Recent Developments in the Theory of Smale Flows

Abstract. We review some recent work by Bin Yu and Elizabeth Haynes concerning which manifolds support certain classes of nonsingular Smale flows.

Spring Topology and Dynamical Systems Conference
Special Session on Dynamics

March 2012

Mexico City

Mike Sullivan

SIU Carbondale

mikesullivan (at) math (dot) siu (dot) edu

http://www.math.siu.edu/sullivan

Smale Flows

Let $\mathcal{M}$ be a smooth compact connected 3-manifold with or without boundary.

A nonsingular Smale flow **(NSF)** on $\mathcal{M}$ is a structurally stable flow with one-dimensional chain recurrent set $\mathcal{R}$. It is hyperbolic on $\mathcal{R}$ and the stable and unstable manifolds only meet transversely. The term was introduced by John Franks in the 1970's.

The chain recurrent set, by a theorem of Smale, consists of a finite number of disjoint **basic sets**, which are compact and transitive. A basic set may be an **attractor**, **repeller** or **saddle set**.

Attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a nontrivial shift of finite type - we call these **chaotic saddle sets**.

Knots and Templates

Basic sets that are closed orbits form knots. While chaotic basic sets contain finitely many knots and knot types. [Franks & Williams]

To get a handle on chaotic saddle sets we use **templates**. These are branched two manifolds with semi-flows formed by taking an isolating neighborhood and collapsing out the local stable manifolds. [Birman & Williams]

Lyapunov Graphs

THEOREM (Conley, 1978): Given a Smale flow ϕ_t on $\mathcal{M}$ there exists a **Lyapunov function**, $\gamma : \mathcal{M} \to \mathbb{R}$. Here γ is decreasing off of $\mathcal{R}$, and takes on distinct constant values on each basic set.

DEFINITION: A **Lyapunov graph** is a finite, connected, oriented graph Γ which possesses no oriented cycles with each vertex labeled by a chain recurrent flow on a compact space (up to topological equivalence).

CONSTRUCTION: One constructs a Lyapunov graph for a flow by identifying to a point components of level sets of a Lyapunov function and labeling the vertices in the obvious way.

Lyapunov graphs were introduced by Franks.

Smale Flows in $\mathbf{S^3}$

THEOREM (Franks 1985): Suppose Γ is a Lyapunov graph whose sinks and sources are labeled with single periodic orbits, and the other vertices are labeled with suspensions of irreducible SFTs. Then Γ is associated with a non-singular Smale flow ϕ_t on S^3 iff the following are satisfied.

(1) The graph Γ is a tree with one edge attached to each source and sink vertex.

(2) Suppose v is any other vertex. Let e^+ be the number of entering edges and e^- be the number of exiting edges. Let A be an incidence matrix for v's SFT, and let $\overline{A}$ be its mod 2 reduction. Let $k = \dim\ker(I - \overline{A}) : \mathbb{Z}/2 \to \mathbb{Z}/2$. Then

$$e^+, e^- \leq k + 1 \leq e^+ + e^-.$$

Embedded Lorenz-Smale Flows in $\mathbf{S^3}$

DEFINITION: A **Lorenz-Smale** flow is a Smale flow with one attracting periodic orbit A, one repelling periodic orbit R, and one saddle set whose template is an embedding of the Lorenz template.

THEOREM (S. 2000): For a Lorenz-Smale flow on S^3, the link $A \cup R$ is either a Hopf link or a trefoil and meridian.

Embedded Lorenz-like Smale Flows in $\mathbf{S^3}$

THEOREM (Bin Yu): For an $\mathcal{L}(1,0)$ Lorenz like Smale flow on S^3 the $A \cup R$ link is either a Hopf link or a trefoil and meridian pair.

THEOREM (Bin Yu): For an $\mathcal{L}(1,1)$ Lorenz like Smale flow on S^3 the $A \cup R$ link is an unknot and a $(p,3)$ torus knot on a torus with core A or R.

Beyond the Lorenz Template

The template U shown below also models the suspended 2-shift. It is known to contain all knots and links [Ghrist].

THEOREM (Liz Haynes): For a simple Smale flow where the saddle is modeled by U the following configurations are realizable. The $A \cup R$ link is either a Hopf link or a figure-8 knot an a meridian. In the latter case U must be standardly embedded (no twists or knotting or linking). In the Hopf link case one of the bands can have any number of full twists and its core will be a torus knot. The bands are still unknotted and unlinked.

