u>TWFE</u> to $B(t)$.

<u>Q</u>: How can we decide when two twist matrices are TWFE?

Here are some invariants I found.

Let $PS^+(A(t)) = PS(A(1))$, and $PS^-(A(t)) \equiv PS(A(-1))$.

Obviously PS^+ is an invariant, but so is PS^- and it is not redundant.

Let $BF^{\pm}(A(t)) = BF(A(\pm 1))$. Both groups are invariants

Next let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Notice $T^2 = I$. Let $A(T)$ be $2n \times 2n$ given by $a_{ij} + b_{ij}t \longmapsto a_{ij} I + b_{ij} T = \begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{bmatrix}$.

Let $BF^{\partial}(A(t)) = BF(A(T))$. It is an inv. and is independent of the others. (However, $PS(A(T)) = PS^+(A(t)) \times PS^-(A(t))$.)

And, let $O(A(t)) = \begin{cases} \text{"orienable" is no Mobius bands in ribbon set} \\ \text{"nonorienable" otherwise.} \end{cases}$

It is computable by checking $\operatorname{tr} A(t)^k$ for $k = 1, \ldots, n$ ($A(t)$ is $n \times n$). ~~But is it independent~~? Are these complete?)

<u>Ex</u> $\begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ are not distinguished by these invariants. Are they TWFE?

I had no way to tell!

V. K-thy

This is joint work with Mike Boyle.

Many details will be skipped!

Let $SL(\underbrace{\mathbb{N}}_{\{1,2,3\dots\}}, \mathbb{Z}\overbrace{G}^{\text{finite gp}})$ = $\mathbb{N} \times \mathbb{N}$ matrices that = infinite identity matrix off an $n \times n$ block and have det = 1.

Let $E_{ij}(g)$ = infinite id matrix but with ij entry $g \in G$, $i \neq j$.

For $G = \mathbb{Z}/2$ or trivial it is known $\underbrace{\text{these}}_{\text{elementary matrices}}$ generate $SL(\mathbb{N}, \mathbb{Z}G)$.

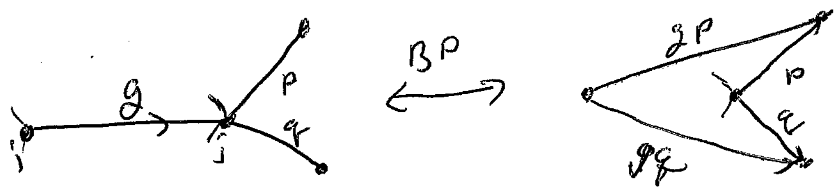
__Notation__: ~~Let A~~ Given A an $n \times n$ matrix let A_∞ be the $\mathbb{N} \times \mathbb{N}$ matrix = to A on the upper left block and 0 elsewhere. We use I for the $\mathbb{N} \times \mathbb{N}$ id.

__Def__ Given twist matrices A and B we say there is a __basic positive move (BP)__ from A to B if $\exists$ $E_{ij}(g)$ $(g = 1 \text{ or } t)$ s.t.

$$E_{ij}(g)(I - A_\infty) = I - B_\infty, \quad (I - A_\infty)E_{ij}(g) = I - B_\infty$$

$$E_{ij}^{-1}(g)(I - A_\infty) = I - B_\infty, \quad \text{or} \quad (I - A_\infty)E_{ij}^{-1}(g) = I - B_\infty.$$

__Picture__:



If $\exists$ a finite seq. of BP moves from A to B we say A is __positive equivalent__ to B (PE)

Thm If G = trivial gp PE $\Leftrightarrow$ FE (Boyle '02)
If $G \cong \mathbb{Z}/2$ PE $\Leftrightarrow$ TWFE (Boyle and S. '05)

Def Let A and B be $\mathbb{N} \times \mathbb{N}$ matrices. If $\exists\, U, V \in SL(\mathbb{N}, \mathbb{Z}G)$ such that $UA_\infty V = B_\infty$, we say A and B are <u>$SL(\mathbb{N}, \mathbb{Z}G)$-equivalent.</u>

Thm Let $G \cong \mathbb{Z}/2$. Let A and B be irreducible SEG matrices over $\mathbb{Z}_+G$ and assume $\mathcal{O}(A) = \mathcal{O}(B)$.
Then $A \overset{TWFE}{\sim} B$ iff there is an $SL(\mathbb{N}, \mathbb{Z}G)$-equivalence from $I - A_\infty$ to $I - B_\infty$, i.e. $\exists\, U, V$ in $SL(\mathbb{N}, \mathbb{Z}G)$ s.t $U(I - A_\infty)V = I - B_\infty$.

The proof is that even though U, V need not be positive they can be factored into BP moves.

Thus, we have that TWFE is determined by a "natural" algebraic equivalence.

__Ex__ $A = \begin{bmatrix} 0 & t \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$. We asked before if they were TWFE. Let $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Then $E(I-A) = I-B$. So they are. Notice this is __not__ a BP-move since there is not a 1 in the A_{12} entry. But if one follows the proof one can find that:

Let $Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $Q_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Then

$$(I, Q_1), (I, Q_2), (E, I), (I, Q_2^{-1}), (I, Q_1^{-1})$$

is a seq. of BP-moves from $I-A$ to $I-B$.

But is $SL(N, \mathbb{Z}G)$ equivalence computable? If $\mathbb{Z}G$ is a PID, then yes, because there exist a Smith normal form. But, sadly, $\mathbb{Z}(\mathbb{Z}/2)$ is not a PID: $(1-t)(1+t) = 0$, so it has zero divisors. It is __unknown__ if $SL(N, \mathbb{Z}\mathbb{Z}/2)$ is computable!!

On my preprints page see joint paper with Boyle and "Twistwise Flow Eq and Beyond..."