Other 3-manifolds

THEOREM (Bin Yu & Liz Haynes): For $n = 0$ (LH) or 1 (BY) $\mathcal{L}(0,n)$ can be realized in a Lorenz-like Smale on only the following 3-manifolds:

S^3, $S^2 \times S^1$, any lens space $L(p,q)$,

the Seifert manifolds $S^2(\frac{1}{2}, \frac{1}{3}, \frac{b}{a})$ & $S^2(\frac{1}{2}, \frac{1}{3}, \frac{b_1}{a_1}, \frac{b_2}{a_2})$

and $L(3,1)\#L(2,1)$.

THEOREM (Bin Yu): $\mathcal{L}(1,1)$ can be realized in a Lorenz-like Smale on only the following 3-manifolds:

S^3, $Y\#L(3,1)$, where Y is S^3, $S^2 \times S_1$ or $L(p,q)$,

and the Seifert manifolds $S^2(\frac{1}{3}, \frac{b_1}{a_1}, \frac{b_2}{a_2})$.

An important tool was a very nice paper by Louise Moser from the 1970's.

Characterizing Level Sets

NOTE: One can define the **weight** of an edge in a Lyapunov graph to be the genus of a corresponding connected level set.

THEOREM (Bin Yu): Let $\mathcal{S}$ be the closed orientable surface of genus $n > 1$.

Then a closed 3-manifold $\mathcal{M}$ admits an NSF with a regular level set homeomorphic to $\mathcal{S}$ iff

$$\mathcal{M} = \mathcal{M}' \#_{n-1} S^2 \times S^1$$

for some closed orientable 3-manifold $\mathcal{M}'$.

Furthermore, the cycle rank of the Lyapunov graph is $\geq n - 1$.

Templates and Manifolds

THEOREM (Bin Yu): Let $\mathcal{T}$ be a template with genus g. Then a closed 3-manifold $\mathcal{M}$ admits an NSF with a basic set modeled by $\mathcal{T}$ iff

$$\mathcal{M} = \mathcal{M}' \#_g S^2 \times S^1$$

for some closed orientable 3-manifold $\mathcal{M}'$.

Furthermore, the cycle rank of the Lyapunov graph is $\geq g$.

These two results build on work of Ketty de Rezende, Vadim Meleshuk and George Frank.

Templates verses Filtrations

DEFINITIONS: A **filtration** for a NSF on $\mathcal{M}$ is a sequence of submanifolds

$$\mathcal{M}_n \subset \mathcal{M}_{n-1} \subset \cdots \subset \mathcal{M}$$

such that the flow is transverse to any boundaries and each $\mathcal{M}_i - \mathcal{M}_{i+1}$ contains one basic set. The sets $\mathrm{cl}(\mathcal{M}_i - \mathcal{M}_{i+1})$ are called **filtrating neighborhoods**.

Using filtrating neighborhoods it is easy to see the manifold but hard to see the dynamics.

Using templates it is easy to see the dynamics but hard to see the manifold.

Bin Yu gives a constructive method for taking a template model to a filtrating neighborhood.

COROLLARY (Bin Yu): There exist at most a finite number of filtrating neighborhoods for a given closed orientable 3-manifold $\mathcal{M}$ and a template $\mathcal{T}$ such that
(1) $\mathcal{T}$ models the invariant sets of these filtrating neighborhoods, and
(2) each of these filtrating neighborhoods can be realized as a filtrating neighborhood of an NSF on $\mathcal{M}$.

Example

Let $\mathcal{T} = \mathcal{L}(1,1)$. Then the filtrating neighborhood must be

$$L(3,1)\#(T^2 \times [0,1])$$

or

$$S^1 \times D^2 - \text{(tubular nbhd of a (3,1)-torus knot)}.$$

This builds on work by Beguin & Bonatti and is similar in spirit to Wada's constructions of "knot flow pieces" for nonsingular Morse Smale flows on S^3.

Open Questions

- Characterize all realizations of the full two shift in S^3.

- Can any two component link be realized as an attractor repeller pair?

  > [I have made some progress here: Given any two knots A and R and $n \geq 0$ there exists a NSF on S^3 with an attractor A and repeller R with linking number n, and up to three saddle sets.]

- Smale flows were originally derived by orbit surgery on Anosov flows. But little has been done to "pull back" Smale flow results to Anosov flows.

- Can template theory answer questions about Plykin-like attractors?

- Templates can also be used to study partially hyperbolic attracting invariant sets but little since Williams has been done here. The symbolic dynamics in this case have infinite alphabets and non isolated fixed points sit on the "edge."

